1.1 DE Models

Mechanics: Consider the motion of a ball thrown into the air near the surface of the earth. Let the x-axis be directed upwards and x is the position of the ball on the x-axis. We assume that $x = 0$ corresponds to the surface of the earth.

By Newton's second law we have $F = -mg$ where g is the earth's acceleration due to gravity.

Then $-mg = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$

Therefore, the mathematical model of the motion of the ball is given by the DE: $\frac{d^2x}{dt^2} = -g$

Consider the planetary motion model $g = \frac{GM}{|x|^2}$, where M is the mass of te sun and x is a 3D vector that defines the location of a planet relative to the sun. If m is the mass of the planet, then by Newton's second law and the law of gravitation

$$m\frac{d^2x}{dt^2} = -\frac{GMm}{|x|^2}\frac{x}{|x|}$$

gives the planetary motion model. x is a 3D vector with 3 components.

Population model.

Let $P(t)$ be a population. If its rate of change is proportional to the population itself, then we get the model

$$\frac{dP}{dt} = rP$$

where the r is called the reproductive rate:

- if r is constant, then

$$P(t) = P(0)e^{rt} \qquad \text{(Malthusian model)}$$

- if r depends on the resources (food and environment), i.e.,

$$r = c\Big(1 - \frac{P}{K}\Big),$$

  where $c =$ constant, $K =$ carrying capacity (the maximum sustainable population), then

$$\frac{dP}{dt} = c\Big(1 - \frac{P}{K}\Big)P, \qquad \text{(logistic model)}$$

  with the solution (using separation of variables)

$$P(t) = \frac{1}{\frac{1}{K} + e^{-ct}\big(\frac{1}{P(0)} - \frac{1}{K}\big)}$$