4. Poisson Processes

4.1 Definition

4.2 Derivation of exponential distribution

4.3 Properties of exponential distribution

- a. Normalized spacings
- b. Campbell's Theorem
- c. Minimum of several exponential random variables
- d. Relation to Erlang and Gamma Distribution
- e. Guarantee Time
- f. Random Sums of Exponential Random Variables

4.4 Counting processes and the Poisson distribution

4.5 Superposition of Counting Processes

4.6 Splitting of Poisson Processes

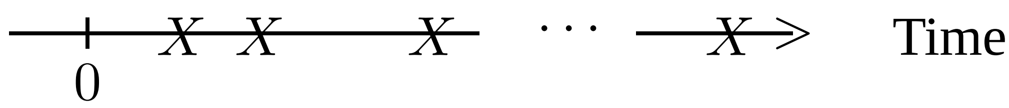
4.7 Non-homogeneous Poisson Processes

4.8 Compound Poisson Processes

4 Poisson Processes

4.1 Definition

Consider a series of events occurring over time, i.e.



Define T_i as the time between the $(i-1)^{st}$ and i^{th} event. Then

$$S_n = T_1 + T_2 + \ldots + T_n \;=\; \text{Time to } n^{th} \text{ event.}$$

Define $N(t) =$ no. of events in $(0, t]$.

Then

$$\boxed{P\{S_n > t\} = P\{N(t) < n\}}$$

If the time for the n^{th} event exceeds t, then the number of events in $(0, t]$ must be less than n.

$$\boxed{P\{S_n > t\} = P\{N(t) < n\}}$$

$$\begin{aligned} p_t(n) = P\{N(t) = n\} &= P\{N(t) < n+1\} - P\{N(t) < n\} \\ &= P\{S_{n+1} > t\} - P\{S_n > t\} \\ \text{where} \quad S_n &= T_1 + T_2 + \ldots + T_n. \end{aligned}$$

Define $Q_{n+1}(t) = P\{S_{n+1} > t\}, \;\; Q_n(t) = P\{S_n > t\}$

Then we can write

$$p_t(n) = Q_{n+1}(t) - Q_n(t)$$

and taking LaPlace transforms

$$p_s^*(n) = Q_{n+1}^*(s) - Q_n^*(s)$$

If $q_{n+1}(t)$ and $q_n(t)$ are respective pdf's.

$$Q^*_{n+1}(s) = \frac{1 - q^*_{n+1}(s)}{s}, \quad Q^*_n(s) = \frac{1 - q^*_n(s)}{s}$$

and

$$p^*_s(n) = \frac{1 - q^*_{n+1}(s)}{s} - \frac{1 - q^*_n(s)}{s} = \frac{q^*_n(s) - q^*_{n+1}(s)}{s}$$

Recall T_1 is time between 0 and first event, T_2 is time between first and second event, etc.

Assume $\{T_i\}$ $i = 1, 2, \ldots$ are independent and with the exception of $i = 1$, are identically distributed with pdf $q(t)$. Also assume T_1 has pdf $q_1(t)$. Then

$$q^*_{n+1}(s) = q^*_1(s)[q^*(s)]^n, \; q^*_n(s) = q^*_1(s)[q^*(s)]^{n-1}$$

and

$$p^*_s(n) = \frac{q^*_n(s) - q^*_{n+1}(s)}{s} = q^*_1(s)q^*(s)^{n-1}\left[\frac{1 - q^*(s)}{s}\right]$$

Note that $q_1(t)$ is a forward recurrence time. Hence

$$q_1(t) = \frac{Q(t)}{m} \quad \text{and} \quad q^*_1(s) = \frac{1 - q^*(s)}{sm}$$

$$p^*_s(n) = \frac{q^*(s)^{n-1}}{m}\left[\frac{1 - q^*(s)}{s}\right]^2$$

Assume $q(t) = \lambda e^{-\lambda t}$ for $t > 0$ $\quad (m = 1/\lambda)$

$= 0$ otherwise.

Then $q^*(s) = \lambda/\lambda + s, \quad \dfrac{1 - q^*(s)}{s} = 1/(\lambda + s)$

and $p_s^*(n) = \left(\dfrac{\lambda}{\lambda + s}\right)^{n-1} \left(\dfrac{1}{\lambda + s}\right)^2 \lambda = \dfrac{1}{\lambda}(\lambda\ /\lambda + s)^{n+1}$.

$$\boxed{p_s^*(n) = \frac{1}{\lambda}(\lambda/\lambda + s)^{n+1}}$$

However $(\lambda/\lambda + s)^{n+1}$ is the LaPlace transform of a gamma distribution with parameters $(\lambda, n+1)$ i.e.

$$f(t) = \frac{e^{-\lambda t}(\lambda t)^{n+1-1}\lambda}{\Gamma(n+1)} \text{ for } t > 0$$

$$\therefore \;\; p_t(n) \; = \; \mathcal{L}^{-1}\{p_s^*(n)\} \; = \; \boxed{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

which is the Poisson Distribution. Hence $N(t)$ follows a Poisson distribution and

$$P\{N(t) < n\} = \sum_{r=0}^{n-1} p_t(r) = P\{S_n > t\}$$

$$P\{S_n > t\} = \sum_{r=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^r}{r!}.$$

We have shown that if the times between events are iid following an exponential distribution the $N(t)$ is Poisson with $E[N(t)] = \lambda t$.

Alternatively if $N(t)$ follows a Poisson distribution, then S_n has a gamma distribution with pdf $f(t) = \dfrac{e^{-\lambda t}(\lambda t)^{n-1}\lambda}{\Gamma(n)}$ for $t > 0$.

This implies time between events are exponential.

Since $P\{S_n > t\} = P\{N(t) < n\}$ we have proved the identity

$$P\{S_n > t\} = \int_t^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n-1}\lambda}{\Gamma(n)} \lambda dx = \sum_{r=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^r}{r!}.$$

This identity is usually proved by using integration by parts.

When $N(t)$ follows a Poisson distribution with $E[N(t)] = \lambda t,$ the set $\{N(t), t > 0\}$ is called a Poisson Process.

