[redacted]

1. Solve each of the following initial value problems. For which values of t is the solution defined? How does the solution behave as $t \to \infty$?

   (a) $\dfrac{dy}{dt} = \dfrac{-4yt}{t^2-1}, \quad y(2) = 1$

   Solution:
   Separating variables gives
   $$\frac{dy}{y} = \frac{-4t\,dt}{t^2-1}$$
   Now integrate:
   $$\int_1^y \frac{dy}{y} = \int_2^t \frac{-4t}{t^2-1}$$
   so
   $$\ln y = -2\ln(t^2-1)\Big|_2^t = -2\ln(t^2-1) + 2\ln 3 = \ln\frac{9}{(t^2-1)^2}$$
   and
   $$y = \frac{9}{(t^2-1)^2}$$
   for $1 < t < \infty$. The solution blows up as $t \to 1^+$, and tends to 0 as $t \to \infty$.

   (b) $(1+t^2)\dfrac{dy}{dt} + 4ty = (1+t^2)^{-1}, \quad y(0) = 3$

   Solution:
   Multiplying by $1+t^2$ gives
   $$(1+t^2)^2 y' + 4t(1+t^2)y = 1.$$
   The left side is the derivative of $(1+t^2)^2 y$, so integrating gives
   $$(1+t^2)^2 y = t + c$$
   and the initial condition gives $c = 3$, so
   $$y = \frac{t+3}{(1+t^2)^2}.$$
   The solution is valid for all t, and $y \to 0$ as $t \to \infty$.

2. A tank initially contains fifty gallons of water in which ten pounds of salt is dissolved. A salt water solution containing one pound of salt per gallon begins to enter the tank at a rate of three gallons per minute. The well-mixed fluid leaves the tank through a pipe at the same rate

   (a) Write down an initial value problem (differential equation and initial condition) for the number of pounds s of salt in the tank after t minutes.

   Solution:

$$\begin{aligned} \frac{ds}{dt} &= 3 - \frac{3}{50}s \\ s(0) &= 10 \end{aligned}$$

   (b) How much salt will be in the tank after 50 minutes?

   Solution:
   First, solve the initial value problem. Write

$$\frac{d}{dt}(s-50) = -\frac{3}{50}(s-50)$$

so

$$s - 50 = (s_0 - 50)e^{-3t/50} = -40e^{-3t/50}$$

and

$$s = 50 - 40e^{-3t/50}$$

After 50 minutes, the amount of salt in the tank is

$$s(50) = 50 - 40e^{-3} \approx 48 \text{ pounds.}$$

3. Solve each of the following differential equations. If no initial conditions are given, find the general solution.

   (a) $y'' + 3y' + 2y = 24e^{2t}$

   Solution:
   The characteristic roots are -1 and -2, so the homogeneous part of the solution is
   $$y_h = c_1 e^{-t} + c_2 e^{-2t}.$$
   Look for a particular solution of the form $y = ce^{2t}$. Substituting into the differential equation gives
   $$(4 + 6 + 2)ce^{2t} = 24e^{2t}$$
   from which it follows that $c = 2$, so $y_p = 2e^{2t}$. The general solution is
   $$y = y_p + y_h = 2e^{2t} + c_1 e^{-t} + c_2 e^{-2t}.$$

   (b) $y'' + 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$

   Solution:
   The general solution is
   $$y = c_1 e^{-2t} + c_2 t e^{-2t}.$$
   Its derivative is
   $$y' = -2c_1 e^{-2t} + c_2(1 - 2t)e^{-2t}$$
   so the initial conditions give
   $$\begin{aligned} c_1 &= 1 \\ -2c_1 + c_2 &= 0 \end{aligned}$$
   and solving gives $c_1 = 1$ and $c_2 = 2$, so
   $$y = e^{-2t} + 2te^{-2t}.$$

   (c) $y'' + 4y' + 5y = 0$

   Solution:
   The characteristic roots are $-2 \pm i$, so the general solution is
   $$y = e^{-2t}(c_1 \cos t + c_2 \sin t).$$

4. (a) Find the general solution to

$$y'' + 3y' + 2y = 260\cos(3t).$$

Solution:
Complexifying gives

$$z'' + 3z' + 2z = 260e^{3it}.$$

Look for a particular solution of the form $z = ce^{3it}$. Substituting into the equation gives

$$(-9 + 9i + 2)c = 260$$

so

$$c = \frac{260}{-7+9i} = \frac{260(-7-9i)}{130} = -14 - 18i.$$

Therefore

$$\begin{aligned} z &= (-14-18i)e^{3it} \\ &= (-14-18i)(\cos 3t + i\sin 3t) \\ &= -14\cos 3t + 18\sin 3t + i(\cdots) \end{aligned}$$

and a particular solution to the original equation is

$$y_p = \Re z = -14\cos 3t + 18\sin 3t.$$

The general solution is

$$y = y_p + y_h = (-14\cos 3t + 18\sin 3t) + c_1e^{-t} + c_2e^{-2t}.$$

(b) As t increases, the solution settles into a periodic steady state oscillation (which does not depend on the initial conditions). Find it's period and amplitude.

Solution:
The steady state oscillation is y_p from part (a). Its period is $\frac{2\pi}{3}$ and its amplitude is

$$\sqrt{(-14)^2 + 18^2} = 2\sqrt{130}.$$

Note: You can find the amplitude from the "complexified" solution without working out its real part. From the beginning of part (a), the complexified oscillation is

$$z = \frac{270}{-7+9i}e^{3it}$$

so the amplitude is

$$\left|\frac{270}{-7+9i}\right| = \frac{270}{|-7+9i|} = \frac{270}{\sqrt{130}}$$