13.2 Separation of variables for the heat equation

- Exercise 5, page 644

Find the temperature $u(x,t)$ in a rod modeled by the initial/boundary value problem

$$\begin{aligned} &u_t(x,t) = k u_{xx}(x,t), && \text{for } x \in (0,L) \text{ and } t > 0, \\ &u(0,t) = T_0 \quad \text{and} \quad u(L,t) = T_L && \text{for } t > 0, && \text{(Dirichlet B.C.)} \\ &u(x,0) = f(x), && \text{for } x \in [0,L], && \text{(I.C.)} \end{aligned}$$

where

$$k = 4, \quad L = 1, \quad T_0 = 0, \quad T_L = 0, \quad \text{and} \quad f(x) = x(1-x).$$

<u>Solution</u>: The steady solution is

$$u^s(x) = \frac{T_L - T_0}{L} x + T_0 \equiv 0$$

and the solution with homogeneous B.C.s $v(x,t)$ satisfies

$$\begin{aligned} &v_t(x,t) = 4 v_{xx}(x,t), && x \in (0,1), t > 0, \\ &v(0,t) = v(L,t) = 0, && t > 0, && \text{(B.C.)} \\ &v(x,0) = x(1-x) - 0, && x \in [0,1]. \end{aligned}$$

Using separation of variables to find $v(x,t) = X(x)T(t)$, denoting by $\omega_n = \frac{n\pi}{L} \equiv n\pi$ the frequency, we have that $X(x)$ and $T(t)$ satisfy

$$\begin{aligned} &XT' = kX''T, \qquad \frac{T'}{kT} = \frac{X''}{X} = -\omega_n^2, \\ &T(t) = e^{-k\omega_n^2 t} C \equiv e^{-kn^2\pi^2 t}, \\ &X(x) = b_n \sin(\omega_n x) \equiv b_n \sin(n\pi x), \end{aligned}$$

hence

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t} \sin(n\pi x),$$

where the Fourier coefficients b_n are

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L \big(x(1-x) - 0\big) \sin(n\pi x)\, dx \equiv 2 \int_0^1 x(1-x) \underbrace{\sin(n\pi x)}_{=\left(-\frac{\cos(n\pi x)}{n\pi}\right)'} dx \\ &= \cancel{-\frac{2}{n\pi}(x - x^2)\cos(n\pi x)\Big|_0^1} + \frac{2}{n\pi} \int_0^1 (1-2x) \underbrace{\cos(n\pi x)}_{=\left(\frac{\sin(n\pi x)}{n\pi}\right)'} dx \\ &= \cancel{\frac{2}{n^2\pi^2}(1-2x)\sin(n\pi x)\Big|_0^1} - \frac{2}{n^2\pi^2} \int_0^1 (-2) \underbrace{\sin(n\pi x)}_{=-\left(\frac{\cos(n\pi x)}{n\pi}\right)'} dx \end{aligned}$$

$$= -\frac{4}{n^3\pi^3}\cos(n\pi x)\Big|_0^1 = -\frac{4}{n^3\pi^3}\big((-1)^n - 1\big) = \frac{4}{n^3\pi^3}\big(1-(-1)^n\big).$$

Finally we have

$$u(x,t) = \underbrace{u^s(x)}_{=0} + v(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3}\big(1-(-1)^n\big)e^{-4n^2\pi^2 t}\sin(n\pi x)$$

- Exercise 6, page 644
  Find the temperature $u(x,t)$ in a rod modeled by the initial/boundary value problem

$$\begin{aligned}
&u_t(x,t) = k u_{xx}(x,t), \qquad \text{for } x \in (0,L) \text{ and } t>0,\\
&u(0,t) = T_0 \quad \text{and} \quad u(L,t) = T_L \quad \text{for } t>0, && \text{(Dirichlet B.C.)}\\
&u(x,0) = f(x), \qquad \text{for } x\in[0,L], && \text{(I.C.)}
\end{aligned}$$

where

$$k=2, \quad L=\pi, \quad T_0 = 0, \quad T_L = 0, \quad \text{and} \quad f(x) = \sin 2x - \sin 4x.$$

