12.3 Fourier Cosine and Sine Series

- Exercise 2, page 617.
  Give a piecewise definition of f_o, the odd expansion for

$$f(x) = 1 - 2x, \qquad [0, 1]$$

as defined on the given interval.
Sketch the graph of f_o. Sketch the graph of f_{op} over three periods.

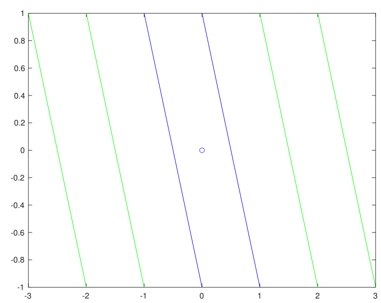

Figure 1: Exercise 2.

Solution: The odd function f_o is symmetric w.r.t. the origin $(0, 0)$, i.e.,

$$f_o(x) = \begin{cases} f(x), & x \in (0, L] \\ 0, & x = 0 \\ -f(-x), & x \in [-L, 0) \end{cases} \equiv \begin{cases} -(1 + 2x), & x \in [-1, 0) \\ 0, & x = 0 \\ 1 - 2x, & x \in (0, 1] \end{cases}$$

- Exercise 4, page 617.
  Give a piecewise definition of f_o, the odd expansion for

$$f(x) = x^2 - 2, \qquad [0, 2]$$

as defined on the given interval.
Sketch the graph of f_o. Sketch the graph of f_{op} over three periods.

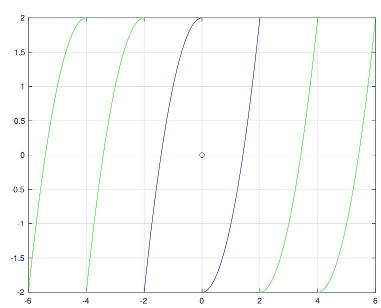

Figure 2: Exercise 4.

Solution: The odd function f_o is symmetric w.r.t. the origin $(0, 0)$, i.e.,

$$f_o(x) = \begin{cases} f(x), & x \in (0, L] \\ 0, & x = 0 \\ -f(-x), & x \in [-L, 0) \end{cases} \equiv \begin{cases} -(x^2 - 2), & x \in [-2, 0) \\ 0, & x = 0 \\ x^2 - 2, & x \in (0, 2] \end{cases}$$

- Exercise 6, page 617.
  Give a piecewise definition of f_e, the even expansion for

$$f(x) = 1 - 2x, \qquad [0, 1]$$

as defined on the given interval.
Sketch the graph of f_e. Sketch the graph of f_{ep} over three periods.

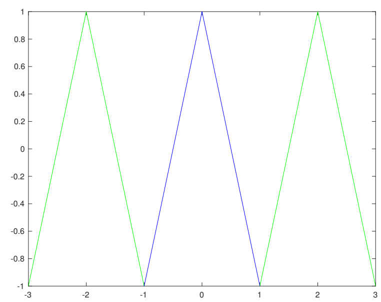

Figure 3: Exercise 6.

Solution: The even function f_e is symmetric w.r.t. the vertical axis, i.e.,

$$f_e(x) = \begin{cases} f(-x), & x \in [-L, 0] \\ f(x), & x \in [0, L] \end{cases} \equiv \begin{cases} 1 + 2x, & x \in [-1, 0] \\ 1 - 2x, & x \in [0, 1] \end{cases}$$

- Exercise 8, page 617.
  Give a piecewise definition of f_e, the even expansion for

$$f(x) = x^2 - 2, \qquad [0, 2]$$

as defined on the given interval.
Sketch the graph of f_e. Sketch the graph of f_{ep} over three periods.

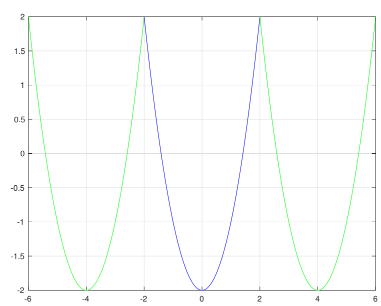

Figure 4: Exercise 8.

