9. Recurrent and Transient States

9.1 Definitions

Define $\quad f_{ii}^{(n)} = P\{X_n = i, X_1 \neq i, \ldots, X_{n-1} \neq i | X_0 = i\}$

= Probability of first recurrence to i is at the n^{th} step.

$$f_i = f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = \text{Prob. of recurrence to } i.$$

Def. A state i is recurrent if $f_i = 1$.

Def. A state i is transient if $f_i < 1$.

Define $T_i =$ Time for first visit to i given $X_0 = 1$. This is the same as Time to first visit to i given $X_k = i$. (Time homogeneous)

$$m_i = E(T_i | X_0 = i) = \sum_{n=1}^{\infty} n f_{ii}^{(n)} = \text{mean time for recurrence}$$

Note: $\quad f_{ii}^{(n)} = P\{T_i = n | X_0 = i\}$

Similarly we can define

$$f_{ij}^{(n)} = P\{X_n = j, X_1 \neq j, \dots, X_{n-1} \neq j | X_0 = i\}$$

= Prob. of reaching state j for first time in n steps starting from $X_0 = i$.

$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} =$ Prob. of ever reaching j starting from i.

Consider $f_{ii} = f_i =$ prob. of ever returning to i.

If $f_i < 1$, $1 - f_i =$ prob. of never returning to i.

i.e.

$$\boxed{\begin{aligned} &1 - f_i = P\{T_i = \infty | X_0 = i\} \\ &f_i = P\{T_i < \infty | X_0 = i\} \end{aligned}}$$

TH. If N is no. of visits to $i|X_0 = i \Rightarrow E(N|X_0 = i) = 1/(1-f_i)$

Proof:

$$\begin{aligned} E(N|X_0 = i) &= E[N|T_i = \infty, X_0 = i]P\{T_i = \infty|X_0 = i\} \\ &\quad + E[N|T_i < \infty, X_0 = i]P\{T_i < \infty|X_0 = i\} \\ E(N|X_o = i) &= 1 \cdot (1-f_i) + f_i[1 + E(N|X_0 = i)] \end{aligned}$$

If $T_i = \infty \Rightarrow$ except for $n = 0$ $(X_0 = i)$, there will never be a visit to i- i.e. $E(N|T_i = \infty, X_0 = i) = 1$. If $T_i < \infty$, there is sure to be one visit, say at X_k $(X_k = i)$. But then

$E(N|T_i < \infty, X_k = i) = E(N|T_i < \infty, X_0 = i)$ by Markov property; i.e.

$$E[N|T_i < \infty, X_0 = i] = 1 + E[N|X_0 = i]$$

$$\therefore \; E[N|X_0 = i] = 1 \cdot (1-f_i) + \{1 + E[N|X_0 = i]\} \cdot (f_i)$$

$$\Rightarrow \boxed{E[N|X_0 = i] = 1/(1-f_i)}$$

Another expression for $E[N|X_0 = i] = \sum_{n=0}^{\infty} p_{ii}^{(n)}$

Relation to Geometric Distribution

Suppose i is transient $(f_i < 1)$ and N_i =no. of visits to i.

$$P\{N_i = k+1 | X_0 = i\} = f_i^k (1 - f_i), \quad k = 0, 1, \ldots$$

$$E(N_i | X_0 = i) = \sum_{k=0}^{\infty} (k+1) f_i^k (1 - f_i) = \sum_{k=0}^{\infty} k f_i^k (1 - f_i) + 1$$

Since

$$(1 - f_i)^{-1} = \sum_{k=0}^{\infty} f_i^k$$

$$(1 - f_i)^{-2} = \frac{d}{df_i}(1 - f_i)^{-1} = \sum_{k=0}^{\infty} k f_i^{k-1}$$

$$\begin{aligned} E(N_i | X_o = i) &= f_i (1 - f_i)(1 - f_i)^{-2} + 1 \\ &= f_i (1 - f_i)^{-1} + 1 = 1/(1 - f_i) \end{aligned}$$

TH. $E[N|X_0 = i] = \sum_{n=0}^{\infty} p_{ii}^{(n)}$

Proof. Let $Y_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{otherwise} \end{cases}$

$$N = \sum_{n=0}^{\infty} Y_n$$

Since $P\{Y_n = 1|X_0 = i\} = P\{X_n = i|X_0 = i\} = p_{ii}^{(n)}$

$$E(N) = \sum_{n=0}^{\infty} E(Y_n) = \sum_{n=0}^{\infty} p_{ii}^{(n)}$$

$E(N)$ may be finite or infinite

Def: A positive recurrent state is defined by $f_i = 1,\ m_i < \infty$.

