2.2 Separable Equations

$y' = g(t) \cdot f(y) \;\; \Leftrightarrow \;\; \frac{dy}{dt} = g(t) \cdot f(y)$

$h(y)\,dy = g(t)\,dt$, where $h(y) = \frac{1}{f(y)}$

The equation $f(y) = 0$ gives possible lost solutions.

$\int h(y)\,dy = \int g(t)\,dt$

Integrate the last equation and find $y(t)$.

Methodology:

- separate
- integrate both sides
- solve for $y(t)$

Examples:

$$\frac{dy}{dt} = y \cdot t$$

$$\frac{dy}{dt} = \frac{t+1}{yt+t} \qquad \left(= \frac{t+1}{t}\frac{1}{y+1}\right)$$

Counterexample:

$$\frac{dy}{dt} = y + t$$

Examples:

(a) Find the general solution to the following differential equation:

$$y' = y^2 \tag{0.1}$$

The equation (0.1) is an autonomous equation (hence separable), and is in a normal form.
We have:

$$\frac{dy}{dt} = y^2$$
$$\Rightarrow \int y^{-2}\,dy = \int dt$$
$$\Rightarrow -\frac{1}{y} = t - C \Leftrightarrow \frac{1}{y} = C - t \Rightarrow y = \frac{1}{C-t}$$

When divided by y^2 the equation could lose the solution $y = 0$. To check if it is a solution or not we plug it into the equation (0.1).

$y = 0$ is solution because (0.1) becomes a true equality $0 = 0$.

(b) Find the general solution to the following differential equation:

$$y' = 2ty^2 \tag{0.2}$$

Similarly

$$\frac{dy}{dt} = 2t \cdot y^2 \Leftrightarrow \int y^{-2}\,dy = \int 2t\,dt \Leftrightarrow -\frac{1}{y} = t^2 + c \Leftrightarrow y = -\frac{1}{t^2+c}$$

A possible lost solution is $y = 0$ (division by y^2 had occurred).
To check we plug it into the equation (0.2) which holds if $y = 0$.

Hence, $y = 0$ is also a solution.

Explicit vs. Implicit Solutions

(c) Find the general solution to the following differential equation:

$$t^2y^2y' + 1 = y \tag{0.3}$$

Normal form:

$$y' = \frac{y-1}{t^2y^2} \Leftrightarrow \frac{dy}{dt} = \frac{1}{t^2}\cdot\frac{y-1}{y^2} \Leftrightarrow \int \frac{y^2}{y-1}\,dy = \int \frac{dt}{t^2}$$
$$\frac{y^2-1+1}{y-1} = \frac{(y-1)(y+1)+1}{y-1} = y+1+\frac{1}{y-1}$$
$$\frac{y^2}{2} + y + \ln|y-1| = -\frac{1}{t} + c$$

is an implicit solution.

When divided by t^2y^2 the equation could lose the solution $y = 0$. Also $y-1 = 0 \;\Leftrightarrow\; y = 1$ could be a solution.
To check if they are solutions or not we plug them into the equation (0.3).

$y = 0$ is not a solution.
$y = 1$ is a solution.

(d) Find the general solution to the following differential equation:

$$y' = \frac{1}{2t^2y}$$

(Remark that this is equation (0.2) where y and t are switched.)

$$\frac{dy}{dt} = \frac{1}{2t^2y} \Rightarrow \int y\,dy = \frac{1}{2}\int t^{-2}\,dt \Leftrightarrow \frac{y^2}{2} = \frac{1}{2}\left(-\frac{1}{t}\right) + c_1$$
$$\Leftrightarrow y^2 = -\frac{1}{t} + c$$

is an implicit solution.

Interval of existence

Initial conditions

(e) **Exercise #13, page 35.** Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y' = \frac{y}{x}, \qquad y(1) = -2.$$

<u>`Solution`</u>:

$$\frac{dy}{y} = \frac{dx}{x} \quad \Leftrightarrow \quad \ln|y| = \ln|x| + c \quad \Leftrightarrow \quad |y| = e^c|x| = A_1|x|,$$

where $A_1 = e^c > 0$. Let $A = \pm A_1$, i.e. the constant A can be positive or negative.

Then the general solution is $y = Ax$.

A possibly lost solution $y = 0$ (not of IVP) is a solution which is easy to check.
Assuming that A can be equal 0 we write the general solution as

$$y = Ax \quad \text{with } A \in (-\infty, \infty)$$

IC: $y(1) = A = -2$

Therefore, the particular solution of the IVP is $y = -2x$.