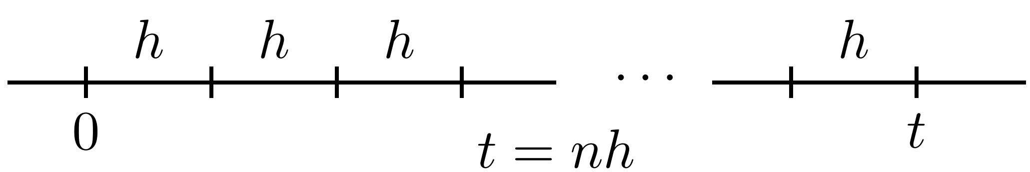
4.2 Derivation of Exponential Distribution

Define $P_n(h) =$ Prob. of n events in a time interval h

Assume

$$P_0(h) = 1 - \lambda h + o(h); \quad P_1(h) = \lambda h + o(h); \quad P_n(h) = o(h) \text{ for } n > 1$$

where $o(h)$ means a term $\psi(h)$ so that $\lim_{h \to 0} \dfrac{\psi(h)}{h} = 0$. Consider a finite time interval $(0, t)$. Divide the interval into n sub-intervals of length h. Then $t = nh$.



The probability of no events in $(0, t)$ is equivalent to no events in each sub-interval; i.e.

$$P_n\{T > t\} = P\{\text{no events in } (0, t)\}$$

$$T = \text{Time for } 1^{st} \text{ event}$$

Suppose the probability of events in any sub interval are independent of each other. (Assumption of independent increments.) Then

$$P_n\{T > t\} = [1 - \lambda h + o(h)]^n = [1 - \frac{\lambda t}{n} + o(h)]^n$$

$$= (1 - \frac{\lambda t}{n})^n + n\, o(h)(1 - \frac{\lambda t}{n})^{n-1} + \dots$$

Since

$$\lim_{n\to\infty} (1 - \frac{\lambda t}{n})^n = e^{-\lambda t}$$

and

$$\lim_{n\to\infty} n\, o(h) = \lim_{h\to 0} \frac{t}{h} o(h) = 0$$

We have $P\{T > t\} = \lim_{h\to 0} P_n\{T > t\} = e^{-\lambda t}$.

$\therefore$ The pdf of T is $-\frac{d}{dt}P\{T > t\} = \lambda e^{-\lambda t}$. (Exponential Distribution)

4.3 Properties of Exponential Distribution

$$q(t) = \lambda e^{-\lambda t} \qquad t > 0$$

$$E(T) = 1/\lambda = m, \quad V(t) = 1/\lambda^2 = m^2$$

$$q^*(s) = \lambda/\lambda + s$$

Consider $r < t$.

Then

$$P\{T > r + t | T > r\} = \text{Conditional distribution}$$

$$= \frac{Q(r+t)}{Q(r)} = \frac{e^{-\lambda(r+t)}}{e^{-\lambda r}} = e^{-\lambda t}$$

i.e. $P\{T > r + t | T > r\} = P\{T > t\}$ for all r and t.

Also $P\{T > r + t\} = e^{-\lambda(r+t)} = Q(r)Q(t) = Q(r+t)$

Exponential distribution is only function satisfying $Q(r+t) = Q(r)Q(t)$

Proof:

$$Q\left(\frac{2}{n}\right) = Q\left(\frac{1}{n}\right)^2 \text{ and in general } Q\left(\frac{m}{n}\right) = Q\left(\frac{1}{n}\right)^m$$

$$Q(1) = Q\left(\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}\right) = \left[Q\left(\frac{1}{n}\right)\right]^n, \quad m = n$$

$$\therefore \; Q\left(\frac{m}{n}\right) = \left[Q\left(\frac{1}{n}\right)^n\right]^{m/n} = Q(1)^{m/n}.$$

If $Q(\cdot)$ is continuous or left or right continuous we can write

$$Q(t) = Q(1)^t.$$

Since $Q(1)^t = e^{t \log Q(1)}$ we have $\log Q(1)$ is the negative of the rate parameter. Hence

$$Q(t) = e^{-\lambda t} \quad \text{where } \lambda = -\log\, Q(1).$$

a. Normalized Spacings

Let $\{T_i\}\ i = 1, 2, \ldots, n$ be iid following an exponential distribution with $E(T_i) = 1/\lambda$.

Define $\quad T_{(1)} \le T_{(2)} \le \ldots \le T_{(n)} \quad$ Order statistics

Then the joint distribution of the order statistics is

$$f(t_{(1)}, t_{(2)}, \ldots, t_{(n)}) dt_{(1)}, t_{(2)}, \ldots t_{(n)} = P\{t_{(1)} < T_{(1)}, \le t_{(1)} + dt_{(1)}, \ldots\}$$

$$= \frac{n!}{1!\,1!\,\ldots 1!}$$

$$= n! \lambda e^{-\lambda t_{(1)}} \cdot \lambda e^{-\lambda t_{(2)}} \ldots \lambda e^{-\lambda t_{(n)}} dt_{(1)} \ldots dt_{(n)}$$

$$f(t_{(1)}, \ldots, t_{(n)}) = n! \lambda^n e^{-\lambda \sum_1^n t_{(i)}} = n! \lambda^n e^{-\lambda S}$$

where $S = \sum_1^n t_{(i)} = \sum_1^n t_i, \quad 0 \le t_{(1)} \le \ldots \le t_{(n)}$

$$
\boxed{\begin{aligned} f(t_{(1)},\ldots,t_{(n)}) &= n!\lambda^n e^{-\lambda S},\ \ 0 \le t_{(1)} \le \ldots \le t_{(n)} \\ S &= \sum_1^n t_{(i)} \end{aligned}}
$$

Consider

$$
Z_1 = nT_{(1)},\ \ Z_2 = (n-1)(T_{(2)} - T_{(1)}), \cdots ,
$$

$$
Z_{(i)} = (n-i+1)(T_{(i)} - T_{(i-1)}), \cdots , Z_{(n)} = T_{(n)} - T_{(n-1)}.
$$

We shall show that $\{Z_i\}$ are iid exponential.

$$
f(Z_1, Z_2, \ldots, Z_n) = f(t_{(1)}, \ldots, t_{(n)}) \left| \frac{\partial(t_{(1)}, \ldots, \partial t_{(n)})}{\partial(Z_1, \ldots, Z_n)} \right|
$$

where $\left| \frac{\partial(t_{(1)}, \ldots, t_{(n)})}{\partial(Z_1, \ldots, Z_n)} \right|$ is the determinant of the Jacobian.

