Time filters in weather and climate models

Paul D. Williams

Department of Meteorology, University of Reading, UK

Time-stepping methods

- Weather and climate models are essentially solving the ODE $\mathrm{d}x/\mathrm{d}t = \mathrm{f}(x)$, where x is a large state vector containing the values of all the variables at all the grid-points and f is a given nonlinear function
- Many current models use the "leapfrog" second-order centred discretisation in time, $x_{n+1} = x_{n-1} + 2\,\Delta t\,\mathrm{f}\,(x_n)$, together with a stabilising filter that reduces the accuracy to first-order
- The large $O(\Delta t)$ numerical errors in this scheme reduce the accuracy of weather forecasts and climate predictions
- There is a need to devise better schemes, analyse their theoretical properties, implement them in a hierarchy of models, and test their performance and ability to reduce errors

Children's Games, Pieter Bruegel

If I could win thee at leapfrog,
Or with vawting with my armour on my backe,
Into my saddle,
Without brag be it spoken,
Ide make compare with any.
– Shakespeare, Henry V

Time-stepping methods

Method	Order	Formula	Storage factor	Efficiency factor	Amplitude error	Phase error	Maximum $\omega\Delta t$
Forward (Adams–Bashforth)	1	$\phi^{n+1} = \phi^n + hF(\phi^n)$	2	0	$1 + \frac{p^2}{2}$	$1 - \frac{p^2}{3}$	0
Backward (Adams–Moulton)	1	$\phi^{n+1} = \phi^n + hF(\phi^{n+1})$	Implicit	∞	$1 - \frac{p^2}{2}$	$1 - \frac{p^2}{3}$	∞
Matsuno	1	$\phi^* = \phi^n + hF(\phi^n)$ $\phi^{n+1} = \phi^n + hF(\phi^*)$	2	.5	$1 - \frac{p^2}{2}$	$1 + \frac{2}{3}p^2$	1
Asselin-filtered leapfrog	1	$\phi^{n+1} = \overline{\phi^{n-1}} + 2hF(\phi^n)$ $\overline{\phi^n} = \phi^n + \gamma(\overline{\phi^{n-1}} - 2\phi^n + \phi^{n+1})$	3	<1	$1 - \frac{\gamma}{2(1-\gamma)}p^2$	$1 + \frac{1+2\gamma}{6(1-\gamma)}p^2$	<1
Leapfrog	2	$\phi^{n+1} = \phi^{n-1} + 2hF(\phi^n)$	2	1	1	$1 + \frac{p^2}{6}$	1
Runge–Kutta (Williamson/Huen)	2	$q_1 = hF(\phi^n), \quad \phi_1 = \phi^n + q_1$ $q_2 = hF(\phi_1) - q_1, \quad \phi^{n+1} = \phi_1 + q_2/2$	2	0	$1 + \frac{p^4}{8}$	$1 + \frac{p^2}{6}$	0
Adams–Bashforth	2	$\phi^{n+1} = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$	3	0	$1 + \frac{p^4}{4}$	$1 + \frac{5}{12}p^2$	0
Adams–Moulton (Trapizoidal)	2	$\phi^{n+1} = \phi^n + \frac{h}{2}[F(\phi^{n+1}) + F(\phi^n)]$	Implicit	∞	1	$1 - \frac{p^2}{12}$	∞
Leapfrog, then Adams–Bashforth (Magazenkov)	2	$\phi^n = \phi^{n-2} + 2hF(\phi^{n-1})$ $\phi^{n+1} = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$	3	.67	$1 - \frac{p^4}{4}$	$1 + \frac{p^2}{6}$	.67
Leapfrog-Trapizoidal (Kurihara)	2	$\phi^* = \phi^{n-1} + 2hF(\phi^n)$ $\phi^{n+1} = \phi^n + \frac{h}{2}[F(\phi^n) + F(\phi^*)]$	3	.71	$1 - \frac{p^4}{4}$	$1 - \frac{p^2}{12}$	1.41
Young's method A	2	$\phi_1 = \phi^n + hF(\phi^n)/2$ $\phi_2 = \phi_1 + hF(\phi_1)/2$ $\phi^{n+1} = \phi^n + hF(\phi_2)$	3	0	$1 + \frac{p^6}{128}$	$1 + \frac{p^2}{24}$	0
Runge–Kutta (Williamson)	3	$q_1 = hF(\phi^n), \quad \phi_1 = \phi^n + q_1/3,$ $q_2 = hF(\phi_1) - 5q_1/9, \quad \phi_2 = \phi^1 + 15q_2/16$ $q_3 = hF(\phi_2) - 153q_2/128, \quad \phi^{n+1} = \phi_2 + 8q_3/15$	2	.58	$1 - \frac{p^4}{24}$	$1 + \frac{p^4}{30}$	1.73
ABM predictor-corrector	3	$\phi^* = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$ $\phi^{n+1} = \phi^* + \frac{5h}{12}[F(\phi^*) - 2F(\phi^n) + F(\phi^{n-1})]$	4	.60	$1 - \frac{19}{144}p^4$	$1 + \frac{1243}{8640}p^4$	1.20
Adams–Moulton	3	$\phi^{n+1} = \phi^n + \frac{h}{12}[5F(\phi^{n+1}) + 8F(\phi^n) - F(\phi^{n-1})]$	Implicit	0	$1 + \frac{p^4}{24}$	$1 - \frac{11}{720}p^4$	0
Adams–Bashforth	3	$\phi^{n+1} = \phi^n + \frac{h}{12}[23F(\phi^n) - 16F(\phi^{n+1}) + 5F(\phi^{n-2})]$	4	.72	$1 - \frac{3}{8}p^4$	$1 + \frac{289}{720}p^4$	0.72
Runge–Kutta (Classical)	4	$k_0 = hF(\phi^n)$ $k_1 = hF(\phi^n + k_0/2)$ $k_2 = hF(\phi^n + k_1/2)$ $k_3 = hF(\phi^n + k_2)$ $\phi^{n+1} = \phi^n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$	3*	.70	$1 - \frac{p^6}{144}$	$1 - \frac{p^4}{120}$	2.82
ABM predictor-corrector	4	$\phi^* = \phi^n + \frac{h}{12}[23F(\phi^n) - 16F(\phi^{n-1}) + 5F(\phi^{n-2})]$ $\phi^{n+1} = \phi^* + \frac{3h}{8}[F(\phi^*) - 3F(\phi^n) + 3F(\phi^{n-1}) - F(\phi^{n-2})]$	5	.59	$1 - \frac{265}{1536}p^6$	$1 - \frac{329}{2880}p^4$	1.18
Adams–Moulton	4	$\phi^{n+1} = \phi^n + \frac{h}{24}[9F(\phi^{n+1}) + 19F(\phi^n) - 5F(\phi^{n-1}) + F(\phi^{n-2})]$	Implicit	0	$1 + \frac{p^6}{48}$	$1 + \frac{19}{720}p^4$	0
Adams–Bashforth	4	$\phi^{n+1} = \phi^n + \frac{h}{24}[55F(\phi^n) - 59F(\phi^{n-1}) + 37F(\phi^{n-2}) - 9F(\phi^{n-3})]$	5	.43	$1 - \frac{13}{24}p^6$	$1 - \frac{251}{720}p^4$	.43

weather and climate models

computational fluid dynamics

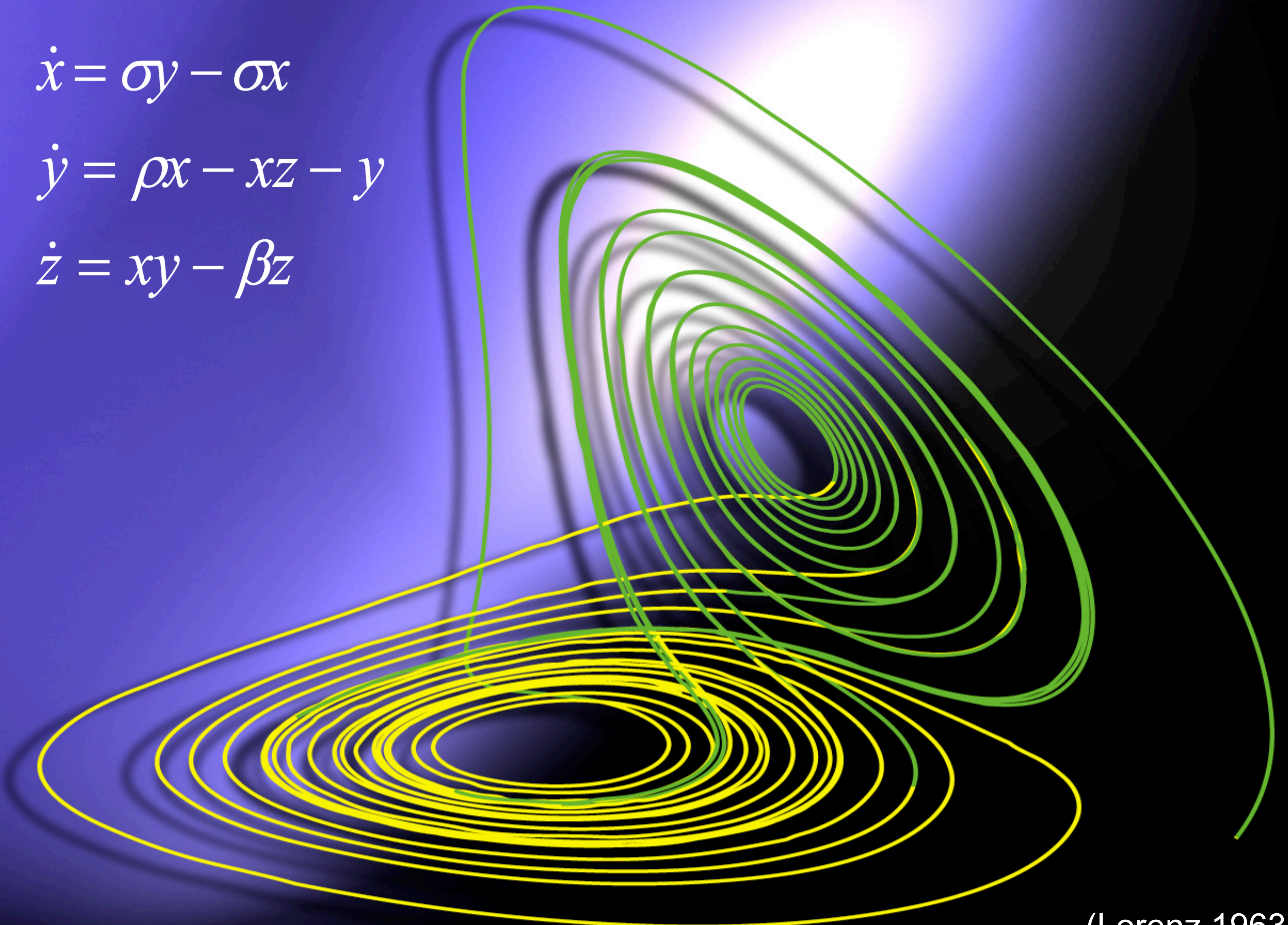
increasing accuracy

(Durran 1991)