A null recurrent state is defined by $f_i = 1,\ \ m_i = \infty$

Ex: $$f_{ii}^{(n)} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$f_i = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1$$

But $m_i = \sum_{n=1}^{\infty} n f_{ii}^{(n)} = \sum_{1}^{\infty} \frac{n}{(n+1)n} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$ as series does not converge.

Classification of States

	f_i	m_i
Positive recurrent state	1	$< \infty$
Null recurrent state	1	∞
Transient	< 1	$< \infty$

where $m_i =$ Expected no. of visits to i given $X_0 = i$.

In addition the recurring and transient states may be characterized by being periodic or aperiodic.

A state is ergodic if it is aperiodic and positive recurrent.

9.2 Relations Between f_i and $p_{ii}^{(n)}$

Consider $p_{ii}^{(n)}$. Starting from $X_0 = i$, the first recurrence to i may be at $k = 1, 2, \ldots, n$. Consider the first visit is at time k and at X_n, $X_n = i$ another visit is made. This probability is $f_{ii}^{(k)} p_{ii}^{(n-k)}$. Summing over all k results in

$$(*) \qquad p_{ii}^{(n)} = \sum_{k=1}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)}$$

where $p_{ii}^{(o)} = 1 = P\{X_0 = i | X_0 = i\}$

Multiplying $(*)$ by s^n and summing

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} s^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)} s^n = \sum_{k=1}^{\infty} f_{ii}^{(k)} s^k \sum_{n=k}^{\infty} p_{ii}^{(n-k)} s^{n-k}$$

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} s^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)} s^n = \sum_{k=1}^{\infty} f_{ii}^{(k)} s^k \sum_{n=k}^{\infty} p_{ii}^{(n-k)} s^{n-k}$$

$$\boxed{P_{ii}(s) - 1 = F_{ii}(s) P_{ii}(s)}$$

where $\quad P_{ii}(s) = \sum_{n=0}^{\infty} p_{ii}^{(n)} s^n, \;\; F_{ii}(s) = \sum_{n=1}^{\infty} f_{ii}^{(n)} s^n$

$$\boxed{P_{ii}(s) = \frac{1}{1 - F_{ii}(s)}}$$

Note:

$$\lim_{s \to 1} F_{ii}(s) = F_{ii}(1) = \sum_{n=1}^{\infty} f_{ii}^{(n)} = f_i$$

$$\lim_{s \to 1} F'_{ii}(s) = F'_{ii}(1) = \sum_{n=1}^{\infty} n f_{ii}^{(n)} = m_i$$

Theorems

(a) If $P_{ii}(1) = \sum_{0}^{\infty} p_{ii}^{(n)} = \infty \Rightarrow f_i = 1.$

Conversely if $f_i = 1, \;\; P_{ii}(1) = \infty$

(b) If $P_{ii}(1) = \sum_{0}^{\infty} p_{ii}^{(n)} < \infty \Rightarrow f_i < 1.$

Conversely if $f_i < 1, \;\; P_{ii}(1) < \infty$

9.3 Limiting Theorems for Generating Functions

Consider $A(s) = \sum_{n=0}^{\infty} a_n s^n, \quad |s| \le 1$ with $a_n \ge 0$.

1. $\lim_{n\to\infty} \sum_{k=0}^{n} a_k = \lim_{s\to 1} A(s)$ where $s \to 1$ means $s \to 1-$.