The function $y = -2x$ is defined for all x but the DE is not defined at $x = 0 \Rightarrow$ the interval of existence of the solution is $(0, \infty)$ since from the IC we have $t = 1$ and $1 \in (0, \infty)$.

(Note: the lost solution $y = 0$ is a solution of the equation not of the IVP).

(f) Find the exact solution of the initial value problem

$$y' = \frac{\cos x}{y},$$

with the following two initial conditions:

$$y(0) = 1 \qquad \text{(IC(1))}$$
$$y(0) = -2 \qquad \text{(IC(2))}$$

<u>`Solution`</u>: Separate the variables:

$$\int y\,dy = \int \cos x\,dx, \qquad \frac{y^2}{2} = \sin x + \frac{c}{2}$$
$$y^2 = 2\sin x + c, \qquad y = \pm\sqrt{2\sin x + c}$$

is the general solution.

(IC(1)): $y(0) = 1 > 0, \quad y(0) = \sqrt{c} = 1, \quad c = 1,$
hence, the particular solution is $y = \sqrt{2\sin x + 1}$

Interval of existence: $2\sin x+1 \geqslant 0 \iff \sin x \geqslant -\frac{1}{2} \iff -\frac{\pi}{6}+2\pi k \leqslant x \leqslant \frac{7\pi}{6}+2\pi k$,

where k is an integer number.

The interval of existence has to contain the point $x=0$. Therefore it is the one when $k=0$, i.e. $\left(-\frac{\pi}{6}, \frac{7\pi}{6}\right)$.

(IC(2)): $y(0)=-2<0, \quad y(0)=-\sqrt{c}=-2, \quad c=4$

And the particular solution is $y=-\sqrt{2\sin x+4}$. Interval of existence: $(-\infty,\infty)$.
(Check & plot your solution with dfield.jar !!!)

Half-life problem:

The DE

$$N'=-\lambda N, \quad \lambda>0$$

is used to model the number of remaining nuclei N in a radioactive substance. This equation is separable.

Its solution is $N(t)=Ae^{-\lambda t}$.

Indeed, $\dfrac{dN}{dt}=-\lambda N \Rightarrow \dfrac{dN}{N}=-\lambda\, dt \Rightarrow \ln N=-\lambda t+C \Rightarrow N=e^{-\lambda t+C}$
$\Rightarrow N(t)=Ae^{-\lambda t}$, where $A=e^C$.

If an initial number of nuclei $N(0)=N_0$ then $N(t)=N_0e^{-\lambda t}$

Let $T_{1/2}$ be the half-life (i.e., $N(t+T_{1/2})=\frac{1}{2}N(t)$ for any t).
Then

$$N(t+T_{1/2})=Ae^{-\lambda(t+T_{1/2})}=Ae^{-\lambda t}e^{-\lambda T_{1/2}}=N(t)\cdot e^{-\lambda T_{1/2}}=N(t)\cdot\frac{1}{2}$$
$$\Rightarrow e^{-\lambda T_{1/2}}=\frac{1}{2}\Rightarrow T_{1/2}=\frac{\ln 2}{\lambda} \quad \left(\text{because } \ln\left(\frac{1}{2}\right)=-\ln 2\right).$$

(*See Example (l) below*.)

(g) **Exercise #4 page 35.** Find the general solution to:

$$y'=(1+y^2)e^x$$

Solution:

$$\frac{dy}{1+y^2}=e^x dx \qquad \text{(separate)}$$
$$\arctan(y)=e^x+C \qquad \text{(integrate)}$$
$$y(x)=\tan(e^x+C) \qquad \text{(solve)}$$

(h) **Exercise #6 page 35.** Find the general solution to:

$$y'=ye^x-2e^x+y-2$$

Solution: Indeed, this is a separable equation, since the right-hand side writes as

$$ye^x - 2e^x + y - 2 = (y-2)e^x + (y-2) = (y-2)(e^x+1),$$

and therefore the equation is equivalent to

$$\frac{dy}{dx} = (y-2)(e^x+1).$$

We separate the variables (divide by $y-2$)

$$\frac{dy}{y-2} = (e^x+1)dx,$$

integrate both sides

$$\int \frac{dy}{y-2} = \int (e^x+1)dx,$$

to obtain

$$\ln(y-2) = e^x + x + C,$$

and taking the exponent of both sides

$$y(x) = 2 + e^{e^x+x+C},$$

or $y(x) = 2 + Ke^{e^x+x}$, where K is a positive constant.
(We can easily check our calculation so far, i.e., that $y(x)$ satisfies the equation $y' = (y-2)(e^x+1)$.)
We note that also the constant function $y(x) = 2$ is a solution.
Check the solution with dfield.jar.
What happens if you try negative initial conditions, e.g., $y(1) = -2$**,** $y(5) = -2$ **or** $y(9.9) = -2$**???**