$$\dot{x} = \sigma y - \sigma x$$

$$\dot{y} = \rho x - xz - y$$

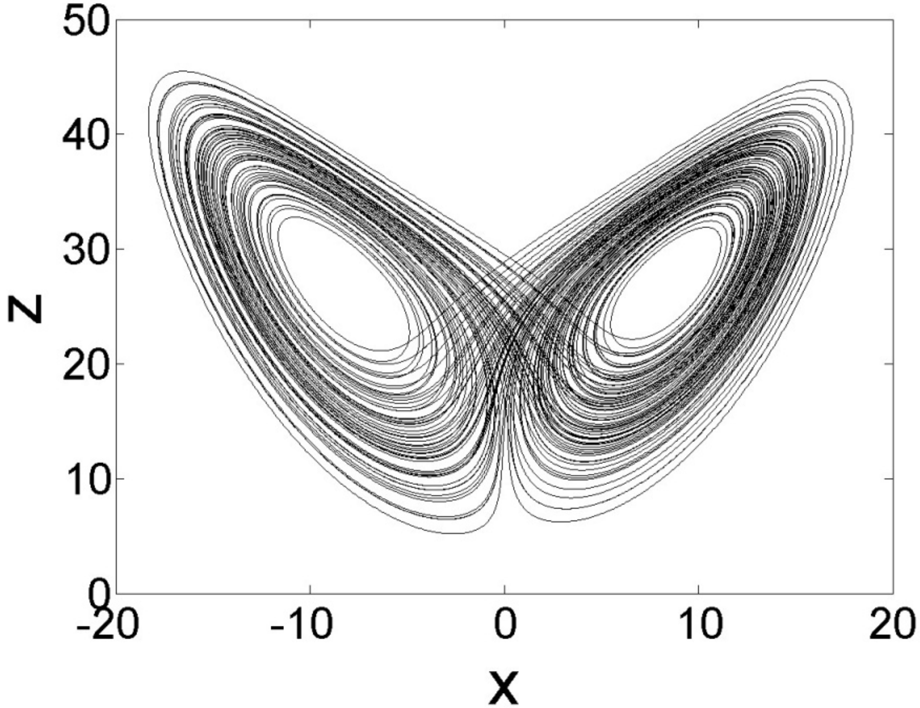
$$\dot{z} = xy - \beta z$$



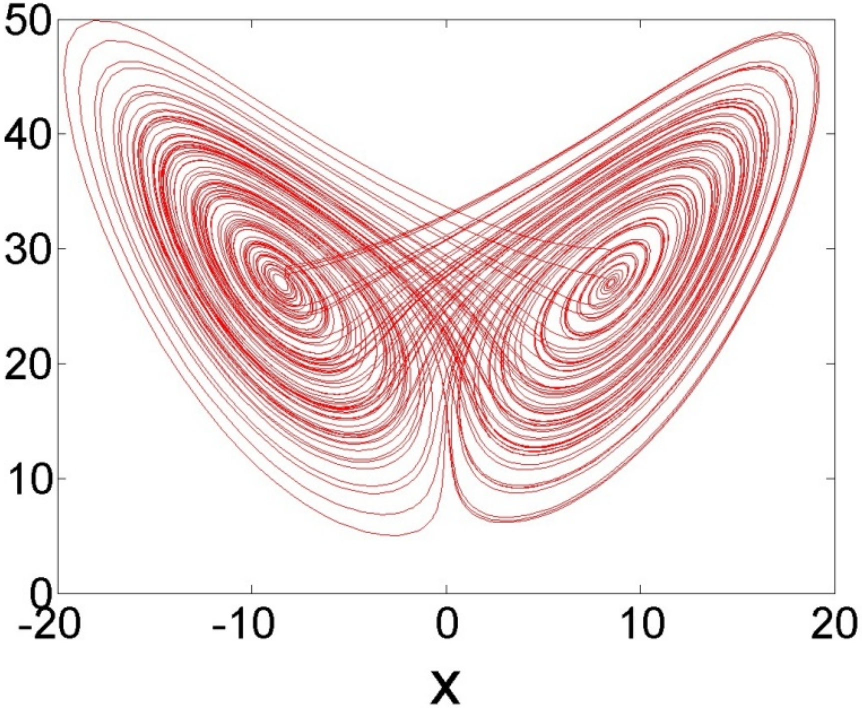
(Lorenz 1963)

Impact of different time steps on the 'climate' of the Lorenz attractor

Δt = 0.001

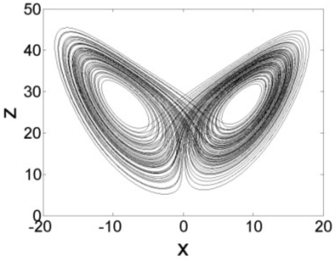


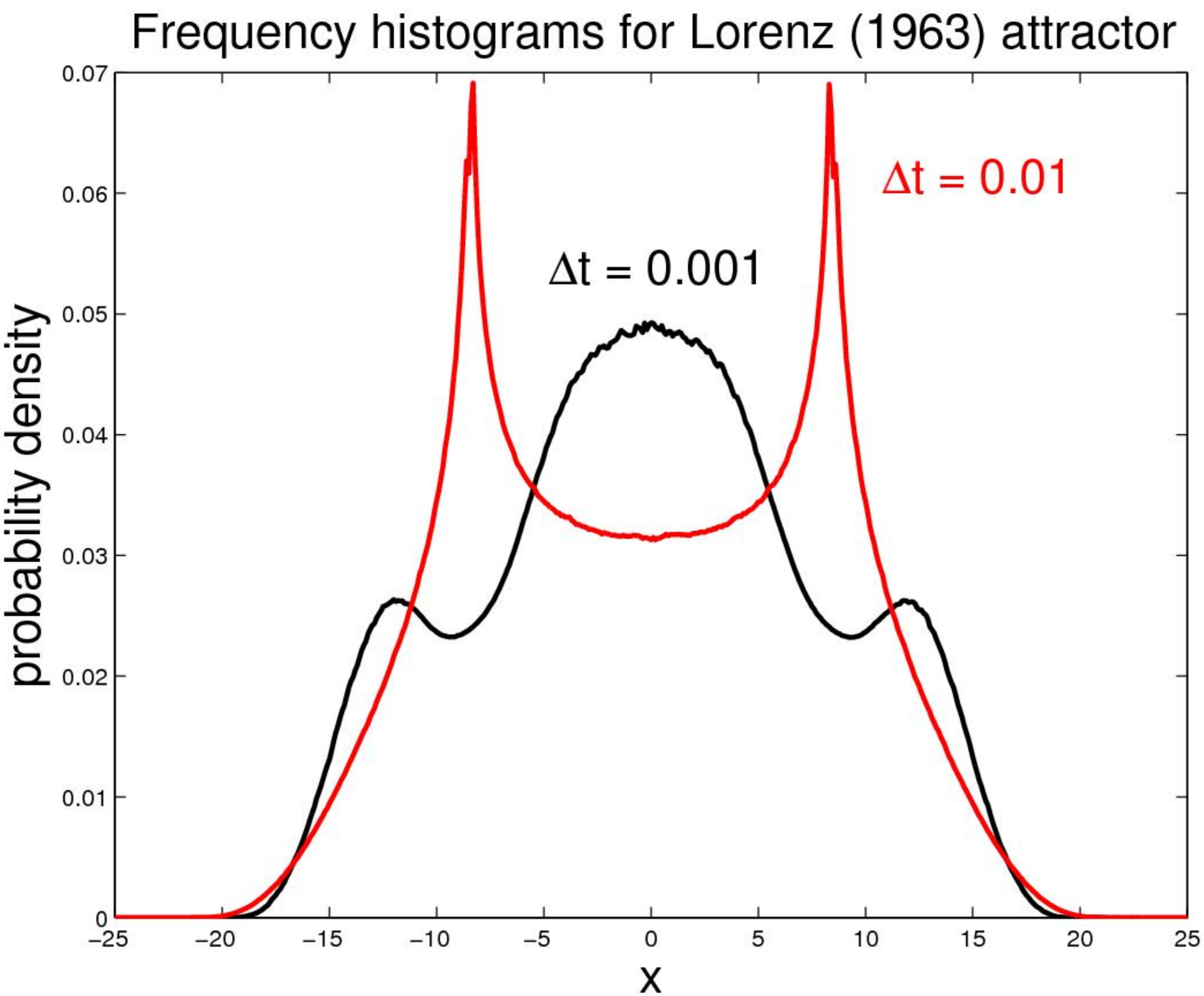
Δt = 0.01

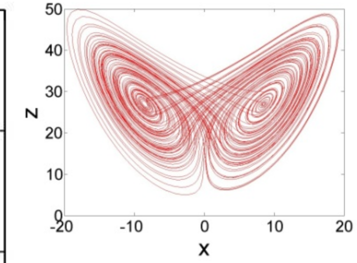


Using the explicit Euler forward scheme

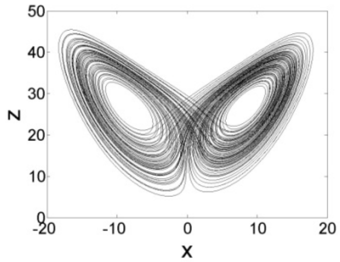
Impact of different time steps on the ‘climate’ of the Lorenz attractor