We shall find the Jacobian by making use of the relation

$$\left|\frac{\partial(t_{(1)}, \ldots, \partial t_{(n)})}{\partial(Z_1, \ldots, Z_n)}\right| = \left|\frac{\partial(Z_1, Z_2, \ldots Z_n)}{\partial t_{(1)}, \ldots, t_{(n)})}\right|^{-1}$$

$$Z_i = (n-i+1)(T_{(i)} - T_{(i-1)}), \quad T_{(0)} = 0$$

$$\frac{\partial Z_i}{\partial T_{(j)}} = \begin{cases} n-i+1 & j = i \\ -(n-i+1) & j = i-1 \\ 0 & \text{Otherwise} \end{cases}$$

$$\frac{\partial(Z_1, \ldots, Z_n)}{\partial(t_{(1)}, \ldots, t_{(n)})} = \begin{bmatrix} n & 0 & 0 & 0 & \ldots & 0 \\ -(n-1) & (n-1) & 0 & 0 & \ldots & 0 \\ 0 & -(n-2) & (n-2) & 0 & \ldots & 0 \\ \vdots & & & & & \\ 0 & 0 & \ldots & \ldots & -1 & 1 \end{bmatrix}$$

Note: The determinant of a trangular matrix is the product of the main diagonal terms

$$\therefore \quad \left| \frac{\partial(Z_1, \dots, Z_n)}{\partial(t_{(1)}, \dots, t_{(n)})} \right| = n(n-1)(n-2)\dots 2 \cdot 1 = n!$$

and

$$f(z_1, z_2, \dots, z_n) = n! \lambda^n \, e^{-\lambda S} \, \frac{1}{n!} = \lambda^n e^{-\lambda \sum_1^n z_i} = \lambda^n e^{-\lambda S}$$

as $S = \sum_{i=1}^{n} t_{(i)} = z_1 + \dots + z_n$.

The spacings $Z_i = (n-i+1)(T_{(i)} - T_{(i-1)})$ are sometimes called normalized spacings.

Homework:

1. Suppose there are n observations which are iid exponential $(T_i = 1/\lambda)$. However there are r non-censored observations and $(n-r)$ censored observations all censored at $t_{(r)}$.
Show $Z_i = (n-i+1)(T_{(i)} - T_{(i-1)})$ for $i = 1, 2, \dots, r$ are iid exponential.

2. Show that

$$T_{(i)} = \frac{Z_1}{n} + \frac{Z_2}{n-1} + \ldots + \frac{Z_i}{n-i+1}$$

and prove

$$E(T_{(i)}) = \frac{1}{\lambda} \sum_{j=1}^{i} \frac{1}{n-j+1}$$

Find variances and covariances of $\{T_{(i)}\}$.

b. Campbell's Theorem

Let $\{N(t), t > 0\}$ be a Poisson Process. Assume n events occur in the interval $(0, t]$. Note that $N(t) = n$ is the realization of a random variable and has probability $P\{N(t) = n\} = e^{-\lambda t} \dfrac{(\lambda t)^n}{n!}$

Define $W_n =$ Waiting time for n^{th} event.

If $\{T_i\}$ $i = 1, 2, \ldots, n$ are the random variables representing the time between events

$$f(t_1, \ldots, t_n) = \Pi_1^n \lambda e^{-\lambda t_i} = \lambda^n e^{-\lambda \sum_1^n t_i}$$

But $\displaystyle\sum_1^n t_i = W_n,$ hence

$$f(t_1, \ldots, t_n) = \lambda^n e^{-\lambda W_n}$$

$$f(t_1, \dots, t_n) = \lambda^n e^{-\lambda W_n}$$

Now consider the transformation

$$W_1 = t_1, \quad W_2 = t_1 + t_2, \dots, \quad W_n = t_1 + t_2 + \dots + t_n$$

The distribution of $\underline{\mathbf{W}} = (W_1, W_2, \dots, W_n)$ is

$$f(\underline{\mathbf{W}}) = f(\underline{\mathbf{t}}) \left| \frac{\partial(\underline{\mathbf{t}})}{\partial \underline{\mathbf{W}}} \right|$$

where $\left| \frac{\partial(\underline{\mathbf{t}})}{\partial \underline{\mathbf{W}}} \right|$ is the determinant of the Jacobian.

Note: $$\frac{\partial(\underline{\mathbf{W}})}{\partial \underline{\mathbf{t}}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & 0 & \ldots & 0 \\ 1 & 1 & 1 & 1 & \ldots & 0 \\ \vdots & & & & & \\ 1 & 1 & 1 & 1 & \ldots & 1 \end{pmatrix}$$

and $$\left|\frac{\partial(\underline{\mathbf{t}})}{\partial \underline{\mathbf{W}}}\right| = \left|\frac{\partial(\underline{\mathbf{W}})}{\partial \underline{\mathbf{t}}}\right|^{-1} = 1$$

$$\therefore \quad f(w_1, \ldots, w_n) = \lambda^n e^{-\lambda w_n} \quad 0 < w_1 \leq \ldots \leq w_n < t.$$

But there are no events in the interval $(w_n, t]$. This carries probability $e^{-\lambda(t-w_n)}$. Hence the joint distribution of the $\underline{\mathbf{W}}$ is

$$f(\underline{\mathbf{W}}) = \lambda^n e^{-\lambda w_n} \cdot e^{-\lambda(t-w_n)} = \lambda^n e^{-\lambda t}$$

$$f\left(\underline{\mathbf{W}}\right) = \lambda^n e^{-\lambda t} \quad 0 \leq w_1 \leq w_2 \leq \ldots \leq w_n < t$$

Consider

$$f(\underline{\mathbf{W}}|N(t) = n) = \frac{\lambda^n e^{-\lambda t}}{e^{\lambda t}(\lambda t)^n / n!} = n!/t^n.$$

This is the joint distribution of the order statistics from a uniform $(0, t)$ distribution; i.e., $f(x) = \dfrac{1}{t} \quad 0 < x \leq t$.

Hence $E(W_i|N(t) = n) = \dfrac{it}{n+1} \quad i = 1, 2, \ldots, n$

We can consider the unordered waiting times, conditional on $N(t) = n$, as following a uniform $(0, t)$ distribution.

Since $w_1 = t_1,\ w_2 = t_1 + t_2, \ldots,\ w_n = t_1 + t_2 + \ldots + t_n$

$$t_i = w_i - w_{i-1} \quad (w_0 = 0)$$

The difference between the waiting times are the original times t_i. These times follow the distribution conditional on $N(t) = n$; i.e.

$$f(t_1, \ldots, t_n | N(t) = n) = n!/t^n$$

Note that if $f(t_i) = 1/t \quad 0 < t_i < t$, the joint distribution for $i = 1, 2, \ldots, n$ of n independent uniform $(0, t)$ random variables is $f(\underline{\mathbf{t}}) = 1/t^n$. If $0 < t_{(1)} \leq t_{(2)} \leq \ldots \leq t_{(n)} < t$ the distribution of the order statistics is

$$f(t_{(1)}, \ldots, t_{(n)}) = n!/t^n$$

which is the same as $f(t_1, \ldots, t_n | N(t))$.

c. Minimum of Several Exponential Random Variables

Let T_i $(i = 1, \ldots, n)$ be ind. exponential r.v. with parameter λ_i and let $T = min(T_1, \ldots, T_n)$

$\rightarrow \quad P\{T > t\} = P\{T_1 > t, T_2 > t, \ldots, T_n > t\} = \pi_{i=1}^{n} P\{T_i > t\}$

$$= \pi_{i=1}^{n} e^{-\lambda_i t} = e^{-\lambda t}, \quad \lambda = \sum_{1}^{n} \lambda_i$$

$\rightarrow \quad T$ is exponential with parameter λ

$$P\{T > t\} = e^{-\lambda t} \qquad \lambda = \sum_{i=1}^{n} \lambda_i$$

$$T = \min(T_1, \ldots, T_n)$$

If all $\lambda_i = \lambda_0, \; \lambda = n\lambda_0, \; P\{T > t\} = e^{-n\lambda_0 t}$

Define N as the index of the random variable which is the smallest failure time.

