10.1 The Linearization of a Nonlinear System

Consider a planar (2-dimensional) autonomous system:

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \tag{1}$$

Let the point (x_0, y_0) be such that

$$f(x_0,y_0) = 0, \quad g(x_0,y_0) = 0 \tag{2}$$

i.e (x_0, y_0) is an equilibrium point of (1).

If f and g are differentiable at (x_0, y_0) we can write their Taylor series expansions at (x_0, y_0) assuming that $x = x_0 + u,\ y = y_0 + v$:

$$\begin{aligned} f(x,y) = f(x_0+u, y_0+v) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)u + \frac{\partial f}{\partial x}(x_0,y_0)v + R_f(u,v) \\ g(x,y) = g(x_0+u, y_0+v) = g(x_0,y_0) + \frac{\partial g}{\partial x}(x_0,y_0)u + \frac{\partial g}{\partial x}(x_0,y_0)v + R_f(u,v) \end{aligned} \tag{3}$$

Reminders have to satisfy the conditions $\lim\limits_{d\to 0} \dfrac{R_f(u,v)}{\sqrt{u^2+v^2}} = 0, \quad \lim\limits_{d\to 0} \dfrac{R_g(u,v)}{\sqrt{u^2+v^2}} = 0.$

Then

$$\begin{aligned} u' = \ & x' = \ f(x,y) \\ v' = \ & y' = \ g(x,y) \end{aligned}$$

Using (2) and ignoring the remainders we obtain the linear system

$$\begin{aligned} u' &= \frac{\partial f}{\partial x}(x_0,y_0)\,u + \frac{\partial f}{\partial x}(x_0,y_0)\,v \\ v' &= \frac{\partial g}{\partial x}(x_0,y_0)\,u + \frac{\partial g}{\partial x}(x_0,y_0)\,v \end{aligned} \tag{4}$$

that is called the linearization of (1) at the equilibrium point (x_0, y_0).

The matrix

$$J = J(x_0,y_0) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix}$$

is called the Jacobian of (f, g) at the point (x_0, y_0).

Linearization can be done correctly only if J is nonsingular, i.e. $\det J \neq 0$.

Theorem If the linearization (4) of (1) at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point for the system (1) at (x_0, y_0) is of the same type.

Recall generic types: Saddle point, Nodal sink, Nodal source, Spiral sink, Spiral source.

Example:

$$\begin{aligned} x' &= \cos x - 4y \\ y' &= \cos 3x - 5y \end{aligned}$$

$$J = \begin{bmatrix} \sin x & -4 \\ -3\sin 3x & -5 \end{bmatrix}$$

$\left(\frac{\pi}{2}, 0\right)$ is an equilibrium point. $J\left(\frac{\pi}{2}, 0\right) = \begin{bmatrix} -1 & -4 \\ 3 & -5 \end{bmatrix}$

Linearzation at $\left(\frac{\pi}{2}, 0\right)$ is

$$\begin{aligned} x' &= -x - 4y \\ y' &= 3x - 5y \end{aligned}$$

$D = \det J = 5 + 12 = 17 > 0,\ T = -6 < 0,\ T^2 - 4D = 36 - 417 < 0 \ \Rightarrow$ spiral sink, generic.

Hence the given nonlinear system has the same type by the previous theorem.

Consider another equilibrium point $\left(\frac{3\pi}{2}, 0\right)$. $J\left(\frac{3\pi}{2}, 0\right) = \begin{bmatrix} 1 & -4 \\ -3 & -5 \end{bmatrix}$

Linearization at $\left(\frac{3\pi}{2}, 0\right)$ is

$$\begin{aligned} x' &= x - 4y \\ y' &= -3x - 5y \end{aligned}$$

$D = -5 - 12 < 0 \ \Rightarrow$ saddle point for the linear system at the origin, generic

$\Rightarrow$ the given system has saddle point at $\left(\frac{3\pi}{2}, 0\right)$.

Example:

$$\begin{aligned} x' &= 4x - y^2 \\ y' &= 8x^2 - 4y \end{aligned}$$

$$J = \begin{bmatrix} 4 & -2y \\ 16x & -4 \end{bmatrix}$$

Equilibrium points: $(0, 0)$ and $(1, 2)$.