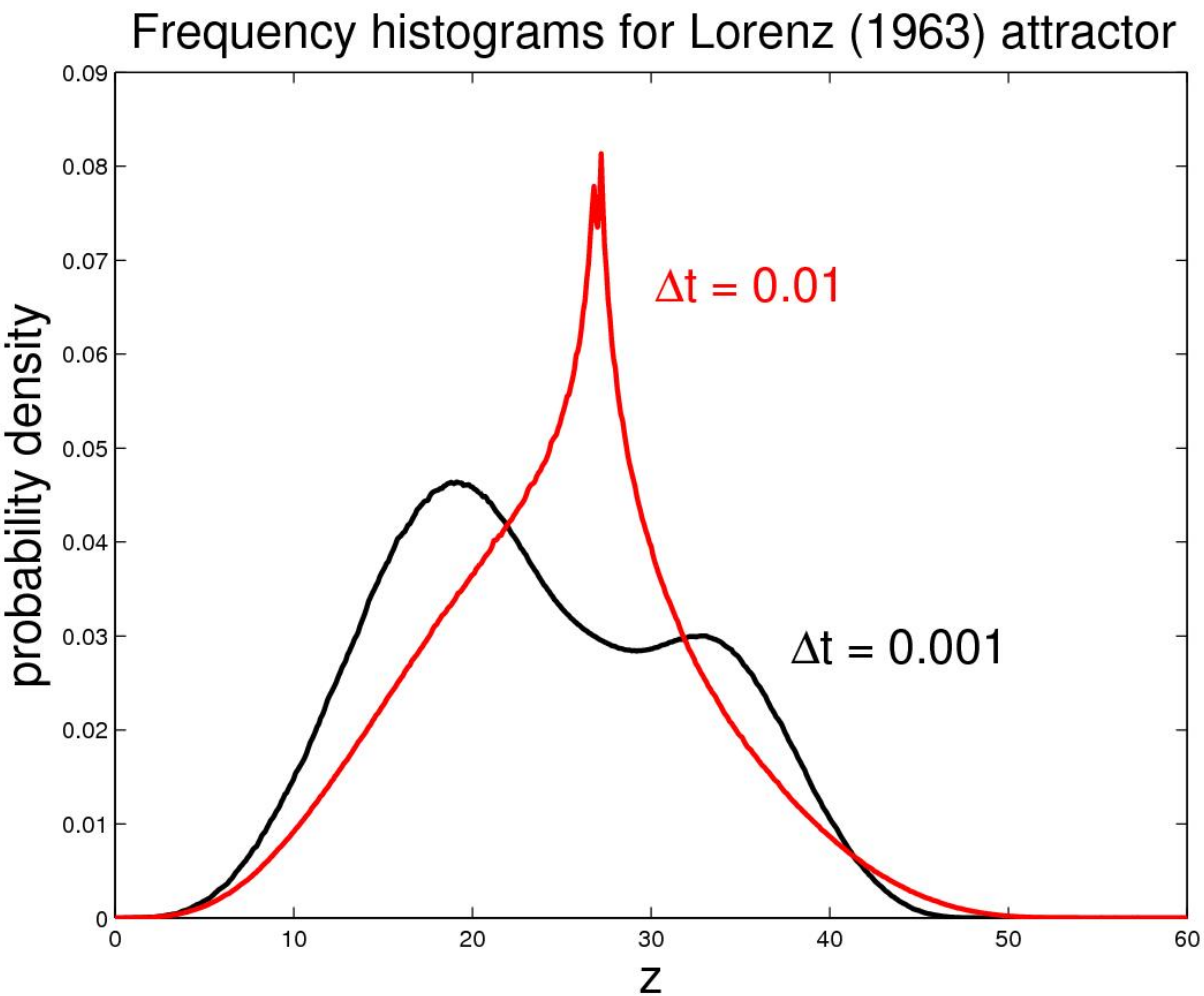


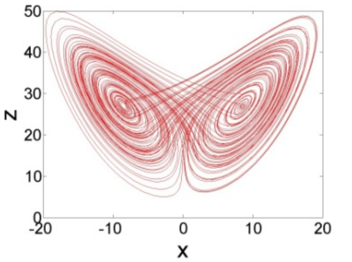




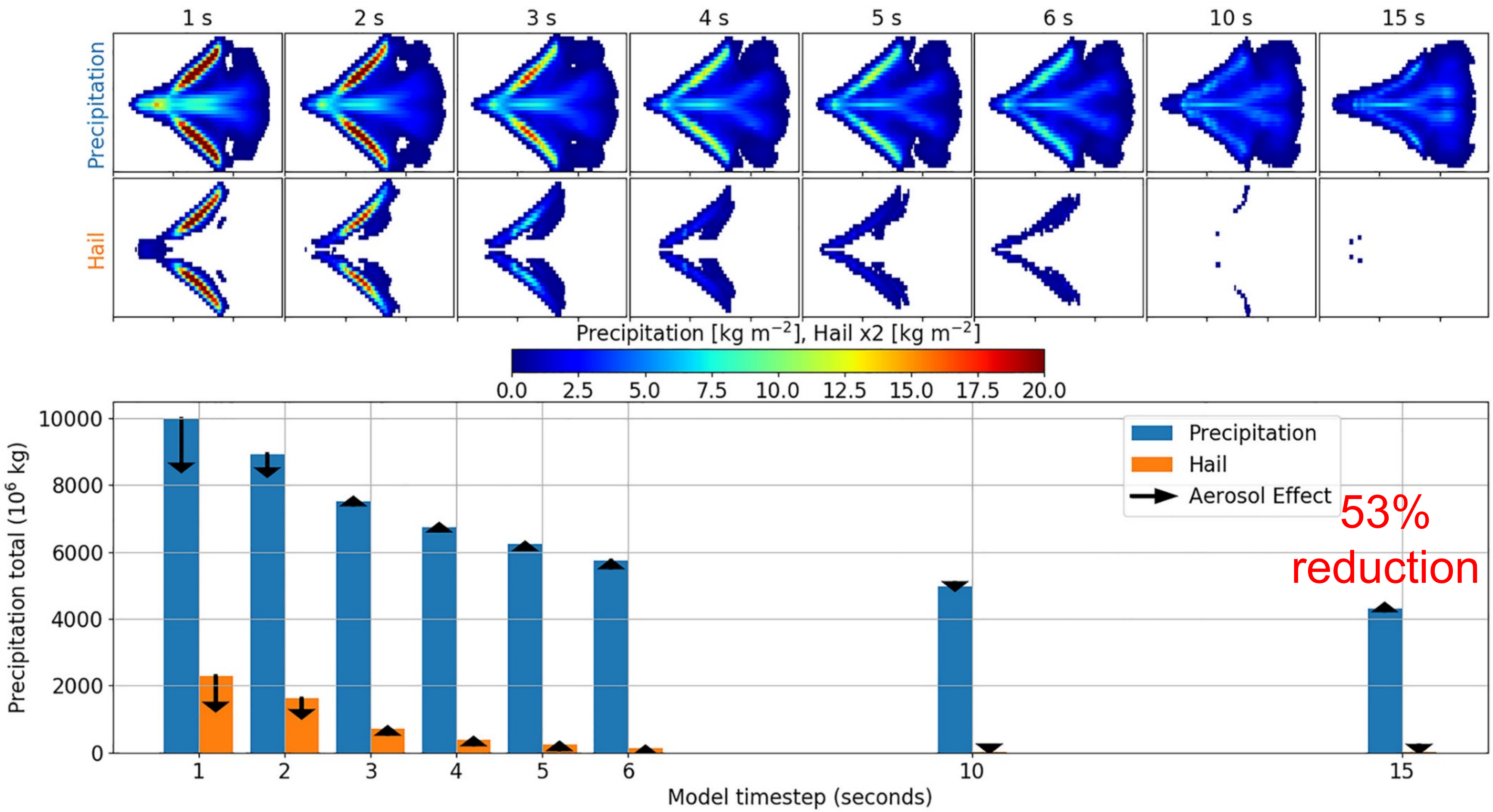
Impact of different time steps on the ‘climate’ of the Lorenz attractor







Impact of different time-step sizes in an atmosphere model

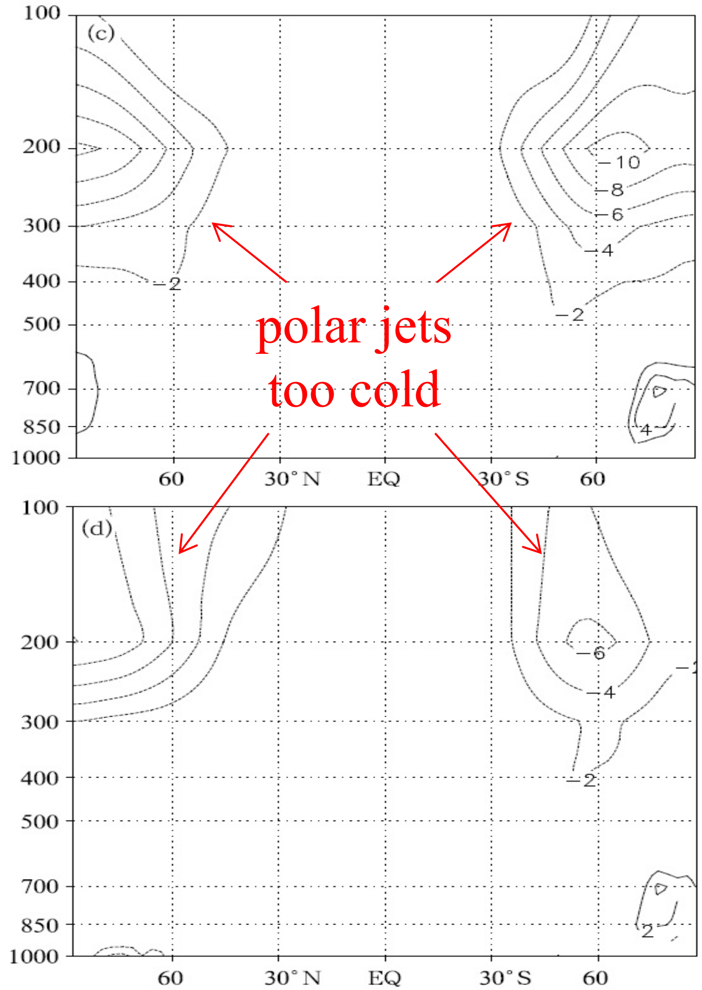


(Barrett et al. 2019)

Impact of time stepping in weather and climate prediction

annual-mean zonal-mean temperature error (°C) in CAM relative to ERA40

pressure (hPa)



pressure (hPa)

First-order
time-stepping scheme

Second-order
time-stepping scheme
(with same Δt)

(Zhao & Zhong 2009)

Impact of different time steps in the ECHAM6-SCM atmosphere model

time step (s)	change in cloud radiative forcing for a warming of 2°C (W m^{-2})
45	-4
90	-4
180	-9
360	-8
720	-3

"Different low-cloud climate feedbacks in current climate models are due to complex interactions between physical parameterizations and numerical artifacts"

(Cheedela et al. 2010)

- “In the weather and climate prediction community, when thinking in terms of model predictability, there is a tendency to associate model error with the physical parameterizations. In this paper, it is shown that time truncation error can be a substantial part of the total forecast error” (Teixeira et al. 2007)

- The sensitivity of the skill of medium-range weather forecasts to the time-stepping method is about the same as to the physics parameterizations (Amezcua 2012)

- “Climate simulations are sensitive not only to physical parameterizations of subgrid-scale processes but also to the numerical methodology employed” (Pfeffer et al. 1992)

- “Many published conclusions on parameter sensitivity, calibrated values and associated uncertainty may be questionable due to numerical artifacts introduced by unreliable time stepping schemes” (Kavetski & Clark 2010)

- “In general, much less concern is given to the temporal accuracy than the spatial accuracy of GCMs” (Thrastarson & Cho 2011)

- Reducing the time step “leads to a statistically significant (at the 5% confidence level) reduction in the number of cyclones over the Northern Hemisphere extratropics” (Jung et al. 2012)

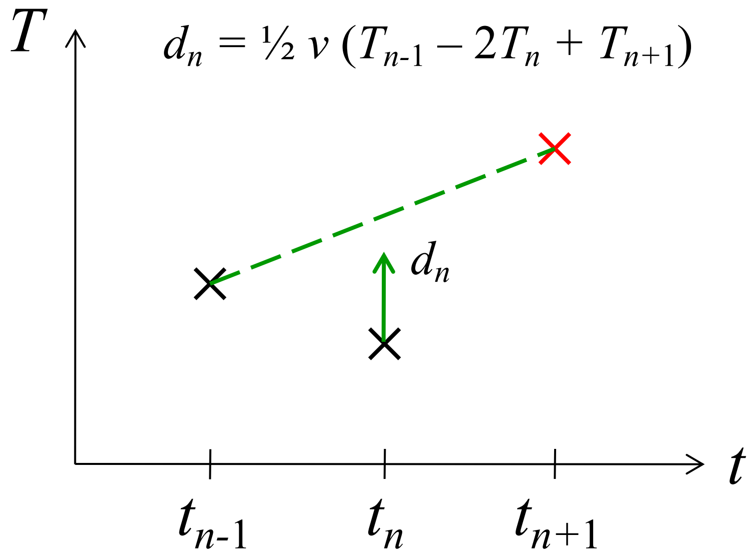
- “Interestingly, both the improvement in near-surface winds in the tropical Pacific and the meridional mean circulation in the tropics found when going from T159 to T511 can also be achieved if the coarser-resolution, T159 model is run with the same, shorter time step used by the T511 model (i.e., 15 min). This suggests that the improvements seen in near-surface tropical winds when going from T159 to T511 are primarily due to the shorter time step required to attain stability rather than increased horizontal resolution.” (Jung et al. 2012)