2. Define

$$a^*(n) = \sum_{k=0}^{n} a_k/(n+1)$$

$$\lim_{n\to\infty} a^*(n) = \lim_{s\to 1}(1-s)A(s)$$

3. Cesaro Limit

   The Cesaro limit is defined by $\lim_{n\to\infty} a^*(n)$ If the sequence $\{a_n\}$ has a limit $\Pi = \lim_{n\to\infty} a_n$ then $\lim_{n\to\infty} a^*(n) = \Pi$.

The Cesaro limit may exist without the existence of the ordinary limit.

Ex. $a_n : \quad 0, 1, 0, 1, 0, 1, \ldots$

$\lim_{n\to\infty} a_n$ does not exist.

However

$$a^*(n) = \begin{cases} \frac{1}{2} & \text{if } n \text{ even} \\ \\ \frac{1}{2}(1 - \frac{1}{n}) & \text{if } n \text{ is odd} \end{cases}$$

$$\lim_{n\to\infty} a^*(n) = \frac{1}{2}$$

9.4 Application to Markov Chains

Consider $p^*_{ii}(n) = \sum_{k=0}^{n} \frac{p^{(k)}_{ii}}{n+1}$

$\sum_{k=0}^{n} p^{(k)}_{ii}$ is expected no. of visits to i starting from $X_0 = i$ $(p^0_{ii} = 1)$.

Dividing by $(n+1)$, $p^*_{ii}(n)$ is expected no. of visits per unit time.

Ex. $n = 29$ days, $p^*_{ii}(29) = 2/30$; i.e. 2 visits per 30 days or 1 visit per 15 days. One would expect mean time between visits = 15 days.

TH. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{p_{ii}^{(k)}}{n+1} = \dfrac{1}{m_i}$, where $m_i =$ expected no. of visits and $f_i = 1$

Proof: Consider $P_{ii}(s) = \dfrac{1}{1 - F_{ii}(s)}$

$$\lim_{s\to 1}(1-s)P_{ii}(s) = \lim_{n\to\infty} p_{ii}^*(n) = \lim_{s\to 1} \frac{(1-s)}{1 - F_{ii}(s)}.$$

Since $F_{ii}(1) = f_i$, if $f_i = 1$, the r.h.s. is indeterminate. Using L'Hopital's rule

$$\boxed{\lim_{n\to\infty} p_{ii}^*(n) = \frac{1}{F'_{ii}(1)} = \frac{1}{m_i}}$$

Recall a positive recurrent state has $m_i < \infty \;\Rightarrow\; \lim\limits_{n\to\infty} p_{ii}^*(n) > 0$

A null recurrent state has $m_i = \infty$

$\Rightarrow \quad \lim\limits_{n\to\infty} p_{ii}^*(n) = 0$ or $\lim\limits_{n\to\infty} p_{ii}^{(n)} = 0$

9.5 Relations Between f_{ij} and $p_{ij}^{(n)}$ $(i \neq j)$

$$f_{ij}^{(n)} = P\{X_n = j, X_r \neq j, r = 1, 2, \dots, n-1 | X_0 = i\}$$

= Prob. of starting from i and reaching j for first time at n^{th} step.

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} \quad i \neq j$$

Proceeding as before $(i \neq j)$

$$\begin{aligned} p_{ij}^{(n)} &= f_{ij}^{(1)} p_{jj}^{(n-1)} + f_{ij}^{(2)} p_{jj}^{(n-1)} + \ldots + f_{ij}^{(n)} \\ &= \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \quad (p_{jj}^{(0)} = 1) \end{aligned}$$

Multiplying by s^n and summing over n

$$\sum_{n=1}^{\infty} p_{ij}^{(n)} s^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} s^n$$

$$P_{ij}(s) = \sum_{k=1}^{\infty} f_{ij}^{(k)} s^k \sum_{n=k}^{\infty} p_{jj}^{(n-k)} s^{n-k}$$

$$\boxed{P_{ij}(s) = F_{ij}(s) P_{jj}(s)} \quad i \neq j$$

$$\lim_{s \to 1} (1-s) P_{jj}(s) = \lim_{n \to \infty} p_{jj}^*(n) = \lim_{s \to 1} (1-s) P_{ij}(s) / F_{ij}(1)$$

$$\lim_{n \to \infty} p_{jj}^*(n) = \lim_{n \to \infty} \frac{p_{ij}^*(n)}{F_{ij}(1)} = \frac{1}{m_i} \text{ or } \boxed{\lim_{n \to \infty} p_{ij}^*(n) = \frac{F_{ij}(1)}{m_i}}$$

