Math 0290 Final Exam Solutions

1. Solve each of the following initial value problems. For which values of t is the solution defined? How does the solution behave as $t \to \infty$?

   (a) $\dfrac{dy}{dt} = \dfrac{-4yt}{t^2 - 1}, \quad y(2) = 1$

   Solution:
   Separating variables gives
   $$\frac{dy}{y} = \frac{-4t\,dt}{t^2 - 1}$$
   Now integrate:
   $$\int_1^y \frac{dy}{y} = \int_2^t \frac{-4t}{t^2 - 1}$$
   so
   $$\ln y = -2\ln(t^2 - 1)\Big|_2^t = -2\ln(t^2 - 1) + 2\ln 3 = \ln\frac{9}{(t^2 - 1)^2}$$
   and
   $$y = \frac{9}{(t^2 - 1)^2}$$
   for $1 < t < \infty$. The solution blows up as $t \to 1^+$, and tends to 0 as $t \to \infty$.

   (b) $(1 + t^2)\dfrac{dy}{dt} + 4ty = (1 + t^2)^{-1}, \quad y(0) = 3$

   Solution:
   Multiplying by $1 + t^2$ gives
   $$(1 + t^2)^2 y' + 4t(1 + t^2)y = 1.$$
   The left side is the derivative of $(1 + t^2)^2 y$, so integrating gives
   $$(1 + t^2)^2 y = t + c$$
   and the initial condition gives $c = 3$, so
   $$y = \frac{t + 3}{(1 + t^2)^2}.$$
   The solution is valid for all t, and $y \to 0$ as $t \to \infty$.

2. A tank initially contains fifty gallons of water in which ten pounds of salt is dissolved. A salt water solution containing one pound of salt per gallon begins to enter the tank at a rate of three gallons per minute. The well-mixed fluid leaves the tank through a pipe at the same rate

   (a) Write down an initial value problem (differential equation and initial condition) for the number of pounds s of salt in the tank after t minutes.

   Solution:

$$\begin{aligned} \frac{ds}{dt} &= 3 - \frac{3}{50}s \\ s(0) &= 10 \end{aligned}$$

   (b) How much salt will be in the tank after 50 minutes?

   Solution:
   First, solve the initial value problem. Write

$$\frac{d}{dt}(s-50) = -\frac{3}{50}(s-50)$$

   so

$$s - 50 = (s_0 - 50)e^{-3t/50} = -40e^{-3t/50}$$

   and

$$s = 50 - 40e^{-3t/50}$$

   After 50 minutes, the amount of salt in the tank is

$$s(50) = 50 - 40e^{-3} \approx 48 \text{ pounds.}$$

3. Solve each of the following differential equations. If no initial conditions are given, find the general solution.

   (a) $y'' + 3y' + 2y = 24e^{2t}$

   Solution:
   The characteristic roots are -1 and -2, so the homogeneous part of the solution is

   $$y_h = c_1e^{-t} + c_2e^{-2t}.$$

   Look for a particular solution of the form $y = ce^{2t}$. Substituting into the differential equation gives

   $$(4 + 6 + 2)ce^{2t} = 24e^{2t}$$

   from which it follows that $c = 2$, so $y_p = 2e^{2t}$. The general solution is

   $$y = y_p + y_h = 2e^{2t} + c_1e^{-t} + c_2e^{-2t}.$$

   (b) $y'' + 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$

   Solution:
   The general solution is

   $$y = c_1e^{-2t} + c_2te^{-2t}.$$

   Its derivative is

   $$y' = -2c_1e^{-2t} + c_2(1 - 2t)e^{-2t}$$

   so the initial conditions give

   $$\begin{aligned} c_1 &= 1 \\ -2c_1 + c_2 &= 0 \end{aligned}$$

   and solving gives $c_1 = 1$ and $c_2 = 2$, so

   $$y = e^{-2t} + 2te^{-2t}.$$

   (c) $y'' + 4y' + 5y = 0$

   Solution:
   The characteristic roots are $-2 \pm i$, so the general solution is

   $$y = e^{-2t}(c_1 \cos t + c_2 \sin t).$$

4. (a) Find the general solution to

$$y'' + 3y' + 2y = 260\cos(3t).$$

Solution:
Complexifying gives

$$z'' + 3z' + 2z = 260e^{3it}.$$

Look for a particular solution of the form $z = ce^{3it}$. Substituting into the equation gives

$$(-9 + 9i + 2)c = 260$$

so

$$c = \frac{260}{-7+9i} = \frac{260(-7-9i)}{130} = -14 - 18i.$$

Therefore

$$\begin{aligned} z &= (-14-18i)e^{3it} \\ &= (-14-18i)(\cos 3t + i\sin 3t) \\ &= -14\cos 3t + 18\sin 3t + i(\cdots) \end{aligned}$$

and a particular solution to the original equation is

$$y_p = \Re z = -14\cos 3t + 18\sin 3t.$$

The general solution is

$$y = y_p + y_h = (-14\cos 3t + 18\sin 3t) + c_1e^{-t} + c_2e^{-2t}.$$

(b) As t increases, the solution settles into a periodic steady state oscillation (which does not depend on the initial conditions). Find it's period and amplitude.