Leapfrog with Robert–Asselin filter

LF+RA
(Robert 1966, Asselin 1972)

$$\frac{dT}{dt} = f(T)$$

$$d_n = \tfrac{1}{2}\,\nu\,(T_{n-1} - 2T_n + T_{n+1})$$



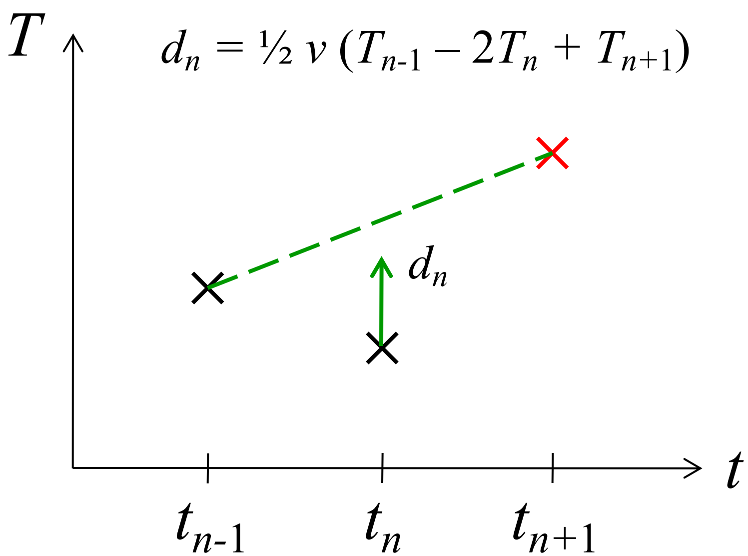
- use leapfrog to calculate $T_{n+1} = T_{n-1} + 2\,\Delta t\, f(T_n)$
- RA filter nudges $T_n = T_n + d_n$
- reduces curvature <u>but does not conserve mean</u>
- amplitude accuracy is 1st order

Leapfrog with Robert–Asselin filter

- Widely used in current numerical models
  - atmosphere: ECHAM, MAECHAM, MM5, CAM, MESO-NH, HIRLAM, KMCM, LIMA, SPEEDY, IGCM, PUMA, COSMO, FSU-GSM, FSU-NRSM, NCEP-GFS, NCEP-RSM, NSEAM, NOGAPS, RAMS, CCSR/NIES-AGCM
  - ocean: OPA, ORCA, NEMO, HadOM3, DieCAST, TIMCOM, GFDL-MOM, POM, MICOM, HYCOM, POSEIDON, NCOM, ICON, OFES, SOM
  - coupled: HiGEM (oce), COAMPS (atm), PlaSim (atm), ECHO (atm), MIROC (atm), FOAM (oce), NCAR-CCSM (atm), BCM (oce), NCEP-CFS (atm/oce), QESM (oce), CHIME (oce), FORTE (atm)
  - others: GTM, ADCIRC, QUAGMIRE, MORALS, SAM, ARPS, CASL, CReSS, JTGCM, ECOMSED, UKMO-LEM, MPI-REMO
- Asselin (1972) has received over 450 citations
- Has many problems
  - "*The Robert–Asselin filter has proved immensely popular, and has been widely used for over 20 years. However, it is not the last word…*" (Lynch 1991)
  - "*Replacement of the Asselin time filter… can be a feasible way to improve the ability of climate models*" (Zhao & Zhong 2009)
  - "*The Robert–Asselin filter can produce slewing frequency as well as the well-known damping and phase errors*" (Thrastarson & Cho 2011)

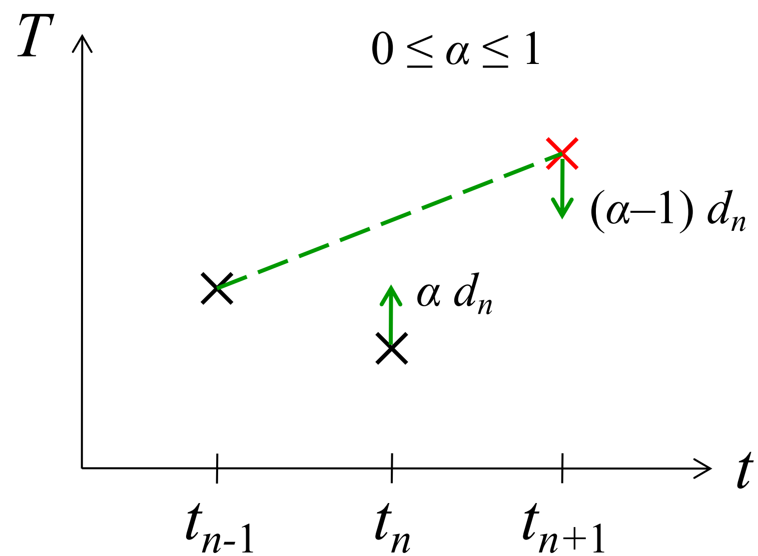
A proposed improvement

<u>LF+RA</u>
(Robert 1966, Asselin 1972)



- use leapfrog to calculate T_{n+1}
- RA filter nudges T_n
- reduces curvature <u>but does not conserve mean</u>
- amplitude accuracy is 1st order

<u>LF+RAW</u>
(Williams 2009, 2011)



- use leapfrog to calculate T_{n+1}
- RAW filter nudges T_n <u>and</u> T_{n+1}
- reduces curvature <u>and conserves mean</u> (for α=½)
- amplitude accuracy is 3rd order

A proposed improvement

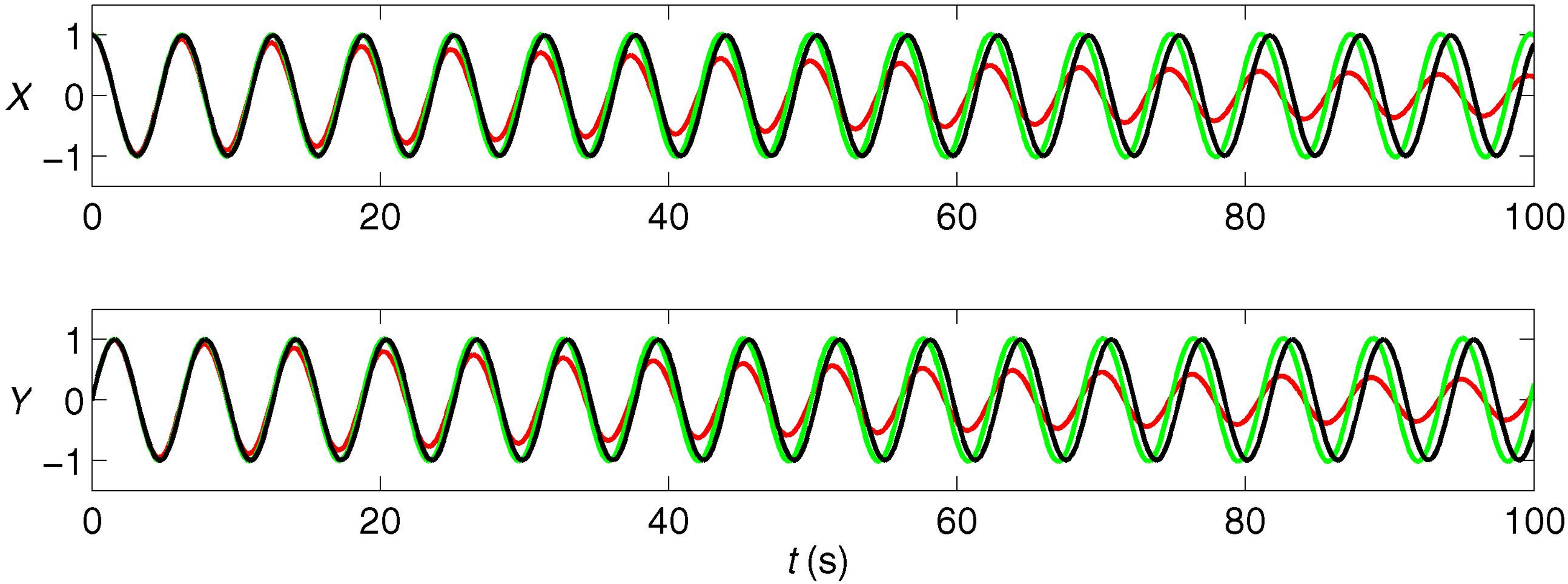
Simple test integration

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\omega Y$$
$$\frac{\mathrm{d}Y}{\mathrm{d}t} = +\omega X$$

$\omega = 1 \text{ rad s}^{-1}$

exact
LF+RA
LF+RAW$_{\alpha=1/2}$

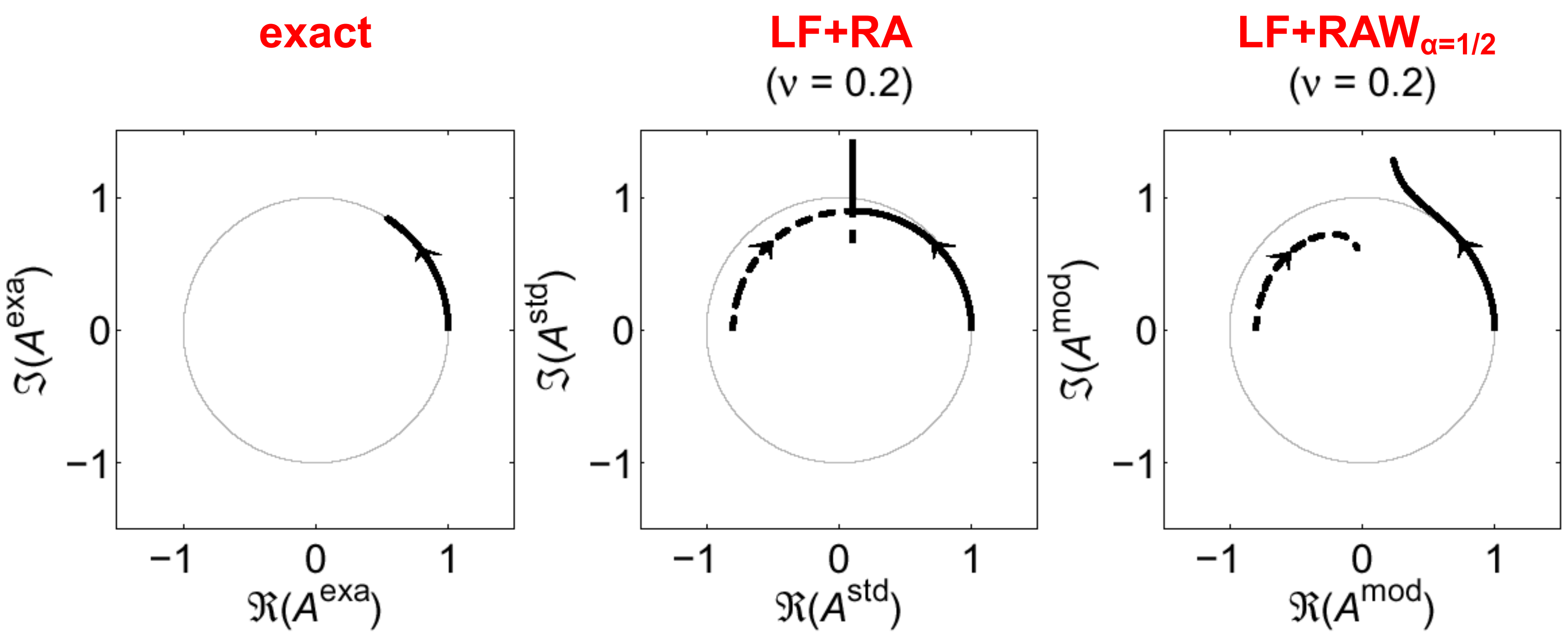
$\left.\right\}$ $\Delta t = 0.2$ s, $\nu = 0.2$



