Mathematical Proofs and Their Importance

MATH 0413, Introduction to Theoretical Mathematics

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In mathematics we use comprehensive arguments, called proofs, to show that certain statements are considered to be true. Before discussing their importance, it is worthwhile to understand exactly what a proof is. A proof can be considered a series of logical arguments and claims that clearly demonstrate that a said statement is true. Often, we construct proofs using theorems, statements that have been previously proved true, and axioms, statements which can be universally accepted without use of proof. In a world without proofs, we would always be working on conjecture. For example, take the Pythagorean Theorem:

$$a^2 + b^2 = c^2$$

Many people are very familiar with this equation but have not given any thought as to why the statement is true. We must recognize that our whole basis in using the above equation comes from the validation of its many proofs. Without the proofs, nothing we construct from the theorem could be logically accepted.

Four common types of proofs include direct proof, proof by mathematical induction, proof by contradiction, and proof by contrapositive. Direct proofs are formed by assuming the hypothesis is true and then progressing through a series of logical arguments, eventually proving that the conclusion is also true. For those proofs which involve the set of natural numbers, namely $\mathbb{N} = \{1, 2, 3, \ldots\}$, we may use the Principle of Mathematical Induction. This principle states that for some statement based on natural numbers, say $P(n)$, we can prove that the statement is true for *all* $n \in \mathbb{N}$ if

i) The base case, $P(1)$ holds
ii) Inductive step: if $P(n)$ is true, then $P(n+1)$ is true

A third type of proof is proof by contradiction, in which we assume the opposite of our hypothesis and then work our way through the proof to eventually find a contradiction in our logic. Lastly, for proofs in the form $P(x) \rightarrow Q(x)$, i.e., some statement $P(x)$ directly implies another statement $Q(x)$, we may use a proof by contrapositive. In this case, we assume the opposite of the concluding statement $Q(x)$, written $\sim Q(x)$. We then give a logical argument that arrives at $\sim P(x)$. I.e., $\sim Q(x) \rightarrow \sim P(x)$; this is the contrapositive. From the nature of these conditional statements, if a contrapositive is true, then so too is its original claim. Examples of each type of proof follow which will aid in the understanding of their definitions.

<u>Direct Proof</u>

Let A be a bounded nonempty subset of $\mathbb{R}$, $\beta \in \mathbb{R}$ and $\beta < 0$. Let $\beta A = \{\beta a : a \in A\}$. Prove that

$$\sup(\beta A) = \beta \inf(A)$$

<u>Proof:</u>

Since A is nonempty and bounded, βA is also nonempty and $\inf(A)$ exists in $\mathbb{R}$. We have, for every $y \in \beta A$, there exists $x \in A$ such that $y = \beta x$. Since $\inf(A)$ is a lower bound of A and $x \in A$, we have

$$x > \inf(A)$$

Since $\beta < 0$, we have

$$\beta x < \beta \inf(A)$$
$$y < \beta \inf(A)$$

for all $y \in \beta A$.

Thus, $\beta \inf(A)$ is an upper bound of βA. There must exist a least upper bound, $\sup(\beta A)$, in $\mathbb{R}$ and we have

$$\sup(\beta A) \geq \beta \inf(A) \tag{1}$$

Now $\forall\, x \in A$, we have $\beta x \in \beta A$. Since $\sup(\beta A)$ is an upper bound of βA and $\beta x \in \beta A$, we have

$$\beta x < \sup(\beta A)$$

Since $\beta < 0$, we have $\frac{1}{\beta} < 0$ and

$$\frac{1}{\beta}(\beta x) > \frac{1}{\beta}\sup(\beta A)$$
$$x > \frac{1}{\beta}\sup(\beta A)$$

$\forall\, x \in A$. Thus, $\frac{1}{\beta}\sup(\beta A)$ is a lower bound of A.

Since $\inf(A)$ is the greatest lower bound of A, we have

$$\frac{1}{\beta}\sup(\beta A) \geq \inf(A)$$

And since $\beta < 0$, we have

$$\beta\left(\frac{1}{\beta}\right)\sup(\beta A) \leq \beta \inf(A)$$
$$\sup(\beta A) \leq \beta \inf(A) \tag{2}$$

Through combining (1) and (2) and utilizing Axiom (O1)[1], we have

$$\sup(\beta A) = \beta \inf(A)$$

Proof by Mathematical Induction

Prove

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 \tag{1}$$

holds for all $n \in \mathbb{N}$.