At $(0,0)$: $J(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$ $\quad \begin{aligned} x' &= 4x \\ y' &= -4y \end{aligned}$

$D = -16 < 0 \quad \Rightarrow \quad$ saddle point.

At $(1,2)$: $J(1,2) = \begin{bmatrix} 4 & -4 \\ 16 & -4 \end{bmatrix}$ $\quad \begin{aligned} x' &= 4x - 4y \\ y' &= 16x - 4y \end{aligned}$

$D = -16 + 4 \cdot 16 = 48 > 0, \ T = 0, \ T^2 - 4D < 0 \quad \Rightarrow \quad$ Center. Non-generic.

$\Rightarrow$ We can not classify the type of the equilibrium point $(1,2)$ of the given non-linear system.

- Exercise 1, page 468.

$$x' = x(6-3x) - 2xy,$$
$$y' = y(5-y) - xy.$$

(i) Sketch the nullclines for each equation. Use a distinctive marking for each nullcline so they can be distinguished.

(ii) Use analysis to find the equilibrium points for the system. Label each equilibrium point on your sketch with its coordinates.

(iii) Use the Jacobian to classify each equilibrium point (spiral source, nodal sink, etc.).

<u>`Solution`</u>:

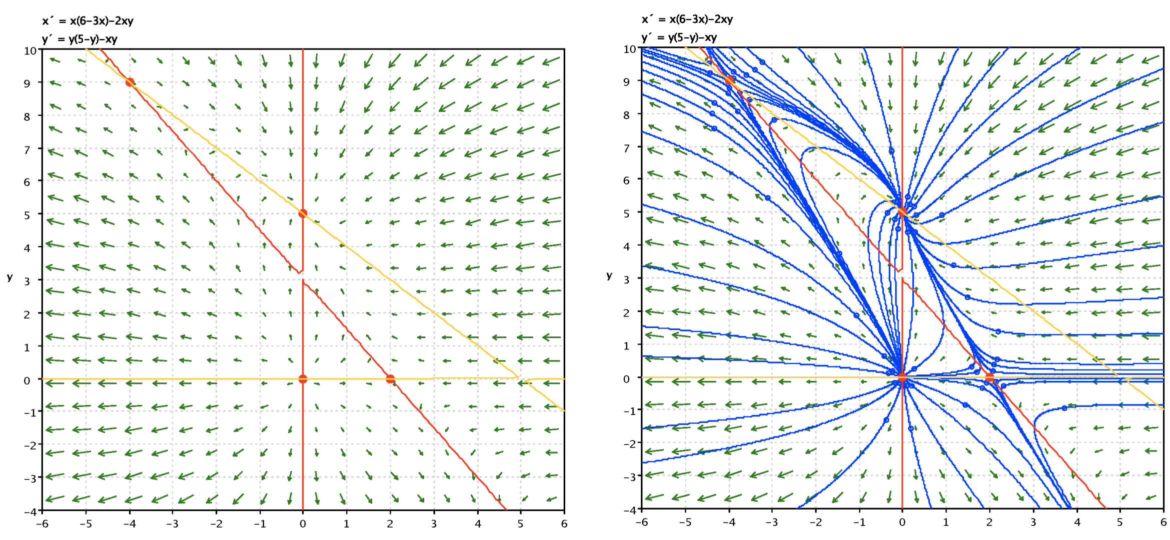

Figure 1: Exercise 1, x-nullclines, y-nullclines and equilibrium points.