(Williams 2009)

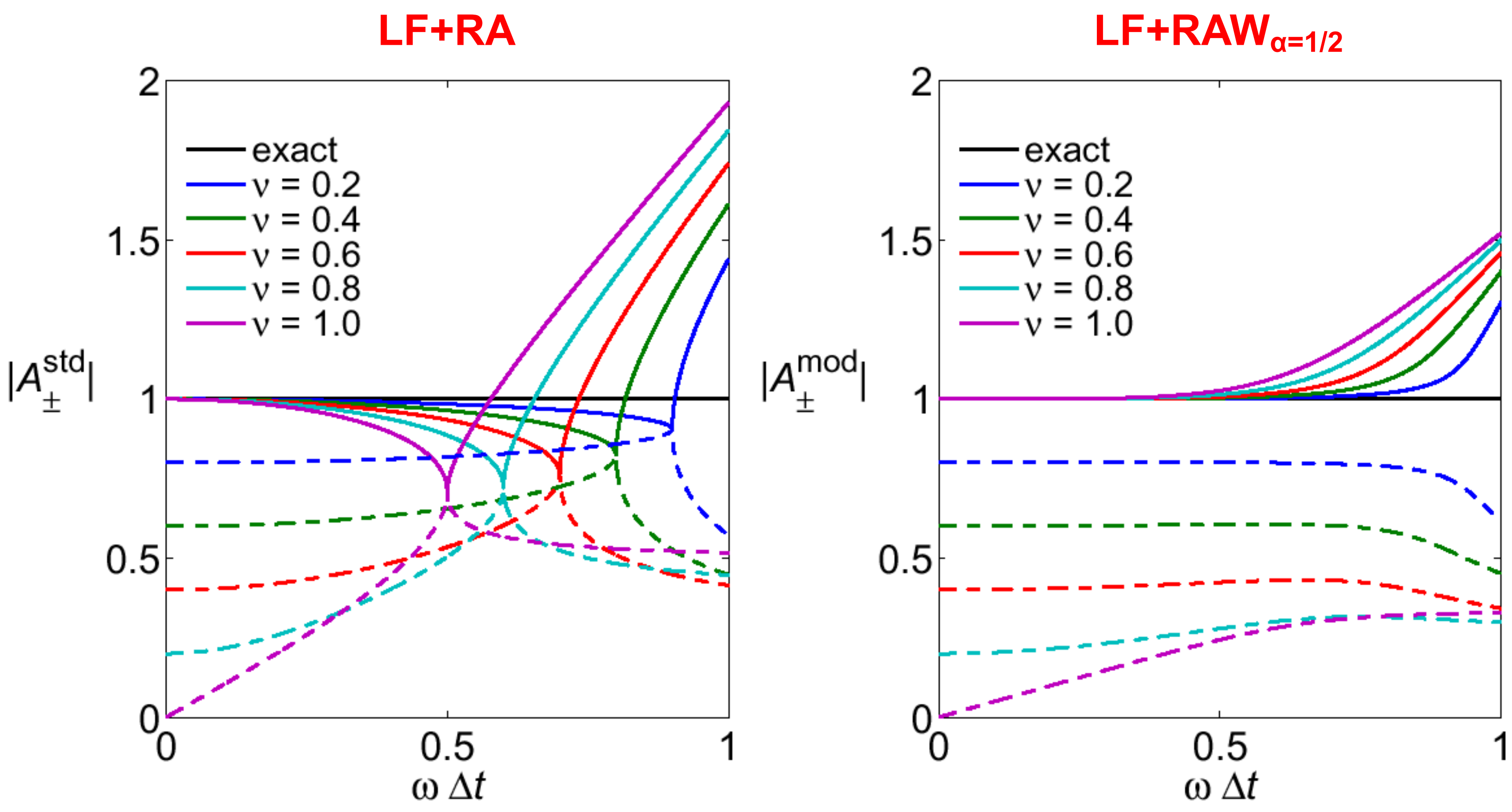
Analysis: numerical amplification

Let $\dot{F} = \mathrm{i}\omega F$ and $A = F(t+\Delta t)\ /\ F(t)$ and trace A as $\omega\Delta t = 0 \rightarrow 1$:



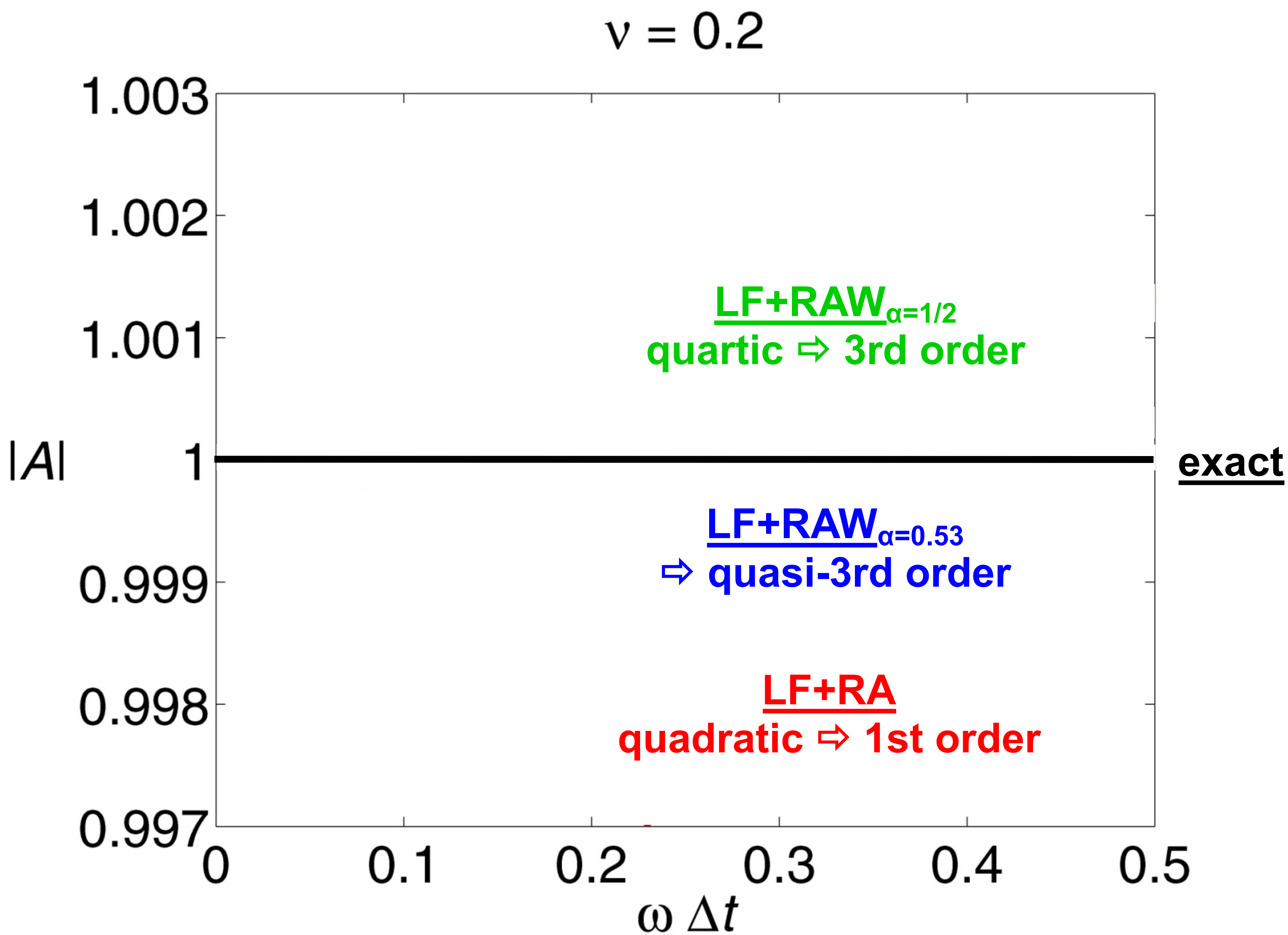
(Williams 2009)

Analysis: numerical amplification



(Williams 2009)

