Modelling errors: "All models are wrong, but some are useful", an aphorism attributed to George Box.

2.3 Models of Motion

Linear motion, Newton's 2nd Law: $F = ma$

$v = x'$, $a = x''$, x is displacement , $x = x(t)$

Gravitational force: $F = -mg$, g is Earth's gravity.

$$mx'' = -mg \quad \Rightarrow \quad x'' = -g, \quad \Rightarrow \quad x' = -gt + v_0 \quad \Rightarrow \quad x(t) = -\frac{g}{2}t^2 + v_0 t + x_0$$

Example 1: A body falls with an initial velocity of 1000 ft/s and is subject to the acceleration of gravity $g \approx 32$ ft/sec^2. What distance does it fall in 3 seconds?

Solution: $v = \dfrac{dx}{dt} = gt + v_0 = 32t + v_0$

$dx = (32t + v_0)\, dt, \quad x = 16t^2 + v_0 t + x_0$

$x(0) = 0 \quad \Rightarrow \quad x_0 = 0, \quad v(0) = v_0 = 1000$ ft/s

Hence $x(t) = 16t^2 + 1000t$ and $x(3) = 16 \cdot 9 + 1000 \cdot 3 = 3,144$ ft.

Air resistance: $R(v) = -r(v) \cdot v$

Case 1 $r(v) = r > 0$ is a constant.

Total force is the sum of the forces of gravity and air resistance:

$F = -mg + R(v) = -mg - rv$

Newton's Second Law gives $mv' = -mg - rv \quad \Leftrightarrow \quad v' = -g - \dfrac{r}{m}v.$

The last equation is separable.

Solution:

$$\frac{dv}{g + \frac{rv}{m}} = -dt, \quad \frac{dv}{g + \frac{rv}{m}} = \frac{m}{r} \cdot \frac{dv}{v + \frac{mg}{r}},$$

Therefore, $\dfrac{m}{r}\displaystyle\int \dfrac{dv}{v+\dfrac{mg}{r}} = -\int dt, \quad \dfrac{m}{r}\ln\left|v+\dfrac{mg}{r}\right| + c_1 = -t$

$\ln\left|v+\dfrac{mg}{r}\right| = -\dfrac{r}{m}t + c_2, \quad v(t) = ce^{-\frac{r}{m}t} - \frac{mg}{r}$

$\displaystyle\lim_{t\to\infty} v(t) = -\dfrac{mg}{r} = v_{term}$ terminal velocity.

Want to get the displacement $x(t)$:

$x' = v \quad \Rightarrow \quad x = -\frac{mc}{r}e^{-\frac{r}{m}t} - \frac{mg}{r}t + A$

Example 2: An 8 lb. weight falls from rest towards the earth. Assuming that the weight is acted upon by air resistance $= -2v$ find the velocity and the distance fallen after 10 seconds. (v is given in ft/sec.)

Solution: $mv' = mg - 2v, \quad v(0) = 0$

$m = \dfrac{8\,\text{lb}}{32\,\text{ft/s}^2} = \frac{1}{4}, \quad g = 32\ \text{ft/s}^2,\ mg = 8.$

$\frac{1}{4}\dfrac{dv}{dt} = 8 - 2v \quad \frac{1}{8}dv = -(v-4)dt \quad \dfrac{dv}{v-4} = -8dt \quad \ln|v-4| = -8t + c \quad v = Ae^{-8t} + 4$

$v(0) = 0 \quad \Rightarrow \quad A = -4, \quad v(t) = 4(1-e^{-8t}) \quad v(10) \approx 4$ ft/sec

$\dfrac{dx}{dt} = 4(1-e^{-8t}) \quad \Rightarrow \quad x = 4t + \frac{1}{2}e^{-8t} + x_0 \quad x(0) = 0 \quad \Rightarrow \quad x_0 = -\frac{1}{2}$

$x(t) = 4t + \frac{1}{2}e^{-8t} - \frac{1}{2}, \quad x(10) \approx 39.5$ ft

Example 3: A ball is released rest towards the earth. How long will it take the ball to reach one-half of its terminal velocity assuming that the air resistance is proportional to velocity.

Solution: $v(t) = ce^{-\frac{r}{m}t} - \frac{mg}{r}, \quad v(0) = 0 \quad \Rightarrow \quad A = \frac{mg}{r}$

$v(t) = \frac{mg}{r}\left(e^{-\frac{r}{m}t} - 1\right).$

Let the time T be such that $v(T) = \frac{1}{2}v_{term}$. Then

$\frac{mg}{r}\left(e^{-\frac{r}{m}T} - 1\right) = -\frac{1}{2}\frac{mg}{r}, \quad e^{-\frac{r}{m}T} - 1 = -\frac{1}{2}, \quad e^{-\frac{r}{m}T} = \frac{1}{2}, \quad -\dfrac{r}{m}T = \ln\left(\frac{1}{2}\right) = -\ln 2$

$T = \dfrac{m}{r}\ln 2.$

Air resistance. **Case 2** (it is not a part of Math 0290 course).

Example 4: A body of mass m is projected upwards in the air with velocity v_0. Air resistance is proportional to the square of the velocity. Find the motion of the body as it rises.