Proof:

We will attempt a proof by mathematical induction. We must first show that the base case, $n = 1$, holds. From (1), with $n = 1$, we have

$$1^3 = (1)^2$$

$$1 = 1 \tag{2}$$

From (2), the base case holds. We will first cite the following formula which holds for all $n \in \mathbb{N}$:

$$1 + 2 + \cdots + n = n(n+1)/2 \tag{3}$$

Now (1) can be written as

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{4}$$

Now assume (4) is true for some $n = k$, where $k \in \mathbb{N}$. Show that (4) is true for any $n = k + 1$ (inductive step). I.e., show that

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{k + 1(k+2)}{2}\right)^2 \tag{5}$$

holds for all $k \in \mathbb{N}$. Take the left-hand side of (5):

$$(1 + 2 + \cdots + k)^2 + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$= (k+1)^2\left[\frac{k^2}{4} + (k+1)\right] = \frac{(k+1)^2}{4}[k^2 + 4k + 4] = \frac{(k+1)^2}{4}[(k+2)^2] = \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left(\frac{k + 1(k+2)}{2}\right)^2$$

Hence, we have shown that (4) is true for any $n = k + 1$. Thus, by Principle of Mathematical, (1) holds for all $n \in \mathbb{N}$.

Proof by Contradiction

For each $n \in \mathbb{N}$, let $I_n = (-\infty, -n]$. Show that $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Proof:

We will attempt a proof by contradiction. Suppose that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. If the set is nonempty and nonzero, then there must exist an $x \in \mathbb{R}$ such that for some $n \in \mathbb{N}$, we have

$$n \leq -x \tag{1}$$

which holds for all $n \in \mathbb{N}$. However, since $-x \in \mathbb{R}$, (1) violates Archimedean's Property. Therefore, we must have $\bigcap_{n=1}^{\infty} I_n = \emptyset$, by contradiction.

Proof by Contrapositive

Suppose that $a_n \geq 0$ for every $n \in \mathbb{N}$. Prove that, if $\sum_{n=1}^{\infty} a_n^2$ is a divergent series, then $\sum_{n=1}^{\infty} a_n$ is also a divergent series.

Proof:

To achieve a proof by contrapositive, we wish to establish that

$$\sum_{n=1}^{\infty} a_n \textit{ is not divergent } \Rightarrow \sum_{n=1}^{\infty} a_n^2 \textit{ is not divergent}$$

First suppose that if $\sum_{n=1}^{\infty} a_n$ is not a divergent series, then $\sum_{n=1}^{\infty} a_n$ is a convergent series. If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then $\lim_{n \to \infty} a_n = 0$, by the Divergence Test for series. Also, since $a_n \geq 0 \; \forall \; n \in \mathbb{N}$, then for some $C \in \mathbb{R}$, we have

$$0 \leq a_n \leq C \tag{1}$$

Therefore a_n is a bounded sequence. Since $a_n \geq 0 \; \forall \; n \in \mathbb{N}$, from (1) we have

$$0 \leq a_n^2 \leq C a_n \tag{2}$$

for $n \in \mathbb{N}$. By the linearity of series, if $\sum_{n=1}^{\infty} a_n$ is a convergent, then $\sum_{n=1}^{\infty} C a_n = C \sum_{n=1}^{\infty} a_n$ is also convergent. Furthermore, by (2) and the comparison test for series, if $\sum_{n=1}^{\infty} C a_n$ converges, then so too does $\sum_{n=1}^{\infty} a_n^2$. Thus, $\sum_{n=1}^{\infty} a_n^2$ is not a divergent series. By contrapositive, if $\sum_{n=1}^{\infty} a_n^2$ is a divergent series, then $\sum_{n=1}^{\infty} a_n$ is also a divergent series.

References

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[1]Axiom (O1) $\mathbb{R}$ is an ordered field. This implies that

For any $a, b \in \mathbb{R}$, *exactly one of* $a < b$, $a = b$, *or* $a > b$ *holds*

For any $a, b, c \in \mathbb{R}$, *if* $a < b$ *and* $b < c$, *then* $a < c$