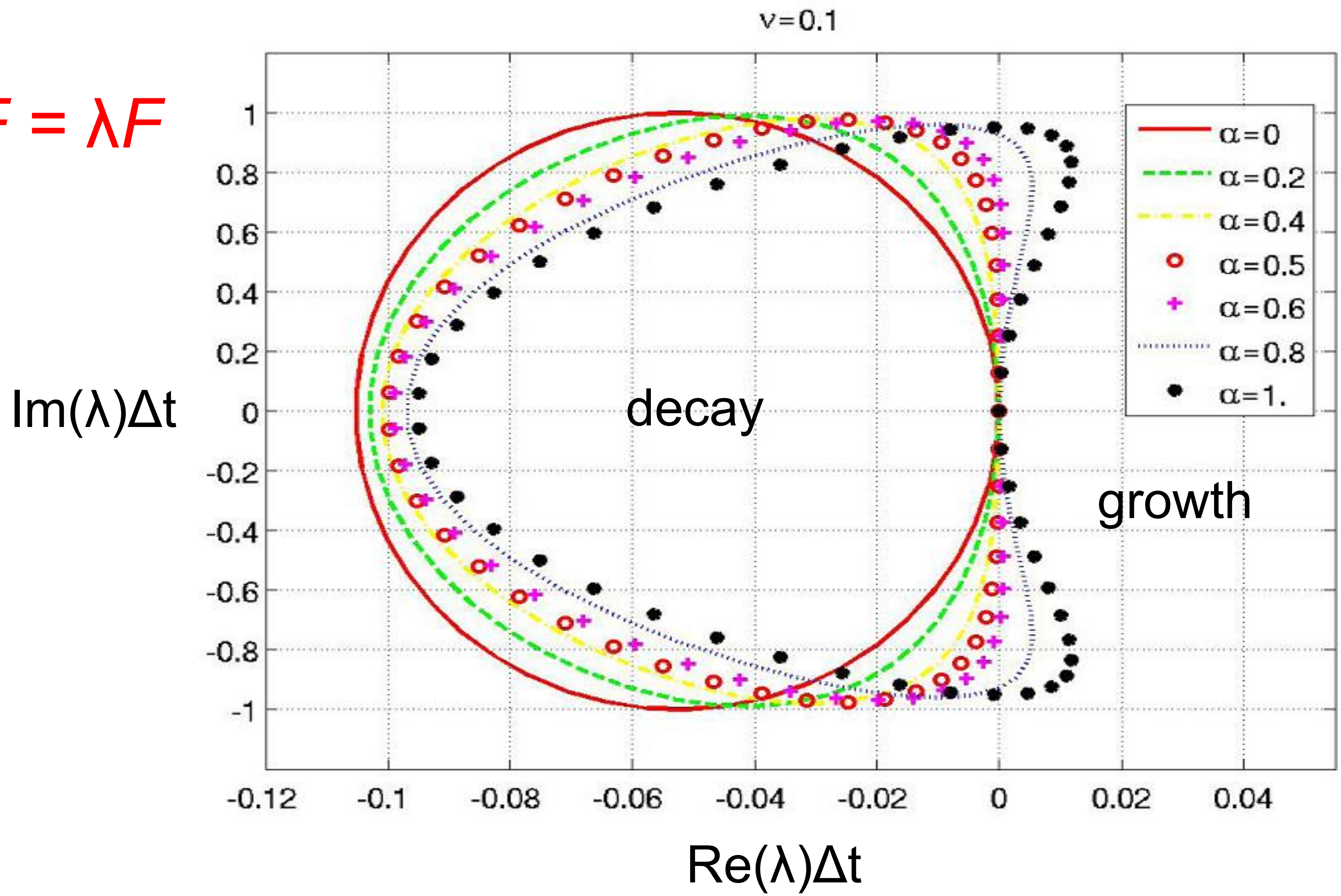
Analysis: numerical amplification



(Williams 2009)

Analysis: numerical stability

$\dot{F} = \lambda F$



(Thanks to Yu-heng Tseng)

Implementation in existing code

```
! Compute tendency at this time step
tendency = f[x_this]

! Leapfrog step
x_next = x_last + tendency*2*delta_t

! Compute filter displacement
d = nu*(x_last - 2*x_this + x_next)/2

! Apply filter
x_this = x_this + d
```

(Williams 2011)

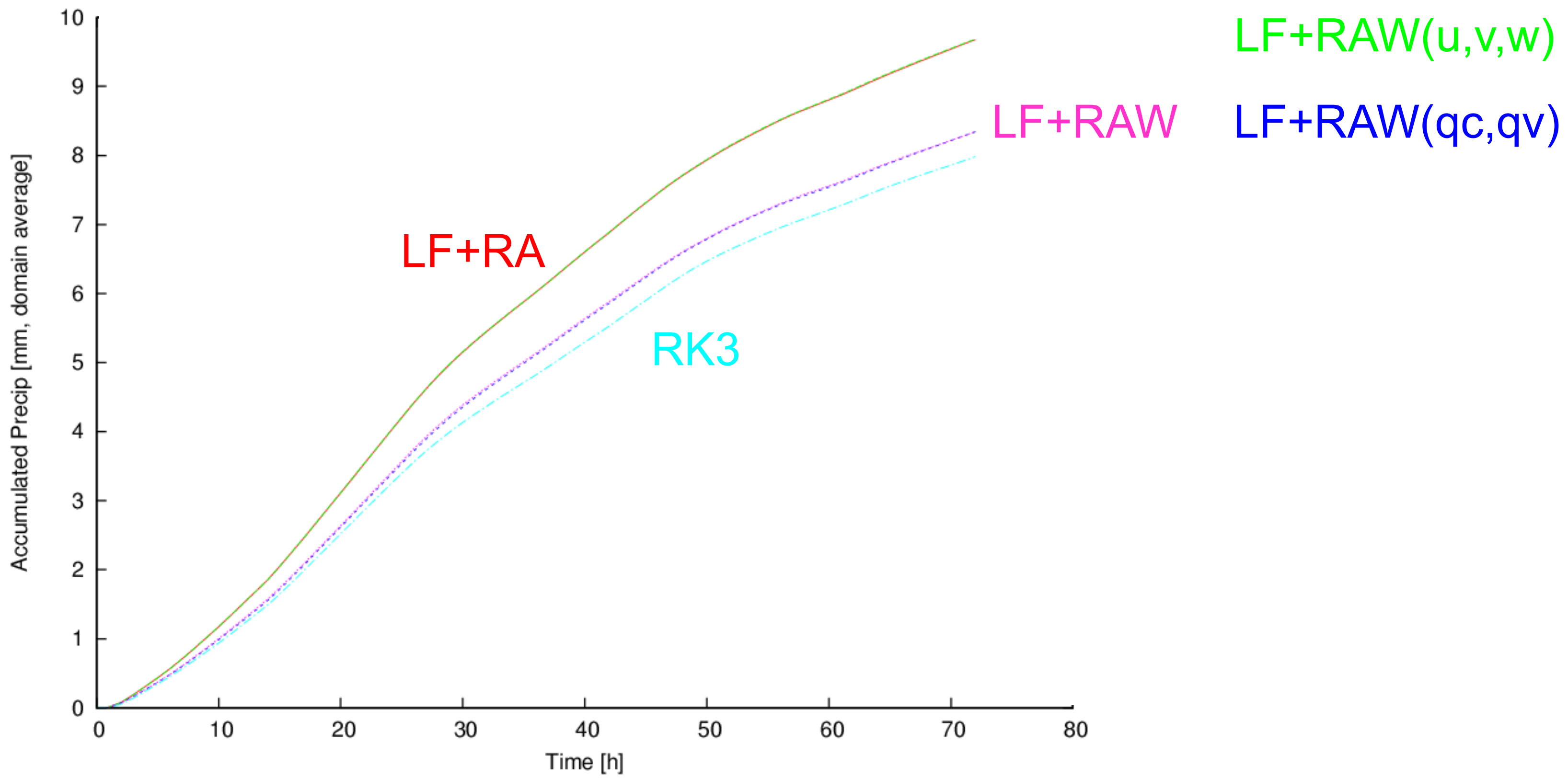
Some recent implementations

The RAW-filtered leapfrog...

- is the default time-stepping method in the atmosphere of MIROC5, the latest version of the Model for Interdisciplinary Research On Climate (Watanabe et al. 2010)
- has been used in the regional climate model COSMO-CLM (CCLM) with α=0.7, and “can lead to a significant improvement, especially for the simulated temperatures” (Wang et al. 2013)
- is the default time-stepping method in TIMCOM, the TaIwan Multi-scale Community Ocean Model, and gives simulations that are in better agreement with observations (Young et al. 2014)
- has been implemented in an ice model, and improves the spin-up and conservation energetics of the physical processes (Ren & Leslie 2011)
- has been implemented in the SPEEDY atmosphere GCM, and significantly improves the skill of medium-range weather forecasts (Amezcua et al. 2011)
- has been found to perform well in various respects in semi-implicit integrations (Durran & Blossey 2012, Clancy & Pudykiewicz 2013)

Implementation in COSMO-7

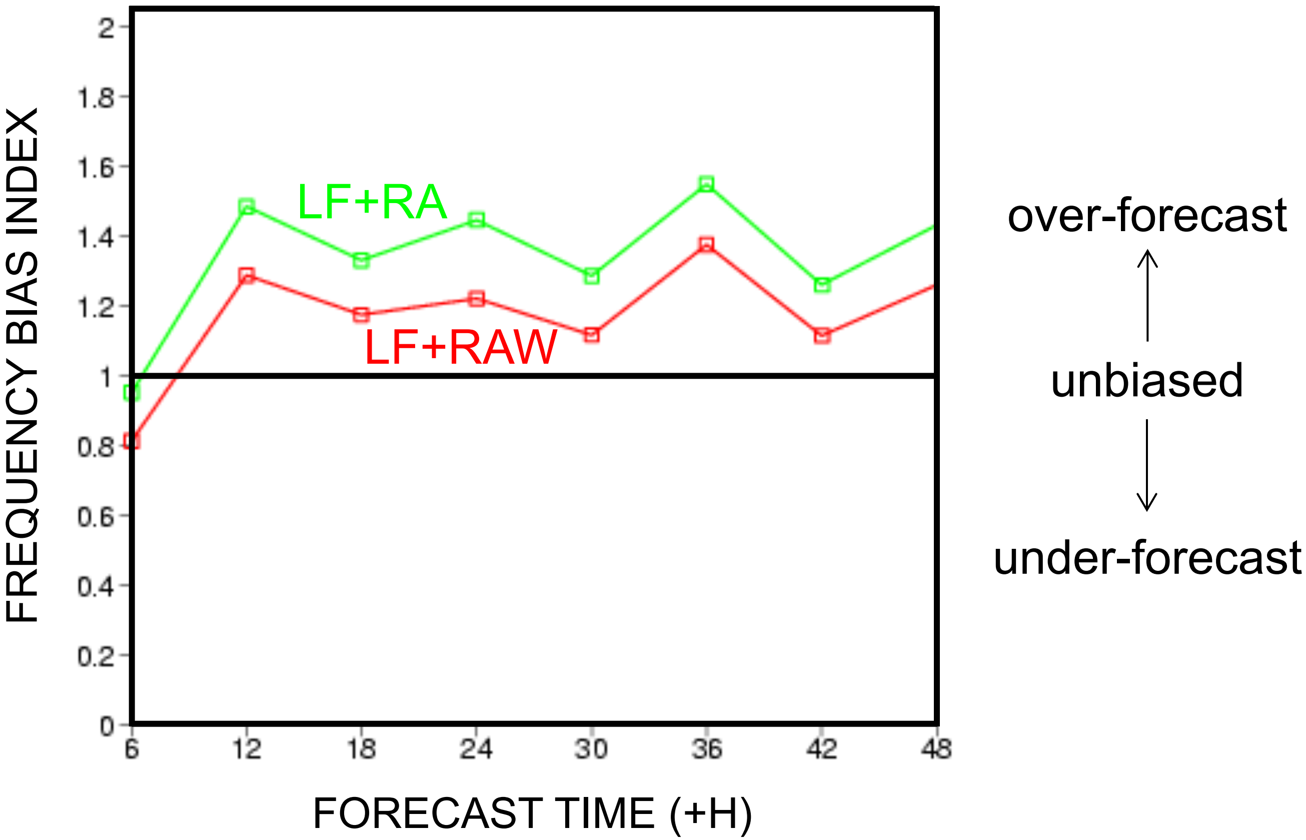
- Test case: 72h forecast over Europe starting at 00Z on 9 Feb 2009
- LF+RAW is almost as accurate as the (more expensive) RK3 integration



