2.1 DE and Solutions

ODE (ordinary DE) is a DE with one independent variable.

Example: $y' = ty^2 - \sin t$, $y = y(t)$

t is an independent and y is dependent variable.

Order of a DE is the order of highest derivative that occurs in the equation.

Example: $y' - ty'' = e^t$ (order is two, the second order equation).

Example: $y''y^{(5)}t^7 = 1$ the fifth order equation.

General form of an ODE of order n

$$\Phi(t, y, y', \ldots, y^{(n)}) = 0$$

Normal form of and ODE of order n:

$$y^{(n)} = \phi(t, y, y', \ldots, y^{(n-1)})$$

Example: Write the DE $(t + y)y'' - y' = 0$ into a normal form.

Solution: $(t + y)y'' = y'$, $y'' = \dfrac{y'}{t + y}$

(Note that a possible solution $y = -t$ is lost in the normal form)

Example: $2x(x + 1) = 3x$, $2(x + 1) = 3$, $x = \dfrac{1}{2}$ and the solution $x = 0$ has been lost.

Example: $(y - t)y' = 0$

Normal form: $y' = 0$

The solution $y = t$ has been lost.

Solutions of ODEs:

Definition A solution of the 1st order ODE

$$\Phi(t, y, y') = 0$$

is a differentiable function $y(t)$ such that $\Phi(t, y(t), y'(t)) = 0$ for all t in the domain of $y(t)$.

Example: $y = \frac{1}{1-cx}$ is a solution of the eq. $xy' + y = y^2$ [where $y = y(x),\ y' = \frac{dy}{dx}$]

Check: $y = (1-cx)^{-1},\ y' = -(1-cx)^{-2}(-c),\ y' = c(1-cx)^{-2}$

$$xy' + y = \frac{cx}{(1-cx)^2} + \frac{1}{1-cx} = \frac{cx}{(1-cx)^2} + \frac{1-cx}{(1-cx)^2} = \frac{cx+1-cx}{(1-cx)^2} = \frac{1}{(1-cx)^2}$$

$$= \left(\frac{1}{1-cx}\right)^2 = y^2$$

Example: $y = c(x+1)e^{-x}$ is a solution for the equation

$$xy + (x+1)y' = 0 \qquad (1)$$

Proof: $y' = ce^{-x} - c(x+1)e^{-x} = ce^{-x} - y$

$xy + (x+1)y' = xy + (x+1)ce^{-x} - (x+1)y = c(x+1)e^{-x} - y = y - y = 0.$

The particular solution $y = (x+1)e^{-x}$ (when $c = 1$) is also a solution.

Another particular solution is $y = 0$ (when $c = 0$)

Both y and y' are defined on $(-\infty, \infty)$ $\Rightarrow$ y is the solution of the equation (1) on $(-\infty, \infty)$.

The equation (1) has infinitely many solutions that all depend on the values of c.

$y(x) = c(x+1)e^{-x}$ is called the general solution.

Definition a graph of a general solution in the xy-plane is called the family of solution curves. A graph of a particular solution (for chosen and fixed value of c) is called a solution curve.

Definition Initial Value Problem (IVP) is a combination of a DE together with initial conditions (ICs). The number of ICs corresponds to the order of a DE.

Solution of an IVP is called a particular solution.

Example: Consider an ODE of order 1:

$$y' = y^2$$

$y = \dfrac{1}{c-t}$ is its general solution.

Let's find a particular solution that in the geometric form passes through the point $(0, 1)$ in the ty-plane, i.e $y(0) = 1$.

$y(0) = 1$ is the IC of the problem. It gives $c = 1$. Then $y = \dfrac{1}{1-t}$ is a particular solution.

Note that the first order equation needs just one IC and the nth order equation needs n ICs to identify all n constants of integration.

Definition A solution of the IVP

$y' = f(t, y), \quad y(t_0) = y_0$ is a differentiable function $y(t)$ such that

1. $y'(t) = f(t, y(t))$ for all t on the interval containing t and where $y(t)$ is defined.

2. $y(t_0) = y_0$.

Definition An interval of existence of a solution to a DE (or an IVP) is defined to be the larges interval over which the solution can be defined and remain a solution.

Example: $y' = y^2, \; y(0) = 1 \quad$ (IVP)

Solution: $y = \dfrac{1}{1-t}$

The left branch of the graph is the solution curve since $t_0 = 0 \in (-\infty, 1)$, where $(-\infty, 1)$ is the interval of existence.

The geometric meaning of a DE and its solution

Consider DE $y' = f(t, y)$ where $f(t, y)$ is defied for (t, y) in the rectangle
b = { (t,y) — a $\leq t \leq b,\ c \leq y \leq d\}$

Let $y(t)$ be a solution. For t_0 in the interval of existence $y(t_0) = y_0 \Rightarrow$ the point (t_0, y_0) is on the solution curve. We have $y'(t_0) = f(t_0, y_0)$ => $f(t_0, y_0)$ is the slope of the tangent line to the solution curve at the point (t_0, y_0).

Consider a set of all small line segments of tangent lines in R.

This set forms a direction field.

Example: $y' = -ty,\ \ R = \{(t, y) | -1 \leq t \leq 1,\ -1 \leq y \leq 1\}$.
Consider points (t, y) made of $t, y \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$

The corresponding slopes are shown in the table below.

yt	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
-1	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
0	0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{2}$