Solution: The even function f_e is symmetric w.r.t. the vertical axis, i.e.,

$$f_e(x) = \begin{cases} f(-x), & x \in [-L, 0] \\ f(x), & x \in [0, L] \end{cases} \equiv \begin{cases} x^2 - 2, & x \in [-2, 0] \\ x^2 - 2, & x \in [0, 2] \end{cases}$$

- Exercise 10, page 617.
  Expand the function

$$f(x) = \sin x,$$

in a Fourier cosine series valid on the interval $0 \le x \le \pi$.
Plot the function and two partial sums of your choice over the interval $0 \le x \le \pi$.
Plot the same sums and the function the series converges to over the interval $-3\pi \le x \le 3\pi$.

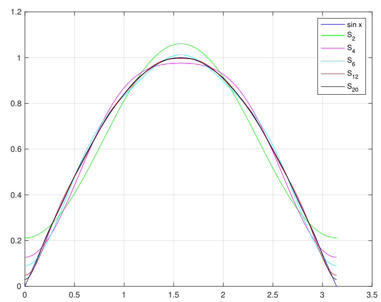

Figure 5: Exercise 10.

<u>**Solution**</u>: The even function f_e is

$$f_e(x) = \begin{cases} f(-x), & x \in [-L, 0] \\ f(x), & x \in [0, L] \end{cases} \equiv \begin{cases} -\sin x, & x \in [-\pi, 0] \\ \sin x, & x \in [0, \pi] \end{cases}$$

gives <u>only cosine terms</u> and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{2\pi} \sin x \cos nx \, dx \qquad\qquad \left(\sin a \cos b = \tfrac{1}{2}\big(\sin(a+b) + \sin(a-b)\big)\right) \\ &= \frac{1}{\pi}\left(-\frac{1}{n+1}\cos(n+1)x\Big|_0^{\pi} + \frac{1}{n-1}\cos(n-1)x\Big|_0^{\pi}\right) \\ &= \frac{1}{\pi}\left(-\frac{1}{n+1}\big((-1)^{n+1} - 1\big) + \frac{1}{n-1}\big((-1)^{n+1} - 1\big)\right) \\ &= \frac{1}{\pi}\left(\frac{(-1)^n}{n+1} + \frac{(-1)^{n+1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right) \\ &= \frac{1}{\pi}\left((-1)^n\left(\frac{1}{n+1} - \frac{}{n-1}\right) + \frac{\cancel{n} - 1 - \cancel{n} - 1}{(n+1)(n-1)}\right) \\ &= \frac{-2}{\pi(n^2-1)}\big(1 + (-1)^n\big), \qquad \forall n \ne 1. \end{aligned}$$

When $n = 1$ we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx = \frac{1}{2\pi}\big(-\cos 2x\big)\Big|_0^{\pi} = 0,$$

also for $n = 0$:

$$a_0 = \frac{2(-2)}{\pi(-1)} = \frac{4}{\pi}.$$

Therefore the Fourier (cosine) expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2}{\pi} \frac{(-1)^{n+1} - 1}{n^2 - 1} \cos nx.$$

- Exercise 11, page 617.
  Expand the function

$$f(x) = \cos x,$$

in a Fourier cosine series valid on the interval $0 \leq x \leq \pi$.
Plot the function and two partial sums of your choice over the interval $0 \leq x \leq \pi$.
Plot the same sums and the function the series converges to over the interval $-3\pi \leq x \leq 3\pi$.
<u>`Solution`</u>: We note that the function $f(x) = \cos x$ is in this case already written in a *Fourier cosine expansion form* with all coefficients $a_n = 0 \quad \forall n \neq 1$, except $a_1 = 1$. !!!

Let us just check that following the technique described in class.
The even function f_e is

$$f_e(x) = \begin{cases} f(-x), & x \in [-L, 0] \\ f(x), & x \in [0, L] \end{cases} \equiv \begin{cases} \cos(-x), & x \in [-\pi, 0] \\ \cos x, & x \in [0, \pi] \end{cases} = \begin{cases} \cos x, & x \in [-\pi, 0] \\ \cos x, & x \in [0, \pi] \end{cases}$$

gives <u>only cosine terms</u> and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \cos x \cos nx \, dx \qquad \qquad \left(\cos a \cos b = \tfrac{1}{2}\big(\cos(a+b) + \cos(a-b)\big)\right) \\ &= \frac{1}{\pi} \left(\frac{1}{n+1} \sin(n+1)x \Big|_0^{\pi} + \frac{1}{n-1} \sin(n-1)x \Big|_0^{\pi} \right) \\ &= 0 \qquad \forall n \neq 1. \end{aligned}$$