For example if $T_r \le T_i$ for all i, then $N = r$.

Consider $P\{T > t, T_r \le T_i \text{ all } i\} = P\{N = r, T > t\}$

$$
\begin{aligned}
P\{N = r, T > t\} &= P\{T > t, T_i \ge T_r, i \ne r\} \\
&= \int_t^\infty P\{T > t_r, i \ne r \mid t_r\} f(t_r) dt_r \\
&= \int_t^\infty e^{-(\lambda - \lambda_r) t_r} \lambda_r e^{-\lambda_r t_r} dt_r \\
&= \lambda_r \int_t^\infty e^{-\lambda t_r} dt_r \\
&= \frac{\lambda_r}{\lambda} e^{-\lambda t}
\end{aligned}
$$

$$\boxed{P\{N = r,\ T > t\} = \frac{\lambda_r}{\lambda} e^{-\lambda t}}$$

$$P\{N = r, T > 0\} = P\{N = r\} = \frac{\lambda_r}{\lambda}, \lambda = \sum_1^n \lambda_i$$

$\rightarrow$ $\boxed{P\{N = r, T > t\} = P\{N = r\} P\{T > t\}}$

$\rightarrow$ N (index of smallest) and T are independent

If $\lambda_i = \lambda_0$

$$P\{N = r\} = \frac{\lambda_0}{n\lambda_0} = \frac{1}{n}$$

(All populations have the same prob. of being the smallest.)

D. Relation to Erlang and Gamma Distribution

Consider $T = T_1 + \ldots + T_n$

Since $q_i^*(s) = \dfrac{\lambda_i}{\lambda_i + s}$, $q_T^*(s) = \prod_{i=1}^{n} \dfrac{\lambda_i}{\lambda_i + s}$

which is L.T. of Erlang distribution. If λ_i are all distinct

$$\boxed{q(t) = \sum_{i=1}^{n} A_i e^{-\lambda_i t}} \quad , A_i = \prod_{j \neq i} \frac{\lambda_i}{\lambda_j - \lambda_i}$$

If $\lambda_i = \lambda$, $\quad q_T^*(s) = \left(\dfrac{\lambda}{\lambda + s} \right)^n$

$$\boxed{q(t) = \frac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{\Gamma(n)}} \quad \text{Gamma Distribution}$$

E. Guarantee Time

Consider the r.v. following the distribution having pdf

$$\begin{aligned} q(t) &= \lambda e^{-\lambda(t-G)} \quad \text{for } t > G \\ &= 0 \quad \text{for } t \leq G \end{aligned}$$

The parameter G is called a guarantee time

If the transformation $Y = T - G$ is made then $f(y) = \lambda e^{-\lambda y}$ for $y > 0$.

$\therefore \; E(Y) = 1/\lambda, \quad V(Y) = 1/\lambda^2, \ldots$

Since $T = Y + G$, $\boxed{E(T) = \frac{1}{\lambda} + G}$

and central moments if T and Y are the same.

F. Random Sums of Exponential Random Variables

Let $\{T_i\}$ $i = 1, 2, \ldots, N$ be iid with $f(t) = \lambda e^{-\lambda t}$ and consider

$$S_N = T_1 + T_2 + \ldots + T_N$$

with $P\{N = n\} = p_n$.

The Laplace Transform of S_N is $\boxed{(\lambda/\lambda + s)^n}$ for fixed $N = n$. Hence $f^*(s) = E\left(\frac{\lambda}{\lambda + s}\right)^N$ resulting in a pdf which is a mixture of gamma distributions.

$$f(t) = \sum_{n=1}^{\infty} \frac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{\Gamma(n)} p_n$$

Suppose $p_n = p^{n-1}q \;\; n = 1, 2, \ldots$ (negative exponential distribution)

$$f^*(s) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda + s} \right)^n p^{n-1} q = \sum_{n=1}^{\infty} \frac{q}{p} \left(\frac{p\lambda}{\lambda + s} \right)^n = \frac{q}{p} \left[\frac{p\lambda/\lambda + s}{1 - \frac{p\lambda}{\lambda+s}} \right]$$

$$= \frac{q}{p} \cdot \frac{p\lambda}{s + \lambda(1-p)} = \frac{q\lambda}{s + q\lambda}$$

$$\boxed{f^*(s) = \frac{q\lambda}{s + q\lambda}}$$

$$\Rightarrow \quad S_N = T_1 + T_2 + \ldots + T_N, \quad P\{N = n\} = p^{n-1}q$$

has exponential distribution with parameter (λq).

4.4 Counting Processes and the Poisson Distribution

Definition: A stochastic process $\{N(t), T > 0\}$ is said to be a counting process where $N(t)$ denotes the number of events that have occurred in the interval $(0, t]$. It has the properties.

(i.) $N(t)$ is integer value

(ii.) $N(t) \geq 0$

(iii.) If $s < t$, $N(s) \leq N(t)$ and $N(t) - N(s) =$ number of events occurring is $(s, t]$.

A counting process has independent increments if the events in disjoint intervals are independent; i.e. $N(s)$ are $N(t) - N(s)$ are independent events.

A counting process has stationary increments if the probability of the number of events in any interval depends only on the length of the interval; i.e.

$$N(t) \text{ and } N(s+t) - N(s)$$

have the same probability distribution for all s. A Poisson process is a counting process having independent and stationary increments.

TH. Assume $\{N(t), t \geq 0\}$ is a Poisson Process. Then the dsitribution of $N_s(t) = N(s+t) - N(s)$ is independent of s and only depends on the length of the interval, i.e.

$$P\{N(t+s) - N(s)|N(s)\} = P\{N(t+s) - N(s)\}$$

for all s. This implies that knowledge of $N(u)$ for $0 < u \leq s$ is also irrelevant.