Also $P_{ij}(1) = \sum_{n=1}^{\infty} p_{ij}^{(n)} = F_{ij}(1) \sum_{n=0}^{\infty} p_{jj}^{(n)}$.

Hence if $\sum_{n=0}^{\infty} p_{jj}^{(n)} = \infty \Rightarrow \sum_{n=0}^{\infty} p_{ij}^{(n)} = \infty \quad (p_{ij}^{(0)} = 0)$.

Similarly if $\sum_{n=0}^{\infty} p_{jj}^{(n)} < \infty \Rightarrow \sum_{n=0}^{\infty} p_{ij}^{(n)} < \infty$

Summary

Transient $\displaystyle\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty, \;\; f_i < 1, \;\; m_i = 1/(1 - f_i)$

$$\lim_{n\to\infty} p_{ii}^{(n)} = 0, \quad \sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty, \quad \lim_{n\to\infty} p_{ij}^{(n)} = 0$$

Positive Recurrent $\displaystyle\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty, \;\; f_i = 1, \;\; m_i < \infty$

$$\lim_{n\to\infty} p_{ii}^{*}(n) > 0 \quad (= 1/m_i)$$

$$\lim_{n\to\infty} p_{ij}^{*}(n) > 0 \quad (= F_{ij}(1)/m_i)$$

Negative Recurrent $\displaystyle\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty, f_i = 1, \;\; m_i = \infty$

$$\lim_{n\to\infty} p_{ii}^{*}(n) = 0, \quad \lim_{n\to\infty} p_{ii}^{(n)} = 0$$

9.6 Periodic Processes

Suppose transition probabilities have period d.
Then

$$p_{ij}^{(n)} = 0, \;\; p_{ii}^{(n)} = 0 \;\text{ if } n \neq rd \;\; r = 1, 2, \ldots$$

$$p_{ij}^{(n)} \geq 0, \;\; p_{ii}^{(n)} \geq 0 \;\text{ if } n = rd$$

$$P_{ii}(s) = \sum_{r=0}^{\infty} p_{ii}^{(rd)} s^{rd} = \sum_{r=0}^{\infty} p_{ii}^{(rd)} z^r, \quad z = s^d$$

$$F_{ii}(s) = \sum_{r=1}^{\infty} f_{ii}^{(rd)} s^{rd} = \sum_{r=1}^{\infty} f_{ii}^{(rd)} z^r$$

We now have a power series in z

$$\begin{aligned}\lim_{z\to 1} P_{ii}(Z) &= \sum_{r=0}^{\infty} p_{ii}^{(rd)} \\ \lim_{z\to 1}(1-z)P_{ii}(z) &= \sum_{n\to\infty} \sum_{r=0}^{n} \frac{p_{ii}^{(rd)}}{n+1}\end{aligned}$$

Note: $E(N|X_0 = i) = \sum_{1}^{\infty} n f_{ii}^{(n)} = d\sum_{r=1}^{\infty} r f_{ii}^{(rd)} = m_i$

However $F'_{ii}(1) = \sum_{r=1}^{\infty} f_{ii}^{(rd)} r = m_i/d$

Since $P_{ii}(Z) = 1/[1 - F_{ii}(z)]$

$\boxed{\lim_{n\to\infty} p^*_{ii}(n) = d/m_i}$ or $\boxed{\lim_{n\to\infty} p_{ii}^{(nd)} = d/m_i}$

9.7 Closed Sets

Def. A set of states C is closed if no state outside C can be reached from any state in C; i.e., $p_{ij} = 0$ if $i \in C$ and $j \notin C$.