(i)

$$0 = x(6-3x) - 2xy, \quad \Rightarrow x = 0, \quad \text{or} \quad y = 3 - 1.5x \qquad (x\text{-nullcline})$$
$$0 = y(5-y) - xy \quad \Rightarrow y = 0, \quad \text{or} \quad y = 5 - x \qquad (y\text{-nullcline})$$

(ii)

$$(0,0), \qquad (0,5), \qquad (2,0), \qquad (-4,9).$$

(iii) The Jacobian is:

$$J(x,y) = \begin{pmatrix} 6-6x-2y & -2x \\ -y & 5-2y-x \end{pmatrix}.$$

At (0,0):

$$J(0,0) = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_1 = 6, \lambda_2 = 5 \quad \Rightarrow \text{nodal source.}$$

At (0,5):

$$J(0,5) = \begin{pmatrix} -4 & 0 \\ -5 & 5 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_1 = -4, \lambda_2 = -5 \quad \Rightarrow \text{nodal sink.}$$

At (2,0):

$$J(2,0) = \begin{pmatrix} -6 & -12 \\ 0 & 3 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_1 = -6, \lambda_2 = 3 \quad \Rightarrow \text{saddle point.}$$

At (-4,9):

$$J(-4,9) = \begin{pmatrix} 12 & 8 \\ -9 & -9 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_{1,2} = \frac{3 \pm \sqrt{9+4\cdot 36}}{2} = 1.5 \pm \sqrt{153}$$

$\Rightarrow$ saddle point.

Exercise 2, page 468.

$$x' = x(6-2x-3y),$$
$$y' = y(1-x-y).$$

(i) Sketch the nullclines for each equation. Use a distinctive marking for each nullcline so they can be distinguished.

(ii) Use analysis to find the equilibrium points for the system. Label each equilibrium point on your sketch with its coordinates.

(iii) Use the Jacobian to classify each equilibrium point (spiral source, nodal sink, etc.).

<u>Solution</u>:

(i)

$$x = 0, \quad \text{or} \quad y = 2 - 2/3x \qquad (x\text{-nullcline})$$
$$y = 0, \quad \text{or} \quad y = 1 - x \qquad (y\text{-nullcline})$$

(ii)

$$(0,0), \qquad (0,1), \qquad (3,0), \qquad (-3,4).$$

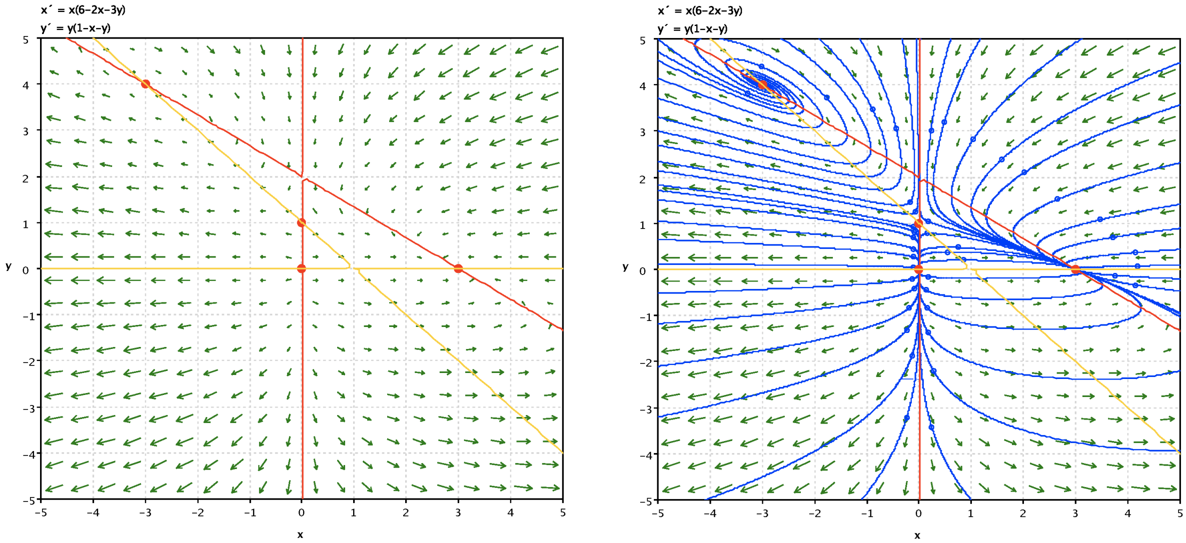

Figure 2: Exercise 2, x-nullclines, y-nullclines and equilibrium points.