Solution:
The steady state oscillation is y_p from part (a). Its period is $\frac{2\pi}{3}$ and its amplitude is

$$\sqrt{(-14)^2 + 18^2} = 2\sqrt{130}.$$

Note: You can find the amplitude from the "complexified" solution without working out its real part. From the beginning of part (a), the complexified oscillation is

$$z = \frac{270}{-7+9i}e^{3it}$$

so the amplitude is

$$\left|\frac{270}{-7+9i}\right| = \frac{270}{|-7+9i|} = \frac{270}{\sqrt{130}}$$

5. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

(a) Find the general solution to the homogeneous system

$$\mathbf{x}' = A\mathbf{x}.$$

Here $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution:
The characteristic polynomial is

$$\lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2)$$

so the eigenvalues are -5 and 2. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The general solution is

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}.$$

(b) Find the general solution to the inhomogeneous system

$$\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

Solution:
Since the forcing term is constant, look for a constant particular solution $\mathbf{x}_p = \mathbf{c}$. Substituting into the equation gives

$$A\mathbf{c} = -\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

which has solution $\mathbf{c} = \begin{bmatrix} -9/5 \\ 2/5 \end{bmatrix}$, so the general solution is

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h = \begin{bmatrix} -9/5 \\ 2/5 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}.$$

6. The matrix

$$B = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$$

has eigenvalues $4 + 3i$ and $4 - 3i$, with corresponding eigenvectors $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

(a) Find the general solution to the system

$$\mathbf{x}' = B\mathbf{x}.$$

Solution:
A complex solution is

$$\begin{aligned}
\mathbf{z} &= e^{(4+3i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} \\
&= e^{4t}(\cos 3t + i \sin 3t) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\
&= e^{4t} \left(\cos 3t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + ie^{4t} \left(\cos 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 3t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} e^{4t} \cos 3t \\ -e^{4t} \sin 3t \end{bmatrix} + i \begin{bmatrix} e^{4t} \sin 3t \\ e^{4t} \cos 3t \end{bmatrix}
\end{aligned}$$

The real and imaginary parts are independent real solutions, so the general solution is

$$\mathbf{x} = c_1 \begin{bmatrix} e^{4t} \cos 3t \\ -e^{4t} \sin 3t \end{bmatrix} + c_2 \begin{bmatrix} e^{4t} \sin 3t \\ e^{4t} \cos 3t \end{bmatrix}$$

(b) Find the solution to the initial value problem

$$\mathbf{x}' = B\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Solution:
The initial condition gives

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

from which it follows that $c_1 = 2$ and $c_2 = 3$. The solution to the initial value problem is therefore

$$\mathbf{x} = 2 \begin{bmatrix} e^{4t} \cos 3t \\ -e^{4t} \sin 3t \end{bmatrix} + 3 \begin{bmatrix} e^{4t} \sin 3t \\ e^{4t} \cos 3t \end{bmatrix}$$

7. Consider the autonomous system

$$\begin{aligned} x' &= y \\ y' &= -\sin x - y \end{aligned}$$

(a) Find all equilibria of the system.

Solution:
The equilibria are solutions to

$$y = 0 \quad (1)$$

$$-\sin x - y = 0 \quad (2)$$

so the equilibria are the points $(n\pi, 0)$, where n is an integer.

(b) Classify each equilibrium as a nodal source/sink, spiral source/sink, saddle, or center.

The Jacobian is

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -\cos x & -1 \end{bmatrix}$$

Evaluating at the equilibrium $(n\pi, 0)$ gives

$$J(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ (-1)^{n+1} & -1 \end{bmatrix}$$

When n is even, this gives

$$J(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

which has eigenvalues $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Since the real parts are negative, the equilibrium $(n\pi, 0)$ is a spiral sink when n is even.
When n is odd,

$$J(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

which has eigenvalues $\frac{-1 \pm \sqrt{5}}{2}$, both of which are positive. Therefore, the equilibrium $(n\pi, 0)$ is a nodal source when n is odd.

8. Using Laplace transforms, solve the following differential equations, and illustrate the behavior of the solution, with a graph.