Absorbing state: Closed set consisting of a single state.

Irreducible Chain: If only closed set is the set of all states. (Every state can be reached from any other state).

This means that we can study the behavior of states in C by omitting all other states.

Th. $i \leftrightarrow j$, i is recurrent $\Rightarrow j$ recurrent

$i \leftrightarrow j$, i is transient $\Rightarrow j$ transient

$$\sum_{r=0}^{\infty} p_{jj}^{(r)} \geq \sum_{r=0}^{\infty} p_{jj}^{(r+n+m)} = \sum_{r=0}^{\infty} \sum_{k \in S} p_{jk}^{(m)} p_{kk}^{(r)} p_{kj}^{(n)}$$

$$\geq \sum_{r=0}^{\infty} p_{ji}^{(m)} p_{ii}^{(n)} p_{ij}^{(n)} = p_{ji}^{(m)} p_{ij}^{(n)} \sum_{r=0}^{\infty} p_{ii}^{(r)}$$

Thus if $\sum_{r=0}^{\infty} p_{ii}^{(r)} = \infty$, $\sum_{r=0}^{\infty} p_{jj}^{(r)} = \infty$

Suppose i is transient, $i \leftrightarrow j$ and assume j is recurrent. By theorem just proved $\underline{i}$ then must be recurrent. However this is a contradiction $\Rightarrow$ $\underline{j}$ is transient.

Summary (Aperiodic, irreducible)

1. If $i \leftrightarrow j$ and j is positive recurrent $\Rightarrow$ i is positive recurrent

$$\lim_{n\to\infty} p_{ij}^{(n)} = \lim_{n\to\infty} p_{jj}^{(n)} = \Pi_j = 1/m_j.$$

2. If $i \leftrightarrow j$ and j is null recurrent $\Rightarrow$ i is null recurrent

$$\lim_{n\to\infty} p_{jj}^{(n)} = 0, \quad \lim_{n\to\infty} p_{ij}^{(n)} = 0$$

$$\text{or } \; P^{(\infty)} = \lim_{n\to\infty} P^{(n)} = \lim_{n\to\infty} P^{n} = 0.$$

3. If $i \leftrightarrow j$ and j is transient $\Rightarrow$ i is transient

$$\lim_{n\to\infty} p_{jj}^{(n)} = 0, \; \lim_{n\to\infty} p_{ij}^{(n)} = 0$$

9.8 Decomposition Theorem

(a) The states of a Markov Chain may be divided into two sets (one of which may be empty). One set is composed of all the recurring states, the other of all the transient states.

(b) The recurrent states may be decomposed uniquely into two closed sets. Within each closed set all states inter-communicate and they are all of the same type and period. Between any two closed sets no communication is possible.

Ex. Decomposition of a Finite Chain

$$
P = \begin{array}{c} \\ C_0 \\ \\ C_1 \\ C_2 \\ C_3 \end{array}
\begin{array}{cc:ccc:ccc:ccc}
\multicolumn{2}{c}{C_0} & & C_1 & & & C_2 & & & C_3 & \\
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
\multicolumn{2}{c:}{O} & & P_1 & & & O & & & O & \\
\multicolumn{2}{c:}{O} & & O & & & P_1 & & & O & \\
\multicolumn{2}{c:}{A} & & B & & & C & & & D &
\end{array}
$$

C_0: Consists of two absorbing states

C_1: Consists of closed recurrent states

C_2: Consists of closed recurrent states

C_3: Consists of transient states

A: Transitions $C_3 \to C_0$ B: Transitions $C_3 \to C_1$

C: Transitions $C_3 \to C_2$ D: Transitions $C_3 \to C_3$

9.9 Remarks on Finite Chains

1. A finite chain cannot consist only of transient states.

   If i, j are transient $\lim_{n\to\infty} p_{ij}^{(n)} = 0$,

   However

   $$\sum_{j\in S} p_{ij}^{(n)} = 1$$

   leading to a contradiction as $n \to \infty$.