Oliver Fuhrer, MeteoSwiss

Implementation in COSMO-ME

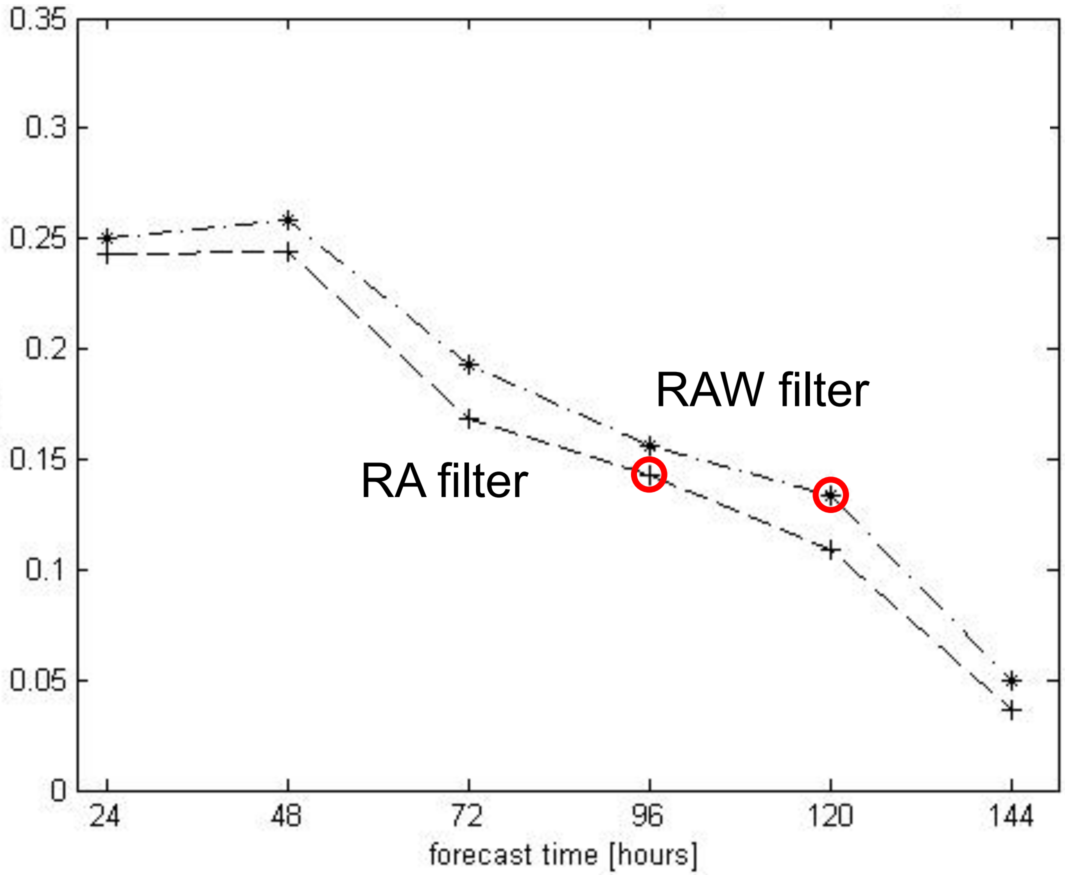
- Statistical analysis: two 48h forecasts per day from 16 Dec 2008 to 18 Jan 2009
- The modified filter significantly improves precipitation forecasts (2-10 mm / 6h)



Lucio Torrisi, CNMCA, Italy

Implementation in SPEEDY

ACC for surface pressure in the tropics (25°S-25°N)



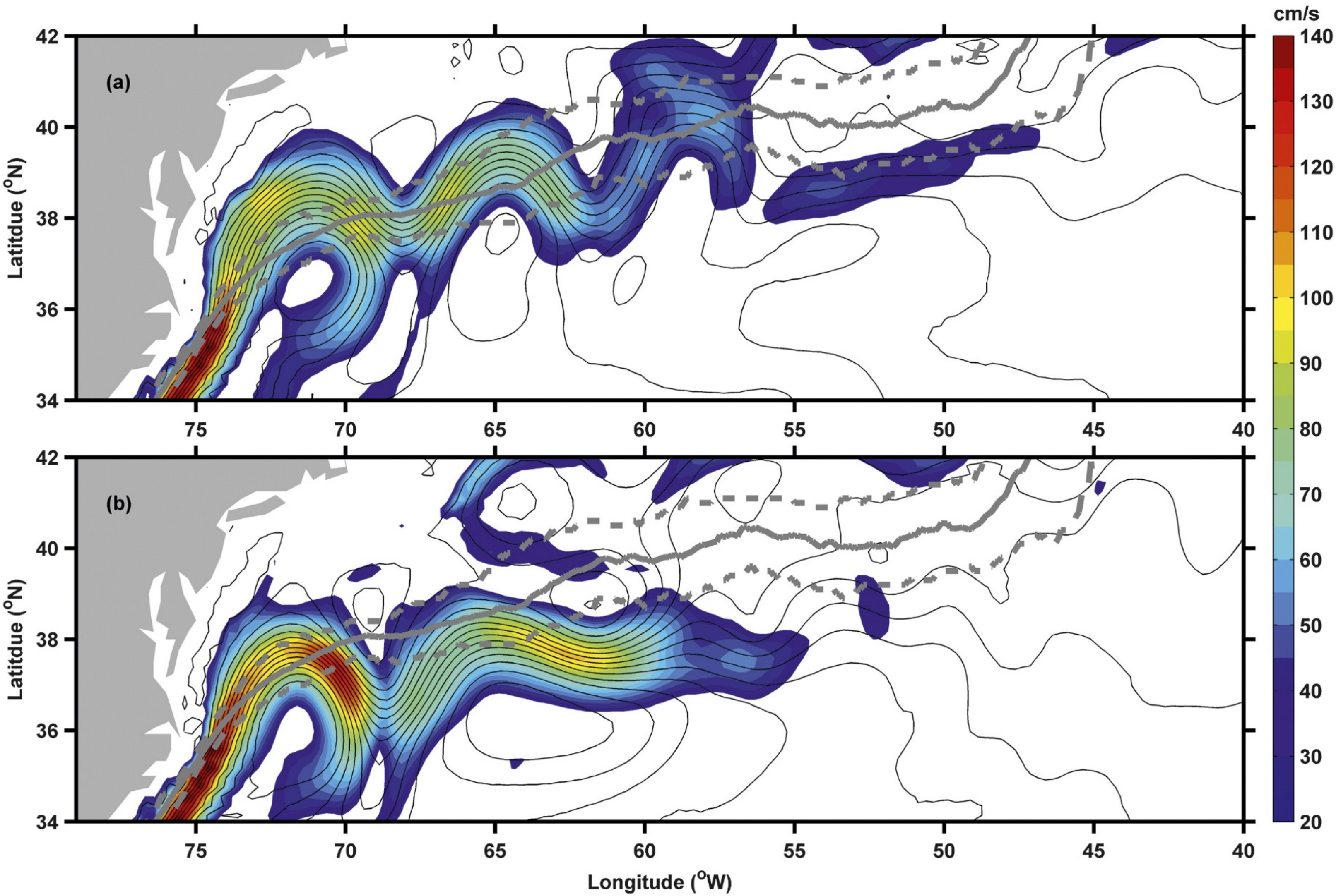
5-day forecasts made using the RAW filter have approximately the same skill as 4-day forecasts made using the RA filter

(Amezcua, Kalnay & Williams 2011)

Implementation in TIMCOM

LF+RAW

LF+RA



(Young et al. 2014)

Composite-tendency leapfrog with (1, -2, 1) filter

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \begin{cases} x_{n+1} = \overline{\overline{x}}_{n-1} + 2\Delta t\left[\gamma f(\overline{x}_n) + (1-\gamma) f(x_n)\right] \\ \overline{\overline{x}}_n = \overline{x}_n + \dfrac{\nu\alpha}{2}\left[\overline{\overline{x}}_{n-1} - 2\overline{x}_n + x_{n+1}\right] \\ \overline{x}_{n+1} = x_{n+1} - \dfrac{\nu(1-\alpha)}{2}\left[\overline{\overline{x}}_{n-1} - 2\overline{x}_n + x_{n+1}\right] \end{cases}
$$

leapfrog (first equation)

filter (second and third equations)

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{i}\omega x \begin{cases} |A_{\mathrm{P}}| - 1 = \dfrac{\nu(1-2\alpha)}{2(2-\nu)}(\omega\Delta t)^2 + \mathcal{O}[(\omega\Delta t)^4] \\ \alpha = \dfrac{1}{2} \qquad |A_{\mathrm{P}}| - 1 = \dfrac{\nu(4\gamma - 3 + \nu - \nu\gamma)}{4(2-\nu)^2}(\omega\Delta t)^4 + \mathcal{O}[(\omega\Delta t)^6] \\ \gamma = \dfrac{3-\nu}{4-\nu} \qquad |A_{\mathrm{P}}| - 1 = \dfrac{\nu}{4(4-\nu)(2-\nu)^2}(\omega\Delta t)^6 + \mathcal{O}[(\omega\Delta t)^8] \end{cases}
$$

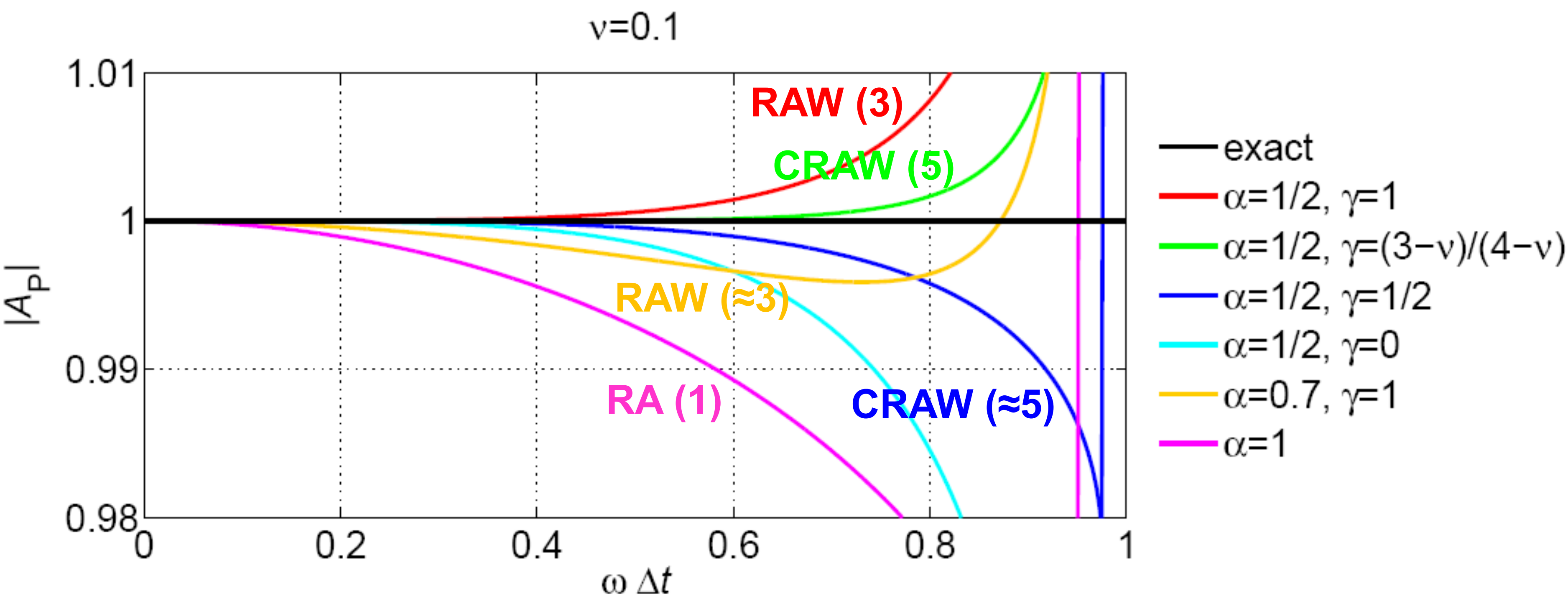
1st order

3rd order

5th order

(Williams 2013)

