PRACTICE TEST – SEQUENCES, LIMITS, AND CONTINUITY

SOLUTIONS

1. (a) Suppose $\delta \le 1$. Then when $|x-4| < \delta$, $3 < x < 5$

Then
$$\begin{aligned} |x^2-16| &= |x+4||x-4| \\ &\le (|x|+4)|x-4| \\ &< (5+4)\,\delta \\ &= 9\delta \end{aligned}$$

We require $|x^2-16| < 1/2$, and so we should choose $\delta = \min\{1, 1/18\} = 1/18$.

(b) Let $\{x_n\} = \{\frac{1}{n\pi}\}$. This sequence converges to 0 (why?), but

$$\begin{aligned} \lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} \cos(n\pi) \\ &= \lim_{n\to\infty} (-1)^n \end{aligned}$$

which does not exist.

Hence, $\lim_{x\to 0} \cos\left(\frac{1}{|x|}\right)$ does not exist.

2. For $d \ne 0, \frac{2}{\pi}$, f is clearly continuous at d regardless of the choice of c (why?)

Note that $\lim_{x\to 0} f(x) = 0$ (why?)

and $f(0) = 0$, so f is continuous at 0.

Next, if $\{x_n\} \subseteq [0, 2/\pi]$ and $\lim_{n\to\infty} x_n = \frac{2}{\pi}$,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} c x_n \cos\left(\frac{1}{x_n}\right)$$

$$= \frac{2c}{\pi} \cos\left(\frac{\pi}{2}\right)$$

$$= 0$$

Also, if $\{x_n\} \subseteq (\frac{2}{\pi}, \infty)$ and $\lim_{n\to\infty} x_n = \frac{2}{\pi}$,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n + c = \frac{2}{\pi} + c.$$

Since $f(\frac{2}{\pi}) = \frac{2c}{\pi} \cos\left(\frac{\pi}{2}\right) = 0$, for f to be continuous at $\frac{2}{\pi}$ we need

$$0 = 0 = \frac{2}{\pi} + c$$

$$\Rightarrow \boxed{c = -\frac{2}{\pi}}$$

3. Let $\varepsilon > 0$ be given. Since each $y_n \geq 0$ and $\lim_{n\to\infty} y_n = 0$, we can choose $M \in \mathbb{N}$ such that

$$n \geq M \Rightarrow y_n = |y_n - 0| < \varepsilon$$

Then, for any $m > n \geq M$, $|x_m - x_n| \leq y_n < \varepsilon$. Hence $\{x_n\}$ is Cauchy and therefore convergent.

4. (a) Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon}{21}$.

For any $x, y \in (0, 10)$ such that $|x - y| < \delta$,

$$
\begin{aligned}
|f(x) - f(y)| &= |(x^2 + x) - (y^2 + y)| \\
&= |(x^2 - y^2) + (x - y)| \\
&\leq |x^2 - y^2| + |x - y| \\
&= (|x + y| + 1)|x - y| \\
&\leq (|x| + |y| + 1)|x - y| \\
&< (10 + 10 + 1)\delta \\
&= \varepsilon.
\end{aligned}
$$

$\therefore$ f is uniformly continuous on $(0, 10)$

(b) Let $\{x_n\} = \{1/n\}$ and $\{y_n\} = \{1/n+1\}$

Then $\lim_{n \to \infty} |x_n - y_n| = 0$ (why?)

However, $\lim_{n \to \infty} |f(x_n) - f(y_n)|$

$$
\begin{aligned}
&= \lim_{n \to \infty} |n^2 - (n+1)^2| \\
&= \lim_{n \to \infty} 2n + 1
\end{aligned}
$$

which does not exist. Since this limit is $\neq 0$,

f is not uniformly continuous on $(0, 10)$.

5. Let $f : [0, \pi/4] \to \mathbb{R}$ by $f(x) = 2\tan x - 1 - x$

Note that f is continuous. (why?)

$f(0) = -1 < 0$

$f(\pi/4) = 1 - \pi/4 > 0$

By the Intermediate Value Theorem, $\exists\, c \in (0, \pi/4)$ such that $f(c) = 0$.

Hence $2\tan(c) - 1 - c = 0$

or $2\tan(c) - 1 = c$.