2. A finite chain cannot have any null recurrent states.

   The one step transition probabilities within a closed set of null recurrent states form a stochastic matrix P such that $P^n \to 0$ as $n \to \infty$. This is impossible as $\sum_{j\in S} p_{ij}^{(n)} = 1.$

9.10 Perron-Frobenius Theorem

Earlier we had seen that if P has a characteristic root (eigenvalue) = 1 of multiplicity 1, and all other $|\lambda_i| < 1$, then

$$P^{(\infty)} = E_1.$$

The conditions under which this is true are proved by the Perron-Frobenius Theorem. The necessary and sufficient conditions are:

$$\begin{aligned} P:&\quad \text{aperiodic} \\ P:&\quad \text{positive recurrent } (m_i < \infty). \end{aligned}$$

Then $P_1^{(\infty)} = P^{\infty} = E_1 = 1y'$

where $\quad y'P = y' \quad$ and $\underline{1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}'$

Def. A state is ergodic if it is aperiodic and positive recurrent.

9.11 Determining Recurrence and Transience when Number of States is Infinite

Compute $f_i = P\{T_i < \infty | X_0 = i\}$ in a closed communicating class.

$\Rightarrow$ All states are recurrent if $f_i = 1$; All states are transient if $f_i < 1$.

Ex. $S = \{0, 1, 2, \ldots\}$, $\quad p_{i,0} = q_i \quad p_{i,i+1} = p_i$,

$$X_{n+1} = \begin{cases} 0 & \text{with } q_i \\ X_n + 1 & \text{with } p_i \end{cases}$$

$$\begin{aligned} P\{T_0 > n | X_0 = 0\} &= P\{X_1 = 1, X_2 = 1, \ldots, X_n = n | X_0 = 0\} \\ &= \prod_{i=0}^{n-1} p_i \end{aligned}$$

$$P\{T_0 < \infty | X_0 = 0\} = 1 - \lim_{n \to \infty} P\{T_0 > n | x_0 = 0\} = 1 - \prod_{i=0}^{\infty} p_i$$

$\therefore$ State 0 (and all states in closed class) are recurrent iff $\prod_{i=0}^{\infty} p_i = 0$

Ex. Random Walk on Integers

$$
\begin{aligned}
S &= \{0, \pm 1, \pm 2, \ldots\} \\
p_{i,i+1} &= p, \quad p_{i,i-1} = q, \quad p + q = 1 \\
p_{00}^{(2n+1)} &= 0, \;\; n \geq 0 \;\; \text{(Go from 0 to 0 in odd number of transitions)} \\
p_{00}^{(2n)} &= \binom{2n}{n} p^n q^n
\end{aligned}
$$

Consider

$$\sum_{n=1}^{\infty} p_{00}^{(n)} = \sum_{n=1}^{\infty} p_{00}^{(2n)} = \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} p^n q^n$$

Note: Ratio Test: If $A = \sum_{n=0}^{\infty} a_n$ and

$$\frac{a_{n+1}}{a_n} < 1 \text{ series converges} \quad \text{as } n \to \infty$$

$$\frac{a_{n+1}}{a_n} > 1 \text{ series diverges} \quad \text{as } n \to \infty$$

$$\frac{p_{00}^{2n+2}}{p_{00}^{2n}} = \frac{(2n+1)(2n+2)}{(n+1)(n+1)} pq \to 4pq \ \text{ as } n \to \infty$$

If $p \neq q \quad 4pq < 1.$

If $p = q = 1/2 \quad 4pq = 1$ and test is inconclusive.

$$\binom{2n}{s} p^s q^{2n-s} \sim N(2np, 2npq) = N(n, n/2) \text{ if } p = 1/2$$

$$\sim e^{-(s-n)^2/n} \Big/ \sqrt{2\pi \frac{n}{2}}$$

Since $p_{00}^{(2n)} \sim \dfrac{1}{\sqrt{\pi n}}$

$$\sum_{n=1}^{\infty} p_{00}^{(2n)} \cong \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} \quad \text{series diverges}$$

$\therefore$ State 0 is recurrent.