Solution: $R(v) = kv^2$

Newton's second law: $m\dfrac{dv}{dt} = -mg - kv^2$

$\dfrac{dv}{dt} = -g - \dfrac{k}{m}v^2$ (separable)

$\dfrac{dv}{dt} = -\dfrac{k}{m}\left(\dfrac{gm}{k} + v^2\right) = -\dfrac{k}{m}\left(a^2 + v^2\right)$, where $a = \sqrt{\dfrac{gm}{k}}$

$\displaystyle\int \frac{dv}{a^2+v^2} = -\frac{k}{m}\int dt$, $\dfrac{1}{a}\tan^{-1}\dfrac{v}{a} = -\dfrac{k}{m}t + c$, $v(0) = v_0 \Rightarrow \dfrac{1}{a}\tan^{-1}\dfrac{v_0}{a} = c$

$v = a\tan\left(-\dfrac{ak}{m}t + ac\right) = a\tan\left(-\sqrt{\dfrac{gk}{m}}t + \tan^{-1}\left(\dfrac{v_0}{a}\right)\right)$

$v(t) = a\tan\left(-\lambda t + \mu\right)$, where $\lambda = \sqrt{\dfrac{gk}{m}}$, $\beta = \tan^{-1}\left(\dfrac{v_0}{a}\right)$

$\dfrac{dx}{dt} = v \Rightarrow x = \int a\tan\left(-\lambda t + \mu\right)\, dt = \dfrac{a}{\lambda}\ln\left|\dfrac{\cos(-\lambda t + \mu)}{\cos\mu}\right| + c$

The body reaches its maximum height when $v = 0 \Leftrightarrow -\lambda t + \mu = 0 \Leftrightarrow t = \dfrac{\mu}{\lambda}$,

Hence $x\left(\frac{\mu}{\lambda}\right) = \dfrac{a}{\lambda}\ln|\sec\mu|$

For the equation $\dfrac{dv}{dt} = -g - \dfrac{k}{m}v^2$ consider the change of variables:

$v = \alpha w$, $t = \beta s \Rightarrow s = \dfrac{t}{\beta}$

Then $\dfrac{dv}{dt} = \dfrac{dv}{dw}\cdot\dfrac{dw}{ds}\cdot\dfrac{ds}{dt} = \alpha\cdot\dfrac{dw}{ds}\cdot\dfrac{1}{\beta} = \dfrac{\alpha}{\beta}\dfrac{dw}{ds}$

$\dfrac{\alpha}{\beta}\dfrac{dw}{ds} = -g - \dfrac{k}{m}\alpha^2 w^2 \Leftrightarrow \dfrac{dw}{ds} = -\dfrac{\beta}{\alpha}g - \dfrac{k}{m}\alpha\beta w^2$

Choose α and β such that $\dfrac{\beta}{\alpha}g = 1$ and $\dfrac{k}{m}\alpha\beta = 1 \Rightarrow \beta = \sqrt{\dfrac{m}{kg}}$, $\alpha = \sqrt{\dfrac{mg}{k}}$

Then $\dfrac{dw}{ds} = -1 - w^2$, $\dfrac{dw}{1+w^2} = -ds$, $\tan^{-1} w = -s + c$, $w = \tan(c - s)$,

$\dfrac{v}{\alpha} = \tan\left(c - \dfrac{t}{\beta}\right)$, $v = \alpha\tan\left(c - \dfrac{t}{\beta}\right)$

Example 5: A body of mass is 4 lb is dropped with no initial velocity and encounters an air resistance that is proportional to v^2. Find the velocity of the body after 2 seconds. What will be its position at this time?

- **Exercise #10, page 45.**
An object having mass 70 kg falls from rest under the influence of gravity. The terminal velocity of the object is −20m/s. Assume the air resistance of the object is proportional to the velocity.
  (a) Find the velocity and distance traveled at the end of 2 seconds.
  (b) How long does it take the object to reach 80% of its terminal velocity?

Solution: From Newton's second law we have

$$mv' = -mg - rv, \qquad \text{(Newton's 2}^{\text{nd}}\text{ law)}$$
$$\frac{v'}{g + \frac{r}{m}v} = -1, \quad \text{or} \quad \frac{\frac{r}{m}dv}{g + \frac{r}{m}v} = -\frac{r}{m}dt, \qquad \text{(separate variables)}$$
$$\ln\left|g + \frac{r}{m}v\right| = -\frac{r}{m}t + C, \qquad \text{(integrate)}$$
$$g + \frac{r}{m}v(t) = e^{-\frac{r}{m}t + C} \implies v(t) = Ke^{-\frac{r}{m}t} - g\frac{m}{r}, \qquad \text{(solve)}$$

hence, when $t \to \infty$ we obtain $v_{\text{terminal}} = -g\frac{m}{r}$.
Since the terminal velocity is -20m/s, then the constant or proportionality is

$$r = -\frac{gm}{v_{\text{terminal}}} \equiv -\frac{9.8 \cdot 70}{-20} = 34.3.$$

Also, since the object starts from rest, we have the initial condition

$$v(0) = 0$$

hence

$$K = g\frac{m}{r} \equiv 9.8\frac{70}{34.3} = 20.$$

and therefore

$$v(t) = g\frac{m}{r}\left(e^{-\frac{r}{m}t} - 1\right) \equiv 20\left(e^{-\frac{34.3}{70}t} - 1\right) = 20\left(e^{-0.49t} - 1\right). \qquad (*)$$

(a) The velocity after 2 seconds:

$$v(2) = 20\left(e^{-0.49*2} - 1\right) \approx -12.4938.$$

For the distance, evaluate the integral

$$x(2) = \int_0^2 20\left(e^{-0.49t} - 1\right)dt$$

(b) We have to find T such that

$$v(T) = 0.8 * v_{\text{terminal}} \equiv -16,$$

i.e.,

$$20\left(e^{-0.49T} - 1\right) = -16$$

which yields

$$T = -\frac{1}{0.49}\ln 0.2 \approx 3.2846.$$