Solution: The steady solution is

$$u^s(x) = \frac{T_L - T_0}{L}x + T_0 \equiv 0$$

and the solution with homogeneous B.C.s $v(x,t)$ satisfies

$$\begin{aligned}
&v_t(x,t) = 2v_{xx}(x,t), \qquad x\in(0,\pi), t>0,\\
&v(0,t) = v(\pi,t) = 0, \qquad t>0, && \text{(B.C.)}\\
&v(x,0) = \sin 2x - \sin 4x - 0, \qquad x\in[0,\pi].
\end{aligned}$$

Using separation of variables to find $v(x,t) = X(x)T(t)$, denoting by $\omega_n = \frac{n\pi}{L} \equiv n$ the frequency, we have that $X(x)$ and $T(t)$ satisfy

$$\begin{aligned}
&XT' = kX''T, \qquad \frac{T'}{kT} = \frac{X''}{X} = -\omega_n^2,\\
&T(t) = e^{-k\omega_n^2 t}C \equiv e^{-2n^2 t},\\
&X(x) = b_n \sin(\omega_n x) \equiv b_n\sin(nx),
\end{aligned}$$

hence

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-2n^2 t}\sin(nx).$$

Recall that the Fourier coefficients b_n are obtained from the initial conditions

$$v(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) \equiv \sin 2x - \sin 4x \qquad \text{(I.C.)}$$

therefore

$$b_2 = 1, \qquad b_4 = -1,$$
$$b_n = 0 \quad \text{for all} \quad n \neq 2 \text{ or } 4.$$

Finally,

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-2n^2 t} \sin(nx) = e^{-8t}\sin(2x) - e^{-32t}\sin(4x).$$

- Exercise 7, page 643
  Find the temperature $u(x,t)$ in a rod modeled by the initial/boundary value problem

$$u_t(x,t) = k u_{xx}(x,t), \qquad \text{for } x \in (0,L) \text{ and } t > 0,$$
$$u(0,t) = T_0 \quad \text{and} \quad u(L,t) = T_L \quad \text{for } t > 0, \qquad \text{(Dirichlet B.C.)}$$
$$u(x,0) = f(x), \qquad \text{for } x \in [0,L], \qquad \text{(I.C.)}$$

where

$$k = 1, \quad L = \pi, \quad T_0 = 0, \quad T_L = 0, \quad \text{and} \quad f(x) = \sin^2 x.$$

Solution: The steady solution is

$$u^s(x) = \frac{T_L - T_0}{L} x + T_0 \equiv 0$$

and the solution with homogeneous B.C.s $v(x,t)$ satisfies

$$v_t(x,t) = v_{xx}(x,t), \qquad x \in (0,\pi), t > 0,$$
$$v(0,t) = v(\pi,t) = 0, \qquad t > 0, \qquad \text{(B.C.)}$$
$$v(x,0) = \sin^2 x - 0, \qquad x \in [0,\pi].$$

Using separation of variables to find $v(x,t) = X(x)T(t)$, denoting by $\omega_n = \frac{n\pi}{L} \equiv n$ the frequency, we have that $X(x)$ and $T(t)$ satisfy

$$T(t) = e^{-k\omega_n^2 t} C \equiv e^{-n^2 t},$$
$$X(x) = b_n \sin(\omega_n x) \equiv b_n \sin(nx),$$

hence

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx).$$

Recall that the Fourier coefficients b_n are obtained from the initial conditions

$$v(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) \equiv \sin^2 x \qquad \text{(I.C.)}$$