<u>Th.</u> An irreducible Markov Chain with $S = \{0, 1, 2, \dots\}$ and Transition Prob. $\{p_{ij}\}$ is transient iff

$$y_i = \sum_{j=1}^{\infty} p_{ij} y_j \qquad i = 1, 2, \dots$$

has a non-zero bounded solution

<u>Proof: P.88</u>

Ex. Random walk on $S = \{0, 1, 2, \dots\}$

$$p_{i,i+1} = p_i, \;\; p_{i,i-1} = q_i \;\;\; p_{i,i} = r_i, \; q_0 = 0 \;\; (p_i + q_i + r_i = 1)$$

Equations:

$$y_1 = p_{11}y_1 + p_{12}y_2 = r_1y_1 + p_1y_2 \Rightarrow y_2 = \left(1 + \frac{q_1}{p_1}\right) y_1$$

$$y_2 = p_{21}y_1 + p_{22}y_2 + p_{23}y_3 = q_2y_1 + r_2y_2 + p_2y_3$$

$$\Rightarrow \;\; y_3 = \left(1 + \frac{q_1}{p_1} + \frac{q_1q_2}{p_1p_2}\right) y_1$$

In general,

$$y_n = \left[1 + \sum_{k=1}^{n-1} \frac{q_1 q_2 \dots q_k}{p_1 p_2 \dots p_k}\right] y_1, \; n \geq 1$$

$$y_n = \left(\sum_{k=0}^{n-1} \alpha_k\right) y_1, \quad \alpha_k = \frac{q_1 q_2 \cdots q_k}{p_1 p_2 \cdots p_k}, \; \alpha_0 = 1$$

Thus the solution is bounded if $\sum_{k=0}^{\infty} \alpha_k < \infty$

9.12 Revisiting Statistical Equilibrium

Assume the Markov Chain has all states which are irreducible positive recurrent aperiodic. (Called Ergodic Chain).

Earlier we had shown

$$\lim_{n\to\infty} p_{ij}^{(n)} = \lim_{n\to\infty} p_{jj}^{(n)} = 1/m_j = \Pi_j$$

$\Rightarrow$ Π_j is given by the solution

$$\Pi_j = \sum_{i\in S} \Pi_i p_{ij} \quad \text{where} \quad \sum_{j\in S} \Pi_j = 1$$

or in matrix notation we can write the linear equations as
$\boxed{\Pi = P\Pi}$ where $\Pi : k \times 1, \quad P : k \times k.$

<u>Proof</u> $a_j(n) = P\{X_n = j\}$ and we will show that $\lim_{n\to\infty} a_j(n) = \Pi_j$

$$a_j(n+m) = \sum_{i\in S} \Pi_i p_{ij}^{(n)}$$

Take $m \to \infty$ $\quad \Pi_j = \sum_{i \in S} \Pi_i p_{ij}^{(n)}$ $\quad$ for any n

If $n = 1$ $\quad \boxed{\Pi_j = \sum_{i \in S} \Pi_i p_{ij}}$

Allow $n \to \infty$ $\quad \Pi_j = \left(\sum_{i \in S} \Pi_i \right) \Pi_j$

which is true if $\sum_{i \in s} \Pi_i = 1$

In the above we made use of

$$\lim_{n \to \infty} p_{ij}^{(n)} = \lim_{n \to \infty} p_{jj}^{(n)} = \Pi_j$$

which holds for ergodic chains.

If chain was transient or null recurrent

$$p_{ij}^{(n)} = p_{jj}^{(n)} \to 0 \text{ as } n \to \infty \text{ and } \Pi_j = \sum_{i \in S} \Pi_i p_{ij}^{(n)} \to 0 \text{ as } n \to \infty,$$

and result does not hold.

Th: If $a_0(j) = P\{X_0 = j\} = \Pi_j$ then $P\{a_n = j\} = \Pi_j$ for all n.

This theorem states that if the initial probabilities correspond to the limiting probabilities, then for any n $p\{X_n = j\} = \Pi_j$.