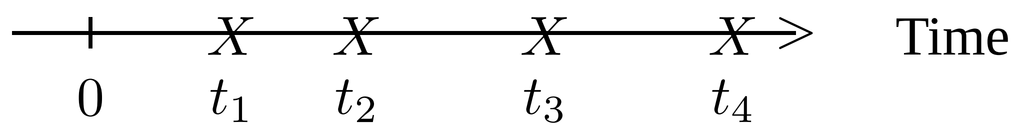
$$P\{N(t+s) - N(s)|N(u), 0 < u \leq s\}$$

$$= P\{N(t+s) - N(s)\}.$$

This feature defines a stationary process.

TH. A Poisson Process has independent increments.

Consider $0 \leq t_1 < t_2 < t_3 < t_4$



Consider events in $(t_3, t_4]$; i.e.

$$N(t_4) - N(t_3)$$

$$P\{N(t_4) - N(t_3) \mid N(u), \;\; o < u \leq t_3\}$$

$$= P\{N(t_4) - N(t_3)\}.$$

Distribution is independent of what happened prior to t_3. Hence if the intervals $(t_1, t_2]$ and (t_2, t_4) are non-overlapping $N(t_2) - N(t_1)$ and $N(t_4) - N(t_3)$ are independent.

TH. $Cov(N(t), N(s+t)) = \lambda t$ (Poisson Process)

Proof $N(s+t) - N(t)$ is independent of $N(t)$

$Cov(N(s+t) - N(t), N(t)) = 0$

$= Cov(N(s+t), N(t)) - V(N(t)) = 0$

$\therefore Cov(N(s+t), N(t)) = V(N(t)) = \lambda t$

as variance of $N(t)$ is λt.

An alternative statement of theorem is

$$\boxed{Cov(N(s), N(t)) = \lambda \min(s, t)}$$

TH. A counting process $\{N(t), t \geq 0\}$ is a Poisson Process if and only if

(i) It has stationary and independent increments

(ii) N(0) = 0 and

$$
\begin{aligned}
P\{N(h) = 0\} &= 1 - \lambda h + 0(h) \\
P\{N(h) = 1\} &= \lambda h + 0(h) \\
P\{N(h) = j\} &= 0(h), \quad j > 1
\end{aligned}
$$

Notes: The notation $0(h)$ "little o of h' refers to some function $\varphi(h)$ for which

$$\lim_{h \to 0} \frac{\varphi(h)}{h} = 0$$

Divide interval $(0, t]$ into $\underline{n}$ sub-intervals of length h; i.e. $nh = t$

$$P\{N(kh) - N(k-1)h)\} = P\{N(h)\}$$

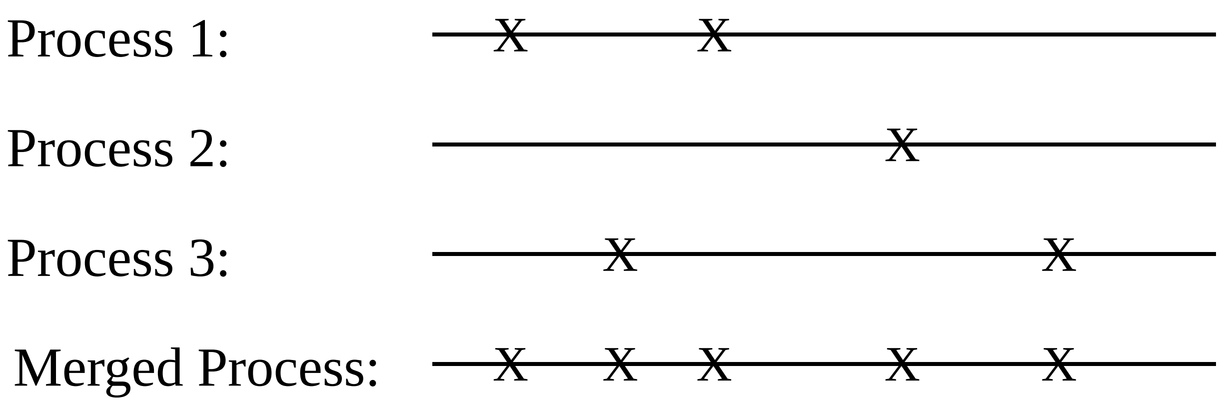
$$
\begin{aligned}
T &= \text{Time to event beginning at } t = 0. \\
P\{T > t\} &= P\{N(t) = 0\} = P\{\text{No events in each sub-interval}\} \\
P\{N(t) = 0\} &= P\{T > t\} = [1 - \lambda h + o(h)]^n \\
&= (1 - \lambda h)^n + n(1 - \lambda h)^{n-1} o(h) + o(h^2) \\
&= (1 - \lambda h)^n \{1 + \frac{n\, o(h)}{1 - \lambda h} + \dots\} \\
&= \left(1 - \frac{\lambda t}{n}\right)^n \left\{1 + \frac{t}{1 - \frac{\lambda t}{n}} \frac{o(h)}{h} + \dots\right\} \\
&\to\ e^{-\lambda t} \ \text{ as } \ n \to \infty, \ \ h \to 0 \\
\Rightarrow &\quad P\{T > t\} = e^{-\lambda t}
\end{aligned}
$$

Hence T is exponential; i.e. Time between events is exponential.

$$\Rightarrow \quad \{N(t), t \geq 0\} \text{ is Poisson Process}$$

4.5 Superposition of Counting Processes

Suppose there are k counting processes which merge into a single counting process; e.g. $k = 3$.



The merged process is called the superposition of the individual counting processes

$$N(t) = N_1(t) + N_2(t) + \ldots + N_k(t)$$

A. Superposition of Poisson Processes

$$N(t) = N_1(t) + \ldots + N_k(t)$$

Suppose $\{N_i(t), t \geq 0\}$ $i = 1, 2, \ldots, k$ are Poisson Processes with $E[N_i(t)] = \lambda_i t$.

Note that each of the counting processes has stationary and independent increments.