therefore

$$b_n = \frac{2}{L}\int_0^L \underbrace{\sin^2 x}_{=\frac{1-\cos(2x)}{2}} \sin nx\,dx \equiv \frac{1}{\pi}\int_0^\pi \big(1-\cos(2x)\big)\sin nx\,dx$$
$$= \frac{1}{\pi}\int_0^\pi \Big(\underbrace{\sin nx}_{=\left(-\frac{\cos(nx)}{n}\right)'} - \cos(2x)\sin nx\Big)\,dx$$
$$= -\frac{1}{n\pi}\cos(nx)\Big|_0^\pi - \frac{1}{\pi}\int_0^\pi \cos(2x)\sin nx\,dx$$
$$= -\frac{1}{n\pi}\big((-1)^n - 1\big) - \frac{1}{\pi}\frac{n}{(n^2-4)}\big((-1)^{n+1}+1\big)$$
$$= \frac{1}{\pi}\big((-1)^{n+1}+1\big)\left(\frac{1}{n} - \frac{n}{n^2-4}\right) = -\frac{4}{n\pi(n^2-4)}\big((-1)^{n+1}+1\big),$$

since

$$\int_0^\pi \cos(2x)\sin nx\,dx \qquad \left(\sin a\cos b = \tfrac{1}{2}\big(\sin(a+b)+\sin(a-b)\big)\right)$$
$$= \frac{1}{2}\int_0^\pi \sin((n+2)x)\,dx + \frac{1}{2}\int_0^\pi \sin((n-2)x)\,dx$$
$$= -\frac{1}{2(n+2)}\cos((n+2)x)\Big|_0^\pi - \frac{1}{2(n-2)}\cos((n-2)x)\Big|_0^\pi$$
$$= -\frac{1}{2(n+2)}\big((-1)^{n+2}-1\big) - \frac{1}{2(n-2)}\big((-1)^{n-2}-1\big)$$
$$= -\left(\frac{1}{2(n+2)} + \frac{1}{2(n-2)}\right)\big((-1)^{n+2}-1\big)$$
$$= \frac{n}{(n^2-4)}\big((-1)^{n+1}+1\big).$$

Finally,

$$v(x,t) = \sum_{n=1}^\infty b_n e^{-n^2 t}\sin(nx)$$
$$= \sum_{n=1}^\infty -\frac{4}{n\pi(n^2-4)}\big((-1)^{n+1}+1\big)e^{-n^2 t}\sin(nx)$$

- Exercise 8, page 644

  Find the temperature $u(x,t)$ in a rod modeled by the initial/boundary value problem

$$u_t(x,t) = ku_{xx}(x,t), \qquad \text{for } x \in (0,L) \text{ and } t > 0,$$
$$u(0,t) = T_0 \quad \text{and} \quad u(L,t) = T_L \quad \text{for } t > 0, \qquad \text{(Dirichlet B.C.)}$$
$$u(x,0) = f(x), \qquad \text{for } x \in [0,L], \qquad \text{(I.C.)}$$

where

$$k = 1, \quad L = 1, \quad T_0 = 0, \quad T_L = 2, \quad \text{and} \quad f(x) = x.$$

<u>`Solution`</u>: The steady solution is

$$u^s(x) = \frac{T_L - T_0}{L} x + T_0 \equiv 2x$$

and the solution with homogeneous B.C.s $v(x,t)$ satisfies

$$\begin{aligned} &v_t(x,t) = v_{xx}(x,t), && x \in (0,\pi), t > 0, \\ &v(0,t) = v(\pi,t) = 0, && t > 0, \\ &v(x,0) = x - 2x \equiv -x, && x \in [0,\pi]. \end{aligned} \tag{B.C.}$$

Using separation of variables to find $v(x,t) = X(x)T(t)$, denoting by $\omega_n = \frac{n\pi}{L} \equiv n\pi$ the frequency, we have that $X(x)$ and $T(t)$ satisfy

$$\begin{aligned} T(t) &= e^{-k\omega_n^2 t} C \equiv e^{-n^2\pi^2 t}, \\ X(x) &= b_n \sin(\omega_n x) \equiv b_n \sin(n\pi x), \end{aligned}$$

hence

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin(n\pi x).$$

Recall that the Fourier coefficients b_n are obtained from the initial conditions

$$v(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \equiv -x \tag{I.C.}$$

therefore

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L (-x) \cdot \sin(n\pi x)\, dx \equiv -2 \int_0^1 x \underbrace{\sin(n\pi x)}_{\left(\frac{-\cos(n\pi x)}{n\pi}\right)'} dx \\ &= \frac{2}{n\pi} x \cos(n\pi x) \Big|_0^1 - \frac{2}{n\pi} \int_0^1 \cos(n\pi x)\, dx = \frac{2}{n\pi}(-1)^n - \cancel{\frac{2}{n^2\pi^2} \sin(n\pi x) \Big|_0^1}. \end{aligned}$$