(iii) The Jacobian is:

$$J(x,y) = \begin{pmatrix} 6-12x-3y & -3x \\ -y & 1-x-2y \end{pmatrix}.$$

At (0,0):

$$J(0,0) = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_1 = 6, \lambda_2 = 1 \quad \Rightarrow \text{nodal source.}$$

At (0,1):

$$J(0,1) = \begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_1 = 3, \lambda_2 = -1 \quad \Rightarrow \text{saddle point.}$$

At (3,0):

$$J(3,0) = \begin{pmatrix} -30 & -9 \\ 0 & -2 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_1 = -30, \lambda_2 = -2 \quad \Rightarrow \text{nodal sink.}$$

At (-3,4):

$$J(-3,4) = \begin{pmatrix} 30 & 9 \\ -4 & -4 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_{1,2} = \frac{26 \pm \sqrt{676+336}}{2} < 0$$

$\Rightarrow$ saddle point.

Exercise 5, page 468.

$$x' = x(4y-5),$$
$$y' = y(3-x).$$

(i) Sketch the nullclines for each equation. Use a distinctive marking for each nullcline so they can be distinguished.

(ii) Use analysis to find the equilibrium points for the system. Label each equilibrium point on your sketch with its coordinates.

(iii) Use the Jacobian to classify each equilibrium point (spiral source, nodal sink, etc.).

Solution:

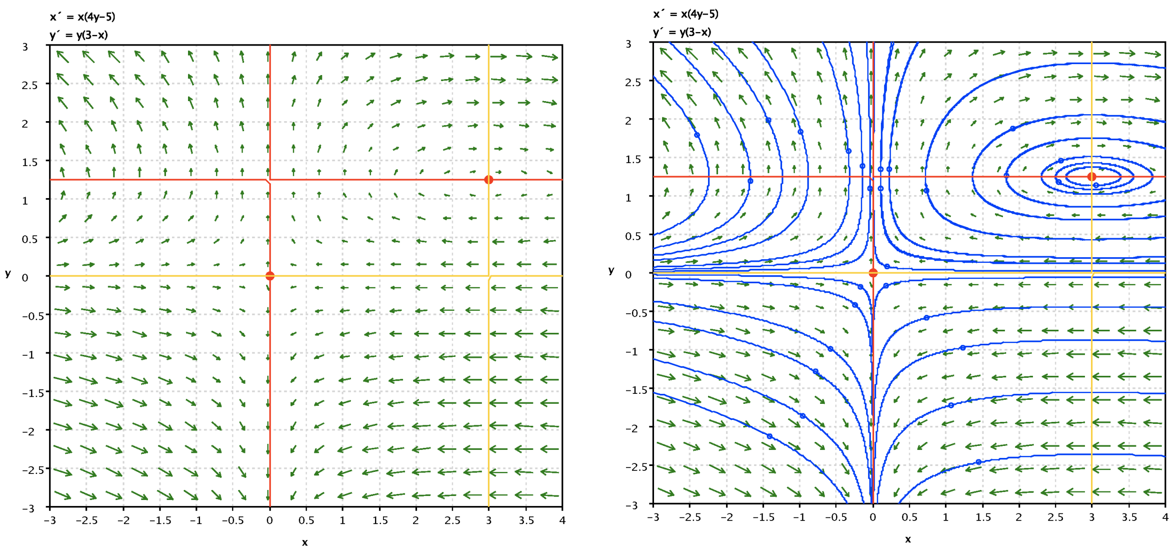

Figure 3: Exercise 5, x-nullclines, y-nullclines and equilibrium points.

(i)

$$x = 0, \quad \text{or} \quad y = \frac{5}{4}x \qquad (x\text{-nullcline})$$
$$y = 0, \quad \text{or} \quad x = 3 \qquad (y\text{-nullcline})$$

(ii)

$$(0,0), \qquad (3, 5/4).$$

(iii) The Jacobian is:

$$J(x,y) = \begin{pmatrix} 4y-5 & 4x \\ -y & 3-x \end{pmatrix}.$$

At (0,0):

$$J(0,0) = \begin{pmatrix} -5 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_1 = -5, \lambda_2 = 3 \quad \Rightarrow \text{saddle.}$$

At (3,5/4):

$$J(3,5/4) = \begin{pmatrix} 0 & 12 \\ -5/4 & 0 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_{1,2} = \pm i\sqrt{15}$$

$\Rightarrow$ linear center, non generic, we cannot classify the equilibrium point!!!