When $n = 1$ we have

$$\begin{aligned} a_1 &= \frac{2}{\pi} \int_0^{\pi} \cos x \cos x \, dx = \frac{2}{\pi} \int_0^{\pi} \frac{\cos 2x + 1}{2} \, dx = \frac{1}{\pi} \int_0^{\pi} \cos 2x \, dx + \frac{1}{\pi} x \Big|_0^{\pi} \\ &= \frac{1}{2\pi} \sin 2x \Big|_0^{\pi} + 1 = 1. \end{aligned}$$

Therefore the Fourier (cosine) expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \equiv a_1 \cos(1 \cdot x) = \cos x.$$

- Exercise 20, page 617.
  Expand the function

$$f(x) = x \sin x,$$

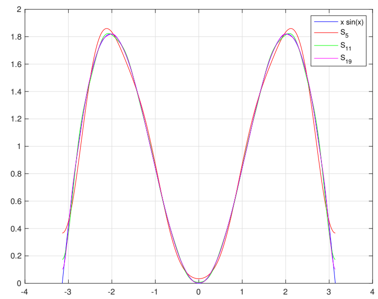

Figure 6: Exercise 20.

in a Fourier cosine series valid on the interval $0 \le x \le \pi$.
Plot the function and two partial sums of your choice over the interval $0 \le x \le \pi$.
Plot the same sums and the function the series converges to over the interval $-3\pi \le x \le 3\pi$.

Solution: We note that the function $f(x) = x\sin x$ is an **even** function on $[0,\pi]$, hence it coincides with its even extension f_e on $[0,\pi]$:

$$f_e(x) = \begin{cases} f(-x), & x \in [-L, 0] \\ f(x), & x \in [0, L] \end{cases} \equiv \begin{cases} x\sin x, & x \in [-\pi, 0] \\ x\sin x, & x \in [0, \pi] \end{cases} = x\sin x \quad \text{on } [-\pi,\pi]$$

giving only cosine series:

$$f(x) \equiv f_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

The Fourier coefficients are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

$$\left(\sin a\cos b = \tfrac{1}{2}\big(\sin(a+b) + \sin(a-b)\big)\right)$$

$$= \frac{2}{\pi}\frac{1}{2}\int_0^{\pi} x\Big(\sin(n+1)x + \sin(1-n)x\Big)\,dx$$

$$= \frac{1}{\pi}\Bigg(-\frac{x}{n+1}\cos(n+1)x\Big|_{x=0}^{x=\pi} + \int_0^{\pi}\frac{\cos(n+1)x}{n+1}\,dx - \frac{x}{1-n}\cos(1-n)x\Big|_{x=0}^{x=\pi} + \int_0^{\pi}\frac{\cos(1-n)x}{1-n}\,dx\Bigg)$$

$$= -\frac{1}{n+1}(-1)^{n+1} - \frac{1}{1-n}(-1)^{n-1} + \int_0^{\pi}\frac{\cos(n+1)x}{\pi(n+1)}\,dx + \frac{1}{\pi(1-n)}\int_0^{\pi}\cos(n-1)x\,dx$$

$$= (-1)^n\left(\frac{1}{n+1} + \frac{1}{1-n}\right) + \frac{1}{\pi}\left(\frac{1}{n+1}\frac{\sin(n+1)x}{(n+1)}\Big|_{x=0}^{x=\pi} + \frac{1}{1-n}\frac{\sin(n-1)x}{n-1}\Big|_{x=0}^{x=\pi}\right)$$

$$= (-1)^n\frac{1-n+n+1}{1-n^2} = \frac{2}{n^2-1}(-1)^{n+1} \qquad \forall n \ne 1.$$

For $n = 1$ we have

$$a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx = \frac{-x}{2\pi}\cos 2x\Big|_{x=0}^{x=\pi} + \frac{1}{2\pi}\int_0^\pi \cos 2x\,dx$$
$$= -\frac{1}{2} + \frac{1}{2\pi}\frac{1}{2}\sin 2x\Big|_0^\pi = -\frac{1}{2}.$$

Therefore the Fourier (cosine) expansion is

$$f(x) = f_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \equiv 1 + \frac{-1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{n^2-1}\cos nx.$$

- Exercise 32, page 617.
  Expand the function

$$f(x) = x\sin x,$$

in a Fourier **sine** series valid on the interval $0 \le x \le \pi$.
Plot the function and two partial sums of your choice over the interval $0 \le x \le \pi$.
Plot the same sums and the function the series converges to over the interval $-3\pi \le x \le 3\pi$.