Therefore,

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} b_n e^{-n^2\pi t} \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} e^{-n^2\pi^2 t} \sin(n\pi x) \end{aligned}$$

and finally

$$u(x,t) = u^s(x) + v(x,t) = 2x + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} e^{-n^2\pi^2 t} \sin(n\pi x) \xrightarrow{t \to \infty} 2x$$

- Exercise 15, page 644
  Find the temperature $u(x,t)$ in a rod modeled by the initial/boundary value problem

$$\begin{aligned} &u_t(x,t) = k u_{xx}(x,t), \qquad \text{for } x \in (0,L) \text{ and } t > 0, \\ &u_x(0,t) = u_x(L,t) = 0 \quad \text{for } t > 0, && \text{(Neumann B.C.)} \\ &u(x,0) = f(x), \qquad \text{for } x \in [0,L], && \text{(I.C.)} \end{aligned}$$

where

$$k = 1, \quad L = 1, \quad \text{and} \quad f(x) = \sin(\pi x).$$

Solution: Using separation of variables to find $u(x,t) = X(x)T(t)$, denoting by $\omega_n = \frac{n\pi}{L} \equiv n\pi$ the frequency, we have that $X(x)$ and $T(t)$ satisfy

$$\begin{aligned} &XT' = kX''T, \qquad \frac{T'}{kT} = \frac{X''}{X} := -\omega_n^2, \\ &T(t) = e^{-k\omega_n^2 t}C \equiv e^{-n^2\pi^2 t}C, \\ &X(x) = a\cos(\omega_n x) + b\sin(\omega_n x), \\ &X'(x) = -a\,\omega_n \sin(\omega_n x) + b\,\omega_n \cos(\omega_n x). \end{aligned}$$

Using the homogeneous Neumann boundary conditions we obtain that

$$\begin{aligned} &X'(0) = X'(L) = 0, \\ &0 = X'(0) = b\,\omega_n, \\ &0 = X'(L) = -a\omega_n \sin(\omega_n L) + \cancel{b\omega_n \cos(\omega_n x)} \\ &\omega_n = \frac{n\pi}{L} \end{aligned}$$

hence

$$\begin{aligned} &X(x) = a_n \cos(\omega_n x), \\ &u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\omega_n^2 t}\cos(\omega_n x) \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\cos(n\pi x). \end{aligned}$$

Recall that the Fourier cosine coefficients a_n are obtained from the initial conditions

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n x) \equiv f(x) \qquad \text{(I.C.)}$$

therefore

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cdot\cos(\omega_n x)\,dx \equiv 2\int_0^1 \sin(\pi x)\cdot\cos(n\pi x)\,dx \\ &= 2\int_0^1 \underbrace{\sin(\pi x)}_{=\left(-\frac{\cos(\pi x)}{\pi}\right)'} \cos(n\pi x)\,dx \end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{\pi}\cos(\pi x)\cos(n\pi x)\Big|_0^1 + \frac{2}{\pi}\int_0^1 \cos(\pi x)(-n\pi)\sin(n\pi x)\,dx \\
&= -\frac{2}{\pi}\big((-1)\cdot(-1)^n - 1\big) - 2n\int_0^1 \underbrace{\cos(\pi x)}_{\left(\frac{\sin(\pi x)}{\pi}\right)'}\sin(n\pi x)\,dx \\
&= \frac{2}{\pi}\big(1+(-1)^n\big) - \cancel{\frac{2n}{\pi}\sin(\pi x)\sin(n\pi x)\Big|_0^1} + \frac{2n}{\pi}\int_0^1 \sin(\pi x)(n\pi)\cos(n\pi x)\,dx \\
&= \frac{2}{\pi}\big(1+(-1)^n\big) + n^2\underbrace{2\int_0^1 \sin(\pi x)\cos(n\pi x)\,dx}_{\equiv a_n} \\
&= \frac{2}{\pi}\big(1+(-1)^n\big) + n^2 a_n,
\end{aligned}
$$

