2.4 Linear Equations

Product rule: $(fg)' = fg' + f'g$

$x = x(t)$, $(tx)' = tx' + x$. On the other hand $tx' + x = (tx)'$

A few more examples: $t^2x' + 2tx = (t^2x)'$, $e^tx' + e^tx = (e^tx)'$, $\sin tx' + \cos tx = (\sin tx)'$

Example: Solve the equation: $tx' + 2x = 4$

Solution: $t^2x' + 2tx = 4t$, $(t^2x)' = 4t$, $\int (t^2x)'\,dt = \int 4t\,dt$, $t^2x = 2t^2 + C$

$x(t) = 2 + Ct^{-2}$

First Order Linear Equation:

$b(t)x' + c(t)x + g(t) = 0$ is a general form of a first order linear differential equation.
It is linear with respect to both x' and x.

[Quasilinear equation: $b(t,x)\,x' + c(t,x) = 0$. It is linear with respect to x'.]

$b(t)\,x' = -c(t)\,x - g(t)$

After division by $b(t)$ we convert it to $x' = a(t)\,x + f(t)$

which is an inhomogeneous equation due to the presence of $f(t)$.

Homogeneous equation:

The equation

$$x' = a(t)\,x$$

is homogeneous. A homogeneous first order linear equation is a separable equation.

Solution:

$\dfrac{dx}{dt} = a(t)x \quad \Rightarrow \quad \int \dfrac{dx}{x} = \int a(t)dt$

$\ln|x| = \int a(t)dt + c \quad \Rightarrow \quad x(t) = Ae^{\int a(t)\,dt}$ (after exponentiating)

Example: $x' = tx$

$\int \dfrac{dx}{x} = \int t\,dt$, $\ln|x| = \dfrac{t^2}{2} + c$, $x(t) = Ae^{t^2/2}$

Example: $tx' = x$

$\int \frac{dx}{x} = \int \frac{dt}{t}, \quad x(t) = At$

Example: $tx^3x' = 1$

The equation is nonlinear but separable

$\int x^3 dx = \int \frac{dt}{t}, \quad \frac{x^4}{4} = \ln|t| + c_1, \quad x = \pm(4\ln|t| + c)^{\frac{1}{4}}$

Alternative solution:

We change the equation and make it linear: $tx^3dx = dt, \quad \frac{dt}{dx} = x^3t$

The last equation linear with respect to the dependent variable t as a function of the independent variable x.

Inhomogeneous equation:

Now let us consider an inhomogeneous first order linear equation

$$x' = a(t)x + f(t) \qquad (1)$$

It can be solved by two different methods.

Method 1. Integrating Factor

To solve the equation we apply the following steps:

1. Rewrite the equation in the form $x' - a(t)\,x = f(t)$.

2. Multiply it by the integrating factor

$$u(t) = e^{-\int a(t)dt}$$

which is solution to the equation $u' = -a(t)\,u$ to get

$u(x' - ax) = ux' - aux = ux' + u'x = (ux)' = uf.$

3. Integrate $(ux)' = uf$ to obtain $ux = \int u(t)f(t)dt + c.$

4. Solve the last equation for x:

$$x(t) = \frac{1}{u(t)} \int u(t)f(t)\,dt + \frac{c}{u(t)}$$

Method 2. Variation of Parameters

1. Find a particular solution $x_h(t)$ to the corresponding homogeneous equation $x' = ax$,

$x_h(t) = e^{\int a(t)dt}$.

2. Substitute $x = v\,x_h$ into (1) to find $v(t)$ or remember that $v' = \dfrac{f}{x_h}$.
Indeed, after substitution $x = v\,x_h$ into (1) we get $v'x_h + vx_h' = avx_h + f$

Recalling that $x' = ax$ we obtain $v'x_h = f$ and finally $v' = \dfrac{f}{x_h}$.
Integrate the last equation and get $v(t)$.

3. Then the general solution is $x(t) = v(t)\,x_h(t)$

Example: $x' + x = t$

$a = -1$

The integrating factor is $u(t) = e^{\int 1dt} = e^t$

After multiplying both sides by $u(t) = e^t$ we get $e^t x' + e^t x = e^t \cdot t$ or $(e^t x)' = te^t$.

Then $e^t\,x = \int te^t dt = te^t - e^t + c$ (integration by parts)

$x(t) = (te^t - e^t + c)\,e^{-t}$, $x(t) = t - 1 + ce^{-t}$ is the solution.

Check: $x' = 1 - ce^{-t}$, $x' + x = 1 - ce^{-t} + t - 1 + ce^{-t} = t$. Correct.

Method 2 (variation of parameters):

The corresponding homogeneous equation is $x' + x = 0$.

Its solution is $x_h = cs^{-t}$ that we write in the form $x_h = v(t)e^{-t}$.

Substitution into inhomogeneous equation gives $(ve^{-t})' + ve^{-t} = t$,

$v'e^{-t} - ve^{-t} + ve^{-t} = t$, $v'e^{-t} = t$, $v' = te^t$, $v(t) = te^t - e^t + c$

Therefore, $x(t) = v(t)e^{-t} = t - 1 + ce^{-t}$.

Example: $tx' = 2x + t^3$

Linear equation is $x' = \dfrac{2}{t}x + t^2, \quad (t \neq 0)$.