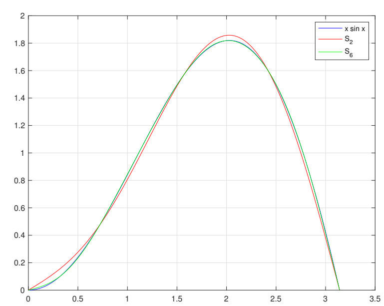

Figure 7: Exercise 32.

<u>`Solution`</u>: We note that the function $f(x) = x\sin x$ is an **even** function on $[0, \pi]$. The <u>sine series</u> expansion is:

$$f(x) \equiv \sum_{n=1}^{\infty} b_n \sin nx.$$

The Fourier coefficients are

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x \sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\sin nx\,dx \\
&\qquad\qquad\qquad\qquad \left(\sin a\sin b = -\tfrac{1}{2}\big(\cos(a+b)-\cos(a-b)\big)\right) \\
&= -\frac{1}{\pi}\frac{1}{2}\int_0^{\pi} x\Big(\cos(n+1)x - \cos(n-1)x\Big)\,dx \\
&= -\frac{1}{\pi}\int_0^{\pi} x\cos(n+1)x\,dx + \frac{1}{\pi}\int_0^{\pi}\cos(n-1)x\,dx \\
&= -\frac{1}{\pi(n+1)}x\sin(n+1)x\Big|_0^{\pi} + \frac{1}{\pi(n-1)}x\sin(n-1)x\Big|_0^{\pi} \\
&\quad + \frac{1}{\pi(n+1)}\int_0^{\pi}\sin(n+1)x\,dx - \frac{1}{\pi(n-1)}\int_0^{\pi}\sin(n-1)x\,dx \\
&= -\frac{1}{\pi(n+1)^2}\cos(n+1)x\Big|_0^{\pi} + \frac{1}{\pi(n-1)^2}\cos(n-1)x\Big|_0^{\pi} \\
&= \frac{1}{\pi(n+1)^2} - \frac{1}{\pi(n-1)^2} - \frac{1}{\pi(n+1)^2}(-1)^{n+1} + \frac{1}{\pi(n-1)^2}(-1)^{n+1} \\
&= -\left(\frac{1}{\pi(n+1)^2} - \frac{1}{\pi(n-1)^2}\right)\big((-1)^{n+1}-1\big) \\
&= -\frac{(-1)^{n+1}-1}{\pi(n^2-1)^2}\big(n^2 - 2n + 1 - n^2 - 2n - 1\big) \\
&= \frac{4n\big((-1)^{n+1}-1\big)}{\pi(n^2-1)^2} \qquad \forall n \neq 1.
\end{aligned}
$$

For $n = 1$ we have

$$
\begin{aligned}
b_1 &= \frac{2}{\pi}\int_0^{\pi} x\sin x\sin x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin^2 x\,dx = \frac{1}{\pi}\int_0^{\pi} x\frac{1-\cos 2x}{2}\,dx \\
&= \frac{1}{2\pi}\frac{x^2}{2}\Big|_0^{\pi} - \frac{1}{2\pi}\int_0^{\pi} x\cos 2x\,dx = \frac{\pi}{4} - \frac{1}{2\pi}\int_0^{\pi} x\left(\frac{\sin 2x}{2}\right)'dx \\
&= \frac{\pi}{4} - \frac{1}{2\pi}x\frac{\sin 2x}{2}\Big|_0^{\pi} + \frac{1}{4\pi}\int_0^{\pi}\sin 2x\,dx
\end{aligned}
$$

Therefore the Fourier (sine) expansion is

$$
f(x) = \sum_{n=1}^{\infty} b_n \sin nx \equiv \frac{\pi}{2}\sin x + \sum_{n=2}^{\infty}\frac{4n\big((-1)^{n+1}-1\big)}{\pi(n^2-1)}\sin nx.
$$