At (3,0):

$$J(3,0) = \begin{pmatrix} -30 & -9 \\ 0 & -2 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_1 = -30, \lambda_2 = -2 \quad \Rightarrow \text{nodal sink.}$$

At (-3,4):

$$J(-3,4) = \begin{pmatrix} 30 & 9 \\ -4 & -4 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_{1,2} = \frac{26 \pm \sqrt{676 + 336}}{2} < 0$$

$\Rightarrow$ saddle point.

Exercise 6, page 468.

$$x' = 1.2x - xy,$$
$$y' = -0.5y + xy.$$

(i) Sketch the nullclines for each equation. Use a distinctive marking for each nullcline so they can be distinguished.

(ii) Use analysis to find the equilibrium points for the system. Label each equilibrium point on your sketch with its coordinates.

(iii) Use the Jacobian to classify each equilibrium point (spiral source, nodal sink, etc.).

Solution:

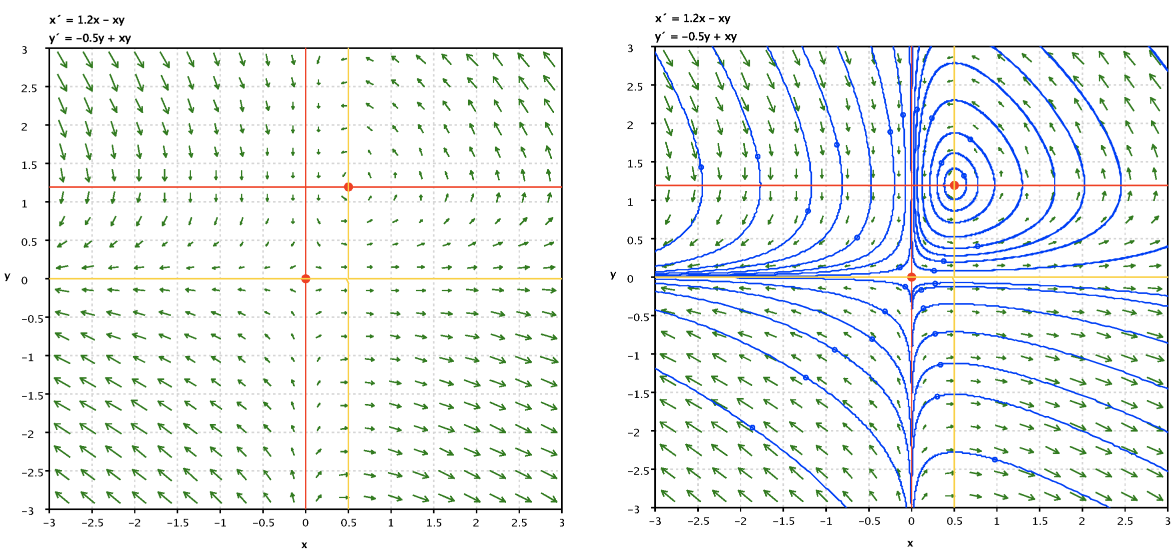

Figure 4: Exercise 6, x-nullclines, y-nullclines and equilibrium points.

(i)

$$x = 0, \quad \text{or} \quad y = 1.2 \qquad (x\text{-nullcline})$$
$$y = 0, \quad \text{or} \quad x = 0.5 \qquad (y\text{-nullcline})$$

(ii)

$$(0,0), \qquad (0.5, 1.2).$$

(iii) The Jacobian is:

$$J(x,y) = \begin{pmatrix} 1.2 - y & -x \\ y & -0.5 + x \end{pmatrix}.$$

At (0,0):

$$J(0,0) = \begin{pmatrix} 1.2 & 0 \\ 0 & -0.5 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_1 = 1.2, \lambda_2 = -0.5 \quad \Rightarrow \text{saddle.}$$

At (0.5 , 1.2):

$$J(3, 5/4) = \begin{pmatrix} 0 & -0.5 \\ 1.2 & 0 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_{1,2} = \pm i\sqrt{0.7}$$

$\Rightarrow$ linear center, non generic, we cannot classify the equilibrium point!!!