so

$$a_n = \frac{2\big(1+(-1)^n\big)}{\pi(1-n^2)} \qquad \forall n \neq 1.$$

Also we have that

$$
\begin{aligned}
a_1 &= 2\int_0^1 \sin(\pi x)\cdot\cos(\pi x)\,dx = \int_0^1 \sin(2\pi x)\,dx = -\frac{1}{2\pi}\cos(2\pi x)\Big|_0^1 = 0, \\
a_0 &= 2\int_0^1 \sin(\pi x)\,dx = -\frac{2}{\pi}\cos(\pi x)\Big|_0^1 = \frac{4}{\pi}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u(x,t) &= \frac{a_0}{2} + \sum_{n=2}^{\infty} a_n e^{-k\omega_n^2 t}\cos(\omega_n x) \\
&= \frac{2}{\pi} - \underbrace{\sum_{n=1}^{\infty} \frac{2\big(1+(-1)^n\big)}{\pi(n^2-1)} e^{-n^2\pi^2 t}\cos(n\pi x)}_{\xrightarrow{t\nearrow\infty} 0} \xrightarrow{t\nearrow\infty} \frac{2}{\pi}.
\end{aligned}
$$

- Exercise 16, page 644

  Find the temperature $u(x,t)$ in a rod modeled by the initial/boundary value problem

$$
\begin{aligned}
&u_t(x,t) = k u_{xx}(x,t), && \text{for } x \in (0,L) \text{ and } t > 0, \\
&u_x(0,t) = u_x(L,t) = 0 && \text{for } t > 0, && \text{(Neumann B.C.)} \\
&u(x,0) = f(x), && \text{for } x \in [0,L], && \text{(I.C.)}
\end{aligned}
$$

  where

$$k = 1, \quad L = 1, \quad \text{and} \quad f(x) = \cos(\pi x).$$

  Solution: Using separation of variables to find $u(x,t) = X(x)T(t)$, denoting by $\omega_n = \frac{n\pi}{L} \equiv n\pi$ the frequency, we have that $X(x)$ and $T(t)$ satisfy

$$XT' = kX''T, \qquad \frac{T'}{kT} = \frac{X''}{X} := -\omega_n^2,$$

$$
\begin{aligned}
T(t) &= e^{-k\omega_n^2 t}C \equiv e^{-n^2\pi^2 t}C,\\
X(x) &= a\cos(\omega_n x) + b\sin(\omega_n x),\\
X'(x) &= -a\,\omega_n \sin(\omega_n x) + b\,\omega_n\cos(\omega_n x).
\end{aligned}
$$

Using the homogeneous Neumann boundary conditions we obtain that

$$
\begin{aligned}
&X'(0) = X'(L) = 0,\\
&0 = X'(0) = b\,\omega_n,\\
&0 = X'(L) = -a\omega_n\sin(\omega_n L) + \cancel{b\omega_n\cos(\omega_n x)}\\
&\omega_n = \frac{n\pi}{L}
\end{aligned}
$$

hence

$$
\begin{aligned}
&X(x) = a_n\cos(\omega_n x),\\
&u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\omega_n^2 t}\cos(\omega_n x) \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\cos(n\pi x).
\end{aligned}
$$

Recall that the Fourier cosine coefficients a_n are obtained from the initial conditions

$$
u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(\omega_n x) \equiv f(x) = \cos(\pi x) \qquad \text{(I.C.)}
$$

therefore

$$
\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x) = \cos(\pi x)
$$

so

$$
\begin{aligned}
a_n &= 0, \quad \forall n \neq 1\\
a_1 &= 1.
\end{aligned}
$$

Finally

$$
\begin{aligned}
u(x,t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-k\omega_n^2 t}\cos(\omega_n x)\\
&\equiv e^{-\pi^2 t}\cos(\pi x).
\end{aligned}
$$