Also $N(t)$ is Poisson with parameter

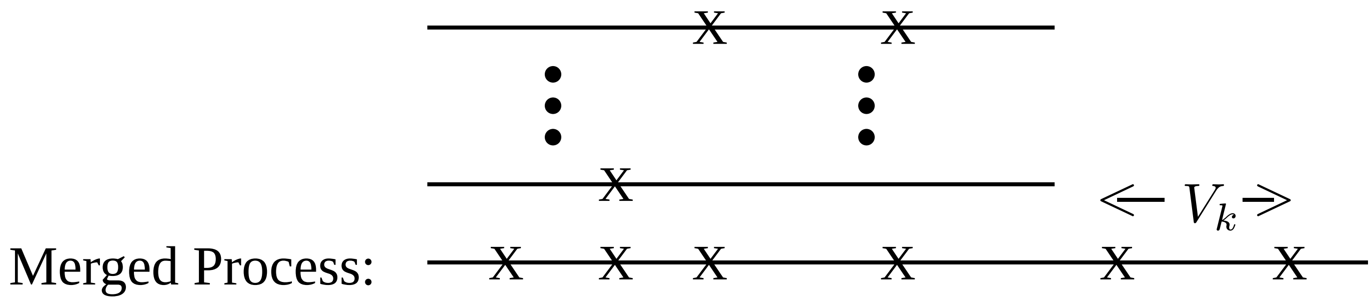
$$E(N(t)) = \sum_{i=1}^{k} (\lambda_i t) = t\lambda, \quad \lambda = \sum_{i=1}^{k} \lambda_i$$

$$\Rightarrow \quad N(t) \text{ is a Poisson Process}$$

Hence $\{N(t), t \geq 0\}$ has stationary and independent increments.

B. <u>General Case of Merged Process</u>

Consider the merged process from k individual processes



The random variable V_k is the forward recurrence time of the merged process. We will show that as $k \to \infty$, the asymptotic distribution of V_k is exponential and hence the merged process is asymptotically a Poisson Process.

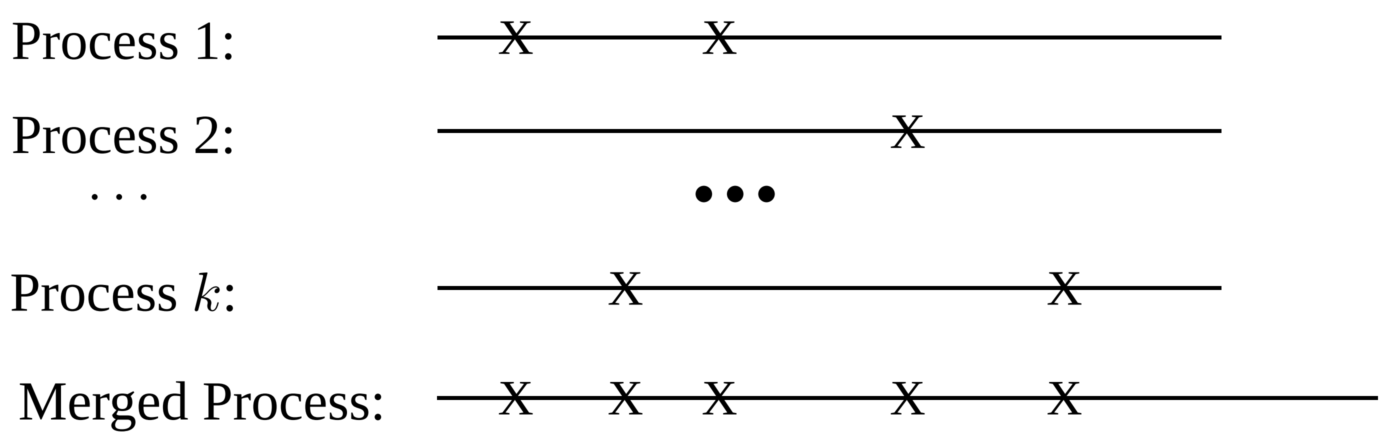
Assume that for each of the processes

- Stationary
- Multiple occurences have 0 probability
- pdf between events of each process is $q(t)$.

If $q(t)$ is pdf of time between events for a single process, then each has the same forward recurrence time distribution with pdf

$$q_f(x) = Q(x)/m$$

With k independent processes there will be
$T_f(1), T_f(2), \ldots, T_f(k)$ forward recurrence time random variables



$$V_k = \min(T_f(1), T_f(2), \ldots, T_f(k))$$

$$P\{V_k > v\} = G_k(v) = P\{T_f(1) > v, T_f(2) > v, \ldots, T_f(k) > v\}$$
$$= Q_f(v)^k$$

$G_k(v) = P\{V_k > v\} = Q_f(v)^k$ where

$$Q_f(v) = \int_v^\infty q_f(x)dx, \quad q_f(x) = Q(x)/m$$

Let $g_k(x) =$ pdf of merged process

$$\begin{aligned} G_k(v) &= \int_v^\infty g_k(x)dx = Q_f(v)^k \\ -\frac{d}{dv}G_k(v) &= g_k(v) = kQ_f(v)^{k-1}q_f(v) \end{aligned}$$

$$\boxed{g_k(v) = kQ_f(v)^{k-1}\frac{Q(v)}{m}}$$

Consider transformation $z = \dfrac{V_k}{m/k} = \dfrac{kV_k}{m}, \quad \dfrac{dz}{dv} = \dfrac{k}{m}$

$$g_k(z) = g_k(v) \left| \frac{\partial V}{\partial z} \right| = \frac{k}{m} Q_f \left(\frac{mz}{k} \right)^{k-1} Q \left(\frac{mz}{k} \right) \frac{m}{k}$$

$$\boxed{g_k(z) = Q \left(\frac{mz}{k} \right) \left[1 - \int_o^{\frac{mz}{k}} \frac{Q(x)}{m} dx \right]^{k-1}}$$

as $\quad Q_f \left(\dfrac{mz}{k} \right) = \displaystyle\int_{\frac{mz}{k}}^{\infty} \frac{Q(x)}{m} dx = 1 - \int_0^{\frac{mz}{k}} \frac{Q(x)}{m} dx$

For fixed z,

as $k \to \infty, \quad \dfrac{zm}{k} \to 0$ and $\quad Q \left(\dfrac{mz}{k} \right) \to 1$

Also

$$\int_0^{\frac{mz}{k}} \frac{Q(x)}{m} dx \to \frac{Q\left(\frac{mz}{k}\right)}{m} \cdot \frac{mz}{k} = Q\left(\frac{mz}{k}\right)\frac{z}{k} \to \frac{z}{k}$$

$\therefore$ as $k \to \infty$

$$\boxed{g_k(z) \to \left(1 - \frac{z}{k}\right)^{k-1} \to e^{-z}}$$

Thus as $k \to \infty$, the forward recurrence time (multiplied by $\frac{m}{k}$) $z = \frac{m}{k}V_k$ is distributed as a unit exponential distribution. Hence for large k, $V_k = \frac{k}{m}z$ has an asymptotic exponential distribution with parameter $\lambda = k/m$. Since the asymptotic forward recurrence time is exponential, the time between events (of the merged process), is asymptotically exponential.