Exercise 8, page 468.

$$x' = y,$$
$$y' = -\cos x - 0.5y.$$

(i) Sketch the nullclines for each equation. Use a distinctive marking for each nullcline so they can be distinguished.

(ii) Use analysis to find the equilibrium points for the system. Label each equilibrium point on your sketch with its coordinates.

(iii) Use the Jacobian to classify each equilibrium point (spiral source, nodal sink, etc.).

<u>Solution</u>:

(i)

$$y = 0, \qquad (x\text{-nullcline})$$
$$y = -2\cos x \qquad (y\text{-nullcline})$$

(ii)

$$\left(\frac{\pi}{2} + n\pi, 0\right) \qquad \forall n \in \mathbb{N}$$

(iii) The Jacobian is:

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ \sin x & -0.5 \end{pmatrix}.$$

At $(\pi/2 + n\pi, 0)$:

$$J(\pi/2 + 2\kappa\pi, 0) = \begin{pmatrix} 0 & 1 \\ 1 & -0.5 \end{pmatrix}, \qquad \text{with (even) eigenvalues} \quad \lambda_{1,2} = \frac{-0.5 \pm \sqrt{4.25}}{2}$$

$\Rightarrow$ saddle point;

$$J(\pi/2 + (2\kappa+1)\pi, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -0.5 \end{pmatrix}, \qquad \text{with eigenvalues} \quad \lambda_{1,2} = -0.125 \pm i\frac{\sqrt{0.7}}{4}$$

$\Rightarrow$ spiral sink

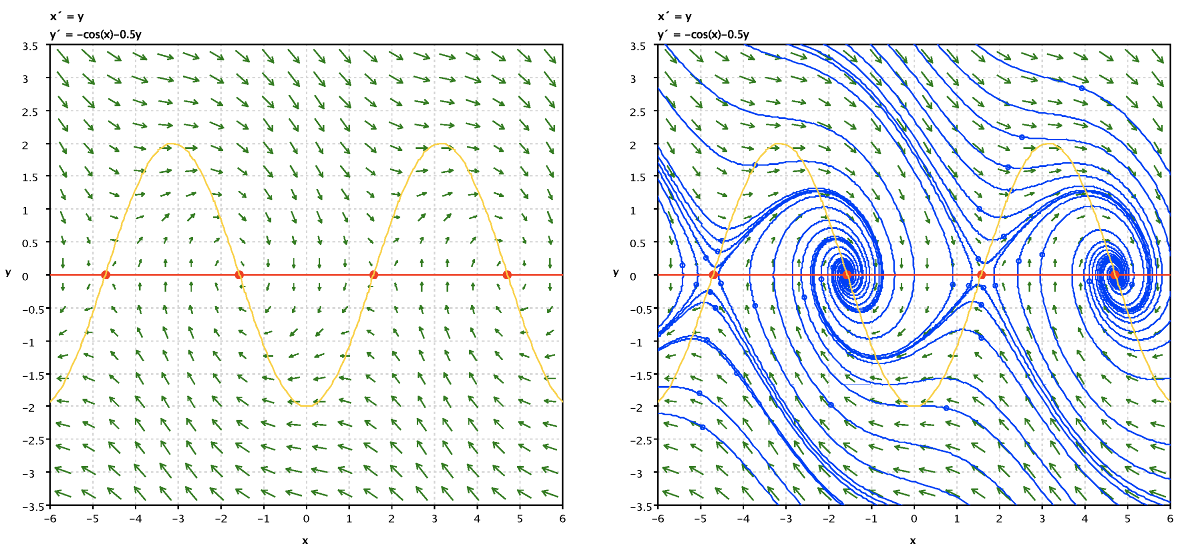

Figure 5: Exercise 8, x-nullclines, y-nullclines and equilibrium points.