(i) **Exercise #10 page 35.** Find the general solution to:

$$xy' - y = 2x^2y.$$

If possible, find an explicit solution.
Solution: We rewrite the equation as $xy' = 2x^2y + y \equiv (2x^2+1)y$ and then divide both sides by xy:

$$\begin{aligned}
&y' = \frac{y(2x^2+1)}{x} \quad \text{or} \quad \frac{dy}{dx} = \frac{y(2x^2+1)}{x} && \text{(normal form)}\\
&\frac{dy}{y} = \frac{2x^2+1}{x}dx \quad \Longrightarrow \quad \int \frac{dy}{y} = \int \left(2x + \frac{1}{x}\right)dx && \text{(separate variables)}\\
&\ln|y| = x^2 + \ln|x| + C && \text{(integrate)}\\
&y(x) = Kxe^{x^2}, && \text{(solve)}
\end{aligned}$$

with $K > 0$.
Also $y(x) = 0$ is a solution.

(j) **Exercise #11 page 35.** Find the general solution to:

$$y^3y' = x + 2y'.$$

If possible, find an explicit solution.
Solution:

$$y' = \frac{x}{y^3-2} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{x}{y^3-2} \qquad \text{(normal form)}$$

$$(y^3-2)dy = xdx \qquad \Longrightarrow \qquad \int (y^3-2)dy = \int xdx \qquad \text{(separate variables)}$$

$$\frac{1}{4}y^4 - 2y = \frac{1}{2}x^2 + C. \qquad \text{(integrate)}$$

(k) **Exercise #14 page 35.** Find the exact solution of the initial value problem:

$$y' = -2t(1+y^2)/y, \qquad y(0) = 1,$$

and find the interval of existence.
Solution:

$$\frac{ydy}{1+y^2} = -2t\,dt \qquad \text{(separate)}$$

$$\frac{1}{2}\ln(1+y^2) = -t^2 + \frac{C}{2} \qquad \text{(integrate)}$$

$$y^2(t) = e^C e^{-2t^2} - 1, \qquad \text{(solve)}$$

or $y^2(t) = K\,e^{-2t^2} - 1$, with $K > 0$. Using the initial condition we have

$$1 = K - 1, \qquad K = 2$$

hence the solution satisfies $y^2(t) = 2\,e^{-2t^2} - 1$ or $y(t) = \pm\sqrt{2\,e^{-2t^2} - 1}$.
The interval of existence is defined through the relation $t^2 \leq \frac{\ln 2}{2}$, i.e., $I = \left[-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right]$.
(Plot your solution with dfield.jar)

(l) **Exercise #25 page 35.** Suppose that 100mg of tritium ^3H decays to 80mg in 4 hours.

Find the half-life of tritium.

Solution: Let $N(t)$ denote the amount of ^3H at time t.

We have $N(0) = 100, \quad N(4) = 80$ (t is given in hours), therefore

$$N(t) = N(0)e^{-\lambda t} = 100e^{-\lambda t}.$$

Let T be the half-life. Then

$$N(T) = \frac{1}{2}N(0) = 50$$

and also we have

$$N(4) = 80 \Rightarrow 100e^{-4\lambda} = 80, \quad (e^{-\lambda})^4 = .8, \quad e^{-\lambda} = (.8)^{1/4}.$$

Then $N(t) = 100e^{-\lambda t} = 100(e^{-\lambda})^t = 100((.8)^{1/4})^t = 100(.8)^{t/4}$

$N(T) = 100(.8)^{T/4} = 50 \ \Rightarrow \ (.8)^{T/4} = \frac{1}{2}, \quad (.8)^T = \frac{1}{16}, \quad T\ln(.8) = -\ln 16$

Hence $T = -\dfrac{\ln 16}{\ln(.8)}$