(a) $y'' + 6y' + 5y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0$

Solutions:
Laplace transforming both sides gives

$$s^2 Y(s) + 6sY(s) + 5Y(s) = 1$$

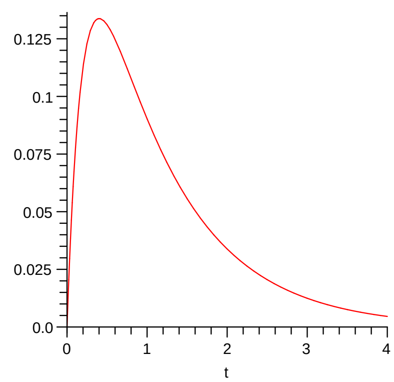
so

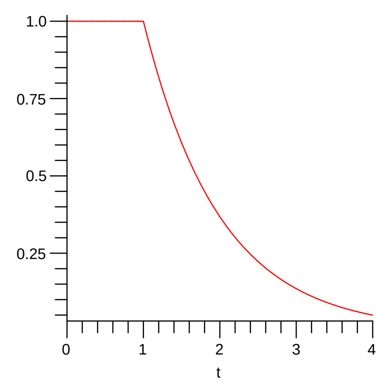
$$Y(s) = \frac{1}{s^2 + 6s + 5} = \frac{1/4}{s+1} - \frac{1/4}{s+5}$$

so

$$y(t) = \frac{1}{4}e^{-t} - \frac{1}{4}e^{-5t}.$$

A plot is below left.





(b) $y' + y = H(t) - H(t-1), \quad y(0) = 1$

Solution:
Laplace transforming gives

$$sY(s) - 1 + Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

so

$$Y(s) = \frac{1}{s(s+1)} - \frac{e^{-s}}{s(s+1)} + \frac{1}{s+1}.$$

Writing $\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$, which is the Laplace transform of $1 - e^{-t}$ gives

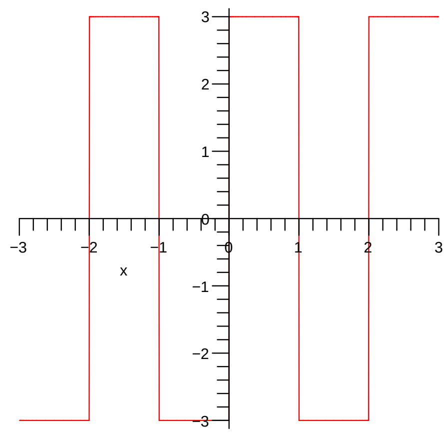
$$\begin{aligned} y(t) &= 1 - e^{-t} - H(t-1)(1 - e^{-(t-1)} + e^{-t} \\ &= 1 - H(t-1)(1 - e^{-(t-1)}) \\ &= \begin{cases} 1, & 0 \le t < 1 \\ e^{-(t-1)}, & t \ge 1 \end{cases} \end{aligned}$$

A plot is above right

9. A periodic function $f(x)$ has period 2 and takes the value 3 on the interval $[0, 1)$ and -3 on the interval $[1, 2)$.

(a) Sketch its graph.

Solution:



(b) Find its Fourier series.

Solution:
Since the function is odd, there are only sine terms in its Fourier series. The coefficients are

$$b_n = 2\int_0^1 3\sin(n\pi x)\,dx = \frac{6}{n\pi}(1-(-1)^n).$$

The last factor on the left vanishes when n is even and is 2 when n is odd. Substituting $n = 2k+1$ for odd values of n gives

$$b_{2k+1} = \frac{12}{(2k+1)\pi}$$

so the Fourier series is

$$\frac{12}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin((2k+1)\pi x)$$

(c) For which values of x does the Fourier series converge to $f(x)$?

Solution:
The Fourier series converges to the function at every point where the function is continuous, in this case, at every non-integer. At integer values, the Fourier series converges to 0 (either by observing that every term is zero, or by appealing to a general principle), which is *not* the function value, so the series does not converge to the function at integer values of x.

10. Solve the initial-boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \qquad \text{for } 0 \le x \le 1,\ t \ge 0 \\ u(t,0) &= 0 \\ u(t,1) &= 0 \\ u(0,x) &= 2\sin(\pi x) - \sin(2\pi x) \end{aligned}$$

Solution:
Separating variables and taking the left boundary condition into account gives solutions of the form

$$u(t,x) = e^{-\lambda x}\sin(\sqrt{\lambda}x)$$

The right boundary condition requires $\sin(\sqrt{\lambda}) = 0$, so $\lambda = (n\pi)^2$ where n is a positive integer. Superpostion of these solutions gives solutions to the boundary value problem of the form

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t}\sin(n\pi x)$$

which has initial value

$$u(0,x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

The given initial condition can be satisfied by taking $b_1 = 2$, $b_2 = -1$, and $b_n = 0$ for $n \ge 3$. Therefore

$$u(t,x) = 2e^{-\pi^2 t}\sin(\pi x) - e^{4\pi^2 t}\sin(2\pi x).$$