Method 1

$u = e^{-\int \frac{2}{t}dt} = e^{-2\ln(t)} = e^{-\ln t^2} = t^{-2}$

$t^{-2}x' = 2t^{-3}x + 1, \quad t^{-2}x' - 2t^{-3}x = 1, \quad (t^{-2}x)' = 1, \quad t^{-2}x = t + c, \quad x(t) = t^3 + ct^2$

Case $t = 0$. If $t = 0$ then $x(0) = 0$ and the equation holds, hence $x(t)$ defined on $(-\infty, \infty)$.

Method 2

The corresponding homogeneous equation of the linear equation is $x' = \dfrac{2}{t}x$

$\int \dfrac{dx}{x} = \int 2\dfrac{dt}{t}, \quad \ln|x| = \ln t^2 + c, \quad x = e^{\ln t^2}e^c = At^2, \quad x = v(t)\,t^2$

$x' = v' \cdot t^2 + 2vt = \dfrac{2}{t} \cdot vt^2 + t^2, \quad v't^2 = t^2, \quad v' = 1, \quad v = t + c, \quad x(t) = t^3 + ct^2$

Example: $x = (2t + x^3)x'$

The equation is linear with respect to t, not x

$x = (2t + x^3)\dfrac{dx}{dt}, \quad dt \cdot x = (2t + x^3)dx, \quad \dfrac{dt}{dx} \cdot x = 2t + x^3$

So we need to find a solution $t(x)$ to the equation $xt' = 2t + x^3$.

It is exactly the previous example with t and x switched.

Therefore, the solution is $t = x^3 + cx^2$. It gives us an implicit solution in terms of $x(t)$.

- Exercise #7, page 55. Find the general solution of the first-order, linear equation:

$$(1 + x)y' + y = \cos x.$$

<u>Solution</u>: First we write the equation in normal form

$$y' = -\frac{1}{1+x}y + \frac{\cos x}{1+x},$$

hence

$$a(x) = -\frac{1}{1+x}.$$

(I) "integrating factor" The integrating factor is

$$u(t) = e^{-\int a(x)dx} = e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1 + x,$$

hence, multiplying the normal form by $1 + x$, and moving the first term in the RHS to the LHS we obtain

$$\begin{aligned} y' &= -\frac{1}{1+x}y + \frac{\cos x}{1+x}, \\ (1+x)y' &= -y + \cos x, \\ \underbrace{(1+x)y' + y}_{\left((1+x)y\right)'} &= \cos x, \\ \left((1+x)y\right)' &= \cos x, \end{aligned}$$

which by integrating both sides gives

$$(1 + x)y = \sin x + C,$$

and therefore the general solution is

$$y = \frac{\sin x}{1+x} + \frac{C}{1+x}.$$

(II) "variation of parameters" The solution to the homogeneous equation

$$z' = -\frac{1}{1+x}z$$

is

$$y_h(x) = e^{\int a(x)dx} \equiv= e^{\int -\frac{1}{1+x}dx} = e^{\,-\ln(1+x)} = \frac{1}{1+x}.$$

Now we seek a solution of the form

$$y(x) = v(x)y_h(x) \equiv \frac{v(x)}{1+x}.$$

Substituting this solution in the equation (imposing it to be a solution to the equation) we obtain (using the normal form of the equation)

$$\begin{aligned} y' &= -\frac{1}{1+x}y + \frac{\cos x}{1+x}, \\ \frac{v'(x)(1+x) - v(x)}{(1+x)^2} &= -\frac{1}{1+x}\frac{v(x)}{1+x} + \frac{\cos x}{1+x}, \end{aligned}$$

and simplifying

$$\begin{aligned} &\frac{v'(x)}{1+x} = \frac{\cos x}{1+x}, \\ &v'(x) = \cos x, \qquad v(x) = \sin x + C, \\ &y(x) = \frac{\sin x + C}{1+x}. \end{aligned}$$

- Exercise #17, page 55. Find the solution of the initial value problem:

$$x' + x\cos t = \frac{1}{2}\sin 2t, \quad x(0) = 1.$$

<u>`Solution`</u>: The normal form of the equation is:

$$x' = -x\cos t + \frac{1}{2}\sin 2t, \quad x(0) = 1,$$

hence

$$a(t) = -\cos t.$$

(I) "integrating factor" The integrating factor is

$$u(t) = e^{\int -a(t)dt} \equiv e^{\int \cos t dt} = e^{\sin t}$$

hence

$$\begin{aligned}
x' &= -x\cos t + \frac{1}{2}\sin 2t && (\cdot e^{\sin t})\\
e^{\sin t}\, x' &= -x\cos t\, e^{\sin t} + \frac{1}{2}\sin(2t)\, e^{\sin t}\\
e^{\sin t}\, x' + x\cos(t)\, e^{\sin t} &= \frac{1}{2}\sin(2t)\, e^{\sin t}\\
\left(e^{\sin t}\, x\right)' &= \frac{1}{2}\sin(2t)\, e^{\sin t} && \text{(integrate)}\\
e^{\sin t}\, x &= \frac{1}{2}\int \sin(2t)\, e^{\sin t} dt \overset{denote}{=\!=} I.
\end{aligned}$$

Here (using $\sin 2t = 2\sin t\cos t$) we have by integration by parts

$$\begin{aligned}
I &= \int \sin(t)\big