Proof:

In general $a_j(n) = \sum_{i \in S} a_0(i) p_{ij}^{(n)}$ and in matrix notation we can write $a_n = P^n a_0$

If $a_o = \Pi$, $a_n = P^n \Pi$. But Π is defined by $\Pi = P\Pi$. $\Rightarrow$ $a_n = P\Pi = \Pi$

This is the reason why Π is sometimes referred to as the *stationary distribution*.

9.13 Appendix. Limit Theorems for Generating Functions

Definition: Let $A(z) = \sum_{n=0}^{\infty} a_z^n$ denote a power series. In our application z will always be real, however all the results also hold if z is a complex number.

Theorem If $\{a_n\}$ are bounded, say $|a_n| \leq B$, then $A(z)$ converges for at least $|z| < 1$.

Proof :
$$A(z) = |A(z)| \leq \sum_{n=0}^{\infty} |a_n| z^n| \leq B \sum_{n=0}^{\infty} |z|^n = B/1-|z|$$

Definition: The number R is called the radius of convergence of the power series $A(z)$ if $A(z)$ converges for $|z| > R$. Without loss of generality the radius of convergence can be taken as $R = 1$. Note if R is the radius of convergence of $A(z)$ we can write $A(z) = \sum_{0}^{\infty} b_n y^n$ and the radius of convergence of y will be unity.

Properties of $A(z)$

(i) If R is the radius of convergence

$$R^{-1} = \lim_{n\to\infty} \sup(a_n)^{1/n}$$

(ii) Within the interval of convergence $(-R < z < R)$, $A(z)$ has derivatives of all orders which may be obtained by term-wise differentiation. Similarly the integral $\int_a^b A(z)dz$ is given by term-wise integration for any (a, b) in $(-R, R)$.

(iii) If $A(z)$ and $B(z) = \sum_0^\infty b_n z^n$ both converge and are equal for all $|z| < R$, then $a_n = b_n$.

(iv) No general statement can be made about the convergence of the series on the boundary $|z| = R$; i.e. $\sum_n a_n R^n$ may or may not be finite.

Two important theorems on power series are:

Abel's Theorem Suppose $A(z)$ has a radius of convergence $R = 1$ and $\sum_0^\infty a_n$ is convergent to s. Then

$$\lim_{z \to 1-} A(z) = \sum_{n=0}^{\infty} a_n$$

If the coefficients $\{a_n\}$ are non-negative, the result continues to hold whether or not the sum on the right is convergent.

Note:

$$A_N(z) = \sum_{n=0}^{N} a_n z^n = \sum_{n=0}^{N} (s_n - s_{n-1}) z^n, \qquad s_n = \sum_{i=0}^{n} a_i$$

$$= \sum_{n=0}^{N} s_n z^n - z \sum_{n=0}^{N-1} s_n z^n$$

$$= (1-z) \sum_{n=0}^{N} s_n z^n + s_N z^N$$

$\lim\limits_{z \to 1-} A_n(z) = s_N$ and taking the limit as $N \to \infty$

$$\lim_{N \to \infty} s_N = s$$

Theorem: If the sequence $\{b_n\}$ converges to a limit b ($\lim\limits_{n\to\infty} = b$), then

$$\lim_{z\to 1-}(1-z)\sum_{n=0}^{\infty} b_n z^n = b$$

Proof: $$(1-z)\sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty}(b_n - b_{n-1})z^n \quad (b_{-1}=0)$$

We can write $$\sum_{n=0}^{N}(b_n - b_{n-1})z^n = (1-z)\sum_{n=0}^{N} s_n z^n + s)nz^N$$

where

$$s_n = \sum_{i=0}^{n}(b_i - b_{i-1}) = b_0 + (b_1 - b_0) + (b_2 - b_1) + \ldots + (b_n - b_{n-1}) = b_n$$

Therefore $$(1-z)\sum_{n=0}^{N} b_n a^n = (1-z)\sum_{n=0}^{N} s_n z^n + b_n z^N$$

and $\lim\limits_{z\to 1^-}(1-z)\sum\limits_{n=0}^{N} b_n z^n = b_n$, so that as $N\to\infty$, $\lim\limits_{N\to\infty} b_N = b$.