Composite-tendency leapfrog with (1, -2, 1) filter



(Williams 2013)

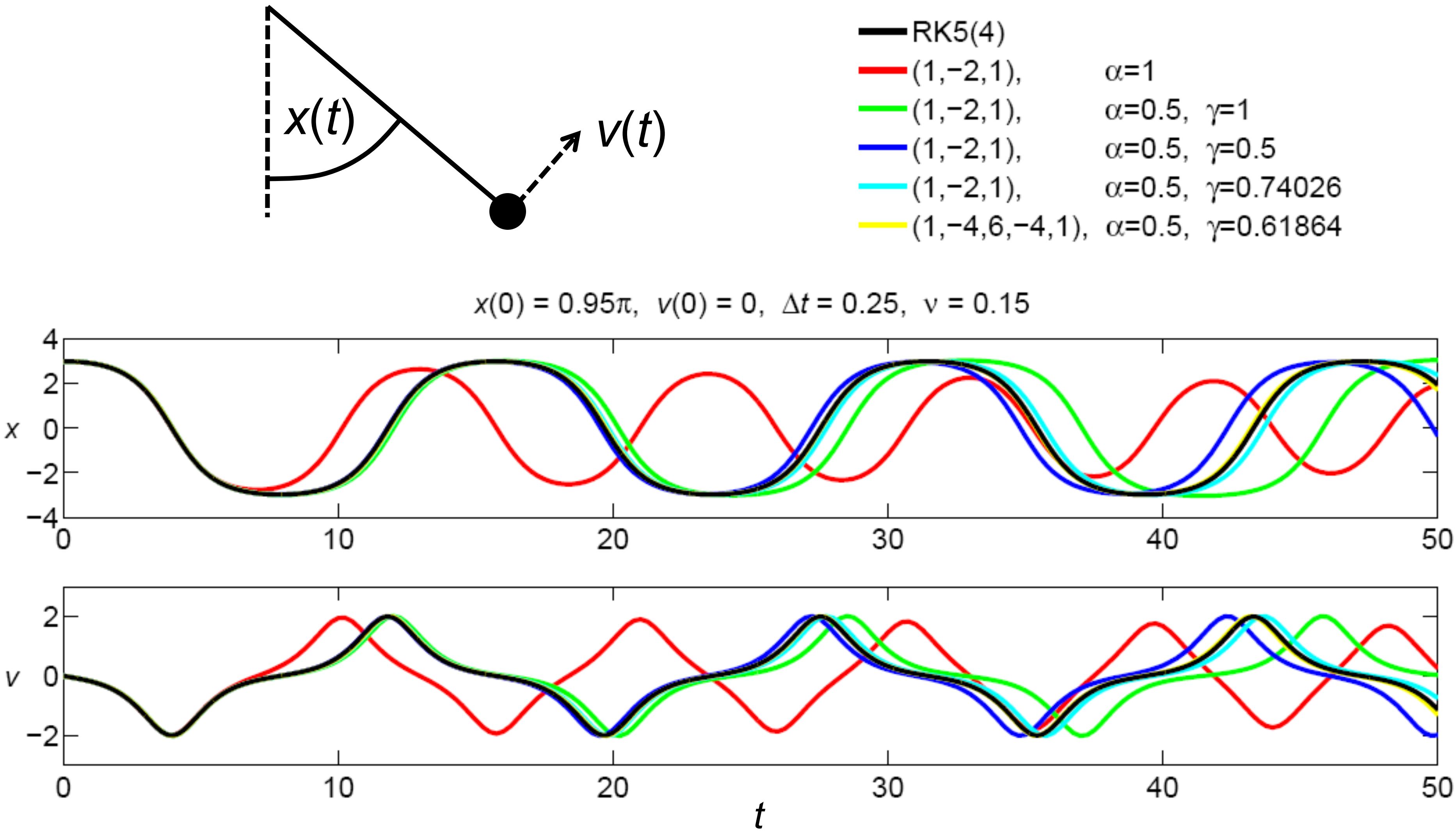
Composite-tendency leapfrog with (1, -4, 6, -4, 1) filter

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \begin{cases} x_{n+1} = \overline{\overline{x}}_{n-1} + 2\Delta t\left[\gamma f(\overline{x}_n) + (1-\gamma) f(x_n)\right] & \text{leapfrog} \\ \overline{\overline{x}}_n = \overline{x}_n + \nu\alpha\left[\overline{\overline{x}}_{n-3} - 4\overline{\overline{x}}_{n-2} + 6\overline{\overline{x}}_{n-1} - 4\overline{x}_n + x_{n+1}\right] & \\ \overline{x}_{n+1} = x_{n+1} - \nu(1-\alpha)\left[\overline{\overline{x}}_{n-3} - 4\overline{\overline{x}}_{n-2} + 6\overline{\overline{x}}_{n-1} - 4\overline{x}_n + x_{n+1}\right] & \text{filter} \end{cases}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{i}\omega x \begin{cases} |A_\mathrm{P}| - 1 = -\dfrac{\nu(1-2\alpha)}{2(1-\nu-2\alpha\nu)}(\omega\Delta t)^4 + \mathcal{O}[(\omega\Delta t)^6] & \text{3rd order} \\ \alpha = \dfrac{1}{2} \qquad |A_\mathrm{P}| - 1 = \dfrac{\nu(5-8\gamma-9\nu+14\nu\gamma)}{8(1-2\nu)^2}(\omega\Delta t)^6 + \mathcal{O}[(\omega\Delta t)^8] & \text{5th order} \\ \gamma = \dfrac{5-9\nu}{2(4-7\nu)} \qquad |A_\mathrm{P}| - 1 = -\dfrac{5\nu(4-13\nu+11\nu^2)}{32(1-2\nu)^2(4-7\nu)}(\omega\Delta t)^8 + \mathcal{O}[(\omega\Delta t)^{10}] & \text{7th order} \end{cases}$$

(Williams 2013)

Nonlinear simple pendulum



(Williams 2013)

Application to image processing!

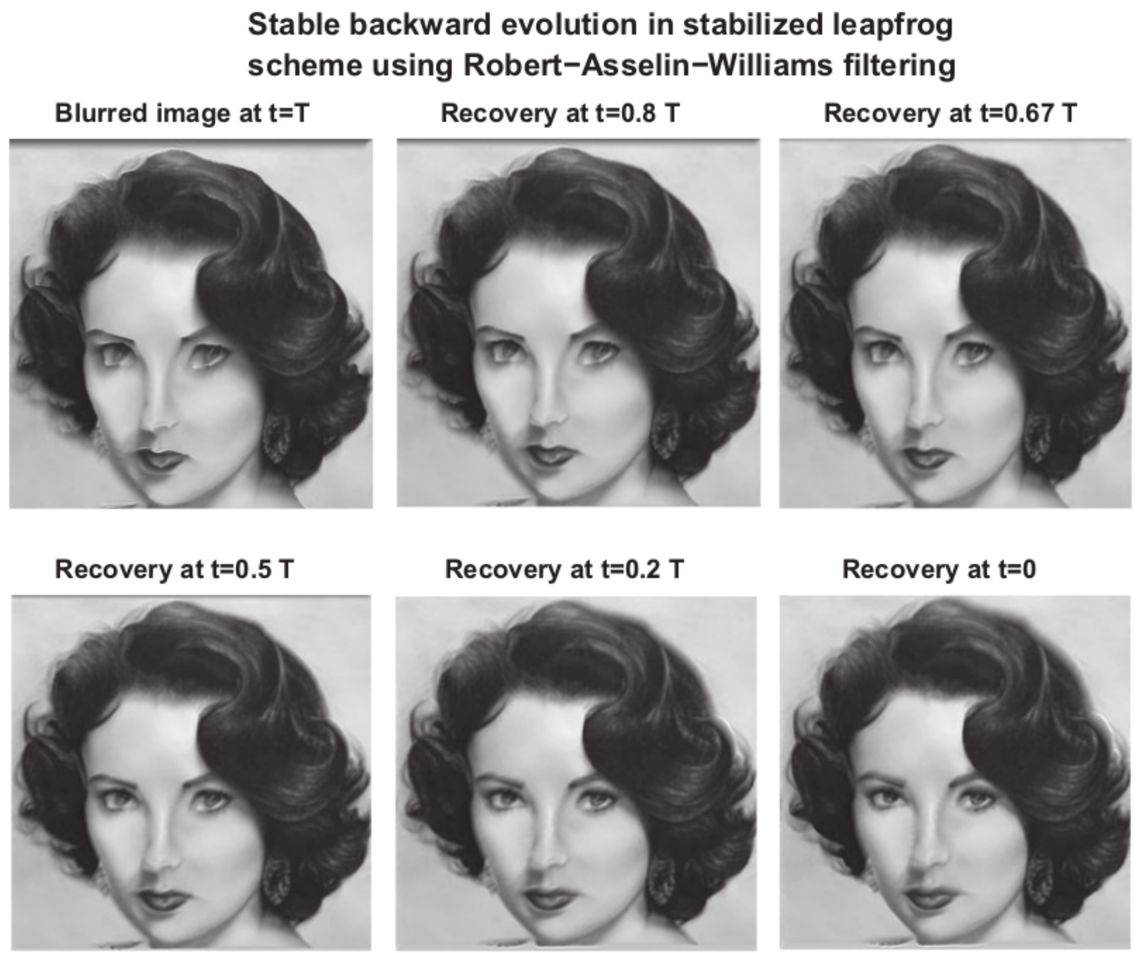


Figure 4. As discussed in Section 7.3, the unsuccessful Liz Taylor deblurring experiment in the previous Figure is repeated with identical parameter values, but with the RAW filter applied at every time step, as described in Equations (6.2), (6.3). This leads to stable backward continuation and successful recovery at $t = 0$.

(Carasso 2017)

Summary

- Time stepping is an important contributor to model error
- The Robert–Asselin filter is widely used but is dissipative and reduces accuracy
- The RAW filter has approximately the same stability but much greater accuracy
- Implementation in an existing code is trivial and there is no extra computational cost
- 5th-order and even 7th-order amplitude accuracy may be achieved, by using a composite tendency and/or a more discriminating filter

Further information

Williams PD (2013) Achieving seventh-order amplitude accuracy in leapfrog integrations. *Monthly Weather Review* **141**(9), 3037-3051.

Williams PD (2011) The RAW filter: An improvement to the Robert–Asselin filter in semi-implicit integrations. *Monthly Weather Review* **139**(6), 1996-2007.

Williams PD (2009) A proposed modification to the Robert–Asselin time filter. *Monthly Weather Review* **137**(8), 2538-2546.

p.d.williams@reading.ac.uk
www.met.reading.ac.uk/~williams