Note: A forward recurrence time is exponential if and only if the time between events is exponential; ie.

$$q_f(x) = \frac{Q(x)}{m} = \lambda e^{-\lambda x} \text{ if } Q(x) = e^{-\lambda x}$$

and if $q_f(x) = \lambda e^{-\lambda x} \Rightarrow Q(x) = e^{-\lambda x}$

Additional Note: The merged process is $N(t) = \sum_{i=1}^{k} N_i(t)$. Suppose $E(N_i(t)) = \nu t$. Units of ν are "no. of events per unit time"

The units of m are "time per event"

Thus $E(N(t)) = (k\nu)t$ and $(k\nu)$ is mean events per unit time. The units of $\left(\frac{1}{k\nu}\right)$ or $\left(\frac{1}{\nu}\right)$ is "mean time per event". Hence $m = 1/\nu$ for an individual process and the mean of the merged process is $1/k\nu$.

Ex. $\nu = 6$ events per year $\Rightarrow m = \frac{12}{6} = 2$ months (mean time between events).

5. Splitting of Poisson Processes

Example: Times between births (in a family) follow an exponential distribution. The births are categorized by gender.

Example: Times between back pain follow an exponential distribution. However the degree of pain may be categorized as the required medication depends on the degree of pain.

Consider a Poisson Process $\{N(t), t \geq 0\}$ where in addition to observing an event, the event can be classified as belonging to one of r possible categories.

Define $N_i(t) \ = \$ no. of events of type i during $(0, t]$ for $i = 1, 2, \ldots, r$

$$\Rightarrow \quad \boxed{N(t) = N_1(t) + N_2(t) + \ldots + N_r(t)}$$

This process is referred to as "splitting" the process.

Bernoulli Splitting Mechanism

Suppose an event takes place in the interval $(t, t+dt]$. Define the indicator random variable $Z(t) = i$ $(i = 1, 2, \ldots, r)$ such that

$$P\{Z(T) = i | \text{event at } (t, t+dt]\} = p_i.$$

Note p_i is independent of time.

Then if $N(t) = \sum_{i=1}^{r} N_i(t)$ the counting processes $\{N_i(t), t \geq 0\}$ are Poisson process with parameter (λp_i) for $i = 1, 2, \ldots, r$.

Proof: Suppose over time $(0, t]$, n events are observed of which s_i are classified as of time i with $\sum_{i=1}^{r} s_i = n$.

$$P\{N_1(t) = s_1, N_2(t) = s_2, \ldots, N_r(t) = s_r | N(t) = n\}$$
$$= \frac{n!}{s_1! s_2! \ldots s_r!}\, p_1^{s_1} p_2^{s_2} \ldots p_r^{s_r}$$

Hence $\quad P\{N_i(t) = s_i, i = 1, \ldots, r \text{ and } N(t) = n\}$

$$= \frac{n!}{\prod_{i=1}^{r} s_i!} p_1^{s_1} p_2^{s_2} \ldots p_r^{s_r} \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$
$$= \prod_{i=1}^{r} \frac{(p_i \lambda t)^{s_i} e^{-p_i \lambda t}}{s_i!} = \prod_{i=1}^{r} P\{N_i(t) = s_i\}$$

which shows that the $\{N_i(t)\}$ are independent and follow Poisson distributions with parameters $\{\lambda p_i\}$.

$\Rightarrow \quad \{N_i(t), t \geq 0\}$ are Poisson Processes.

Example of Nonhomogenous Splitting

Suppose a person is subject to serious migraine headaches. Some of these are so serious that medical attention is required. Define

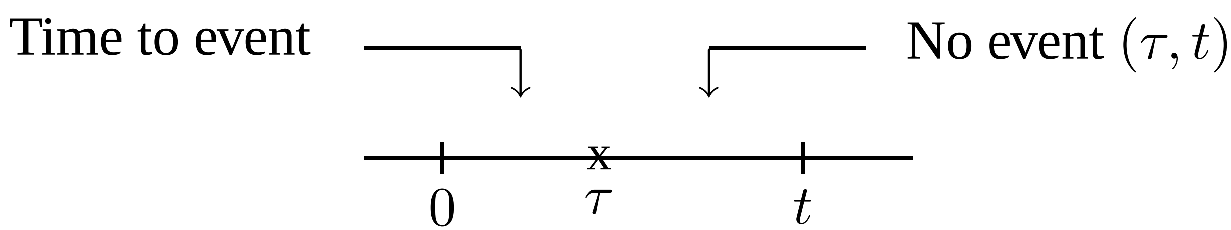
$$\begin{aligned} N(t) &= \text{ no. of migraine headaches in } (0, t] \\ N_m(t) &= \text{ no. of migraine headaches requiring medical attention} \\ p(\tau) &= \text{ prob. requiring medical attention if} \\ &\quad \text{ headache occurs at } (\tau, \tau + d\tau). \end{aligned}$$

Suppose an event occurs at $(\tau, \tau + d\tau)$; then Prob.of requiring attention $= p(\tau)d\tau$.

Note that conditional on a single event taking place in $(0, t], \tau$ is uniform over $(0, t]$; i.e.

$$f(\tau|N(t) = 1) = 1/t \quad 0 < \tau \le t \quad \text{and} \quad \alpha = \frac{1}{t}\int_0^t p(\tau)d\tau$$

$$\alpha = \frac{1}{t}\int_0^t p(\tau)d\tau$$



$$\therefore P\{N_m(t) = k | N(t) = n\} = \binom{n}{k}\alpha^k(1-\alpha)^{(n-k)}$$

$$P\{N_m(t) = k, N(t) = n\} = \binom{n}{k}\alpha^k(1-\alpha)^{n-k}\frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$P\{N_m(t) = k\} = \sum_{n=k}^{\infty} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \frac{\alpha^k}{k!} e^{-\lambda t} \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{(n-k)!} (1-\alpha)^{n-k}$$

$$= \frac{\alpha^k}{k!} e^{-\lambda t} (\lambda t)^k \sum_{n=k}^{\infty} \frac{(\lambda t)^{n-k} (1-\alpha)^{n-k}}{(n-k)!}$$

$$= \frac{\alpha^k}{k!} (\lambda t)^k e^{-\lambda t} \cdot e^{\lambda t(1-\alpha)}$$

$$\boxed{P\{N_m(t) = k\} = e^{-\alpha \lambda t} \frac{(\alpha \lambda t)^k}{k!}}$$

4.7 Non-homogeneous Poisson Processes

Preliminaries

Let $N(t)$ follow a Poisson distribution; i.e.

$$P\{N(t) = k\} = e^{-\lambda t}(\lambda t)^k / k!$$

Holding t fixed, the generating function of the distribution is

$$\phi_{N(t)}(s) = E[e^{-sN(t)}] = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} e^{-sk}$$

$$= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(e^s \lambda t)^k}{k!} = e^{-\lambda t} e^{e^{-s} \lambda t}$$

$$\boxed{\phi_{N(t)}(s) = e^{\lambda t[e^{-s} - 1]}} \quad = e^{\lambda t(z-1)} \text{ if } z = e^{-s}$$

The mean is $E[N(t)] = \lambda t$

Consider the Counting Process $\{N(t), t \geq 0\}$ having the Laplace Transform

(*) $$\boxed{\phi_{N(t)}(s) = e^{\Lambda(t)[e^{-s}-1]} = e^{\Lambda(t)[z-1]}}$$

$$\Rightarrow \qquad E[N(t)] = \Lambda(t), \quad P\{N(t) = k\} = e^{-\Lambda(t)}\,[\Lambda(t)]^k/k!$$

For the Poisson Process $\Lambda(t) = \lambda t$ and the mean is proportional to t. However when $E[N(t)] \neq \lambda t$ we call the process $\{N(t), t \geq 0\}$ a non-homogenized Poisson Process and $\boxed{E(N(t)] = \Lambda(t)}$

$\Lambda(t)$ can be assumed to be continuous and differentiable

$$\frac{d}{dt}\Lambda(t) = \Lambda'(t) = \lambda(t).$$

The quantity $\lambda(t)$ is called intensity function. $\Lambda(t)$ can be represented by

$$\boxed{\Lambda(t) = \int_0^t \lambda(x)dx}$$

If $N(t)$ has the Transform given by $(*)$ then

$$P\{N(t) = k\} = e^{-\Lambda(t)}\Lambda(t)^k/k!$$

Since $P\{S_n > t\} = P\{N(t) < n\}$

We have $P\{S_1 > t\} = P\{N(t) < 1\} = P\{N(t) = 0\}$

$$\boxed{P(S_1 > t) = e^{-\Lambda(t)}}$$

Thus pdf of time between events is

$$f(t) = \lambda(t)e^{-\int_0^t \lambda(x)dx}, \quad \Lambda(t) = \int_0^t \lambda(x)dx$$

Note that if $H = \Lambda(t)$, then H is a random variable following a unit exponential distribution.

Assume independent increments; i.e. $N(t+\mu) - N(\mu)$ and $N(\mu)$ are independent

L.T. Transform $\boxed{\psi(z,t) = e^{\Lambda(t)[z-1]}}$ $z = e^{-s}$

Generating function $= E[e^{-sN(t)}] = E[z^{N(t)}]$

$$e^{\Lambda(t+u)(z-1)} = E[z^{N(t+u)}] \quad = E[z^{N(t+u)-N(u)+N(u)}]$$

$$= E[z^{N(t+u)-N(u)}] \cdot E[z^{N(u)}]$$

$$= \psi e^{\Lambda(u)[z-1]}$$

$$\therefore \quad \psi = E[z^{N(t+u)-N(u)]}] = \frac{e^{\Lambda(t+u)(z-1)}}{e^{\Lambda(u)(z-1)}} = e^{[\Lambda(t+u)-\Lambda(u)][z-1]}$$

where $\Lambda(t+u) - \Lambda(u) = \int_u^{t+u} \lambda(x)dx$

$$\therefore \quad \boxed{P\{N(t+u) - N(u) = k\} = \frac{e^{-[\Lambda(t+u)-\Lambda(u)]}[\Lambda(t+u) - \Lambda(u)]^k}{k!}}$$

Axiomatic Derivation of Non-Homogenized Poisson Distribution

Assume counting process $\{N(t), t \geq 0\}$

(i) $N(0) = 0$

(ii) $\{N(t), t \geq 0\}$ has independent increments; i.e. $N(t+s) - N(s)$ and $N(s)$ are independent

(iii) $P\{N(t+h) = k | N(t) = k\} = 1 - \lambda(t)h + 0(h)$
$P\{N(t+h) = k+1 | N(t) = k\} = \lambda(t) + 0(h)$
$P\{N(t+h) = k+j | N(t) = k\} = o(h) \;\; j \geq 2$

$$\Rightarrow \quad p\{N(t+s) - N(s) = k\} = e^{-[\Lambda(t+s) - \Lambda(s)]^2} \frac{[\Lambda(t+s) - \Lambda(s)]^k}{k!}$$

4.8 Compound Poisson Process

Example. Consider a single hypodermic needle which is shared. The times between use follow a Poisson Process. However at each use, several people use it. What is the distribution of total use?

Let $\{N(t), t \geq 0\}$ be a Poisson process and $\{Z_n, n \geq 1\}$ be iid random variables which are independent of $N(t)$. Define

$$Z(t) = \sum_{n=1}^{N(t)} Z_n$$

The process $Z(t)$ is called a Compound Poisson Process. It will be assumed that $\{Z_n\}$ takes on integer values.

Define $A^*(s) = E[e^{-sz_n}]$. Then

$$\phi(s|N(t) = r) = E[e^{-sZ(t)}] = A^*(s)^r$$

$$\phi(s|N(t) = r) = A^*(s)^r$$

$$\phi(s) = \sum_{r=0}^{\infty} \phi(s|N(t) = r) P(N(t) = r)$$

$$= \sum_{r=0}^{\infty} A^*(s)^r \frac{e^{-\lambda t}(\lambda t)^r}{r!} = e^{-\lambda t} \sum_{r=0}^{\infty} \frac{(A^*(s)\lambda t)^r}{r!}$$

$$\phi(s) = e^{-\lambda t} e^{A^*(s)\lambda t} = \boxed{e^{-\lambda t(1-A^*(s))}}$$

$$A^*(s) = E(e^{-sz_N}) = 1 - sm_1 + \frac{s^2}{2} m_2 + \ldots$$

$$-\lambda t(1 - A^*(s)) = -\lambda t[sm_1 - \frac{s^2}{2} m_2 + \ldots], \;\; m_i = E(z_n^i)$$

Cumulant function $= K(s) = \log \phi(s)$

$$K(s) = -sm + \frac{s^2}{2}\sigma^2 + \ldots$$

where (m, σ^2) refer to $Z(t)$.

$$K(s) = -\lambda t[sm_1 - \frac{s^2}{2}m_2 + \ldots]$$

$$\boxed{\begin{aligned} E[Z(t)] &= \lambda t m_1 \\ V[Z(t)] &= \lambda t m_2 \end{aligned}} \quad m_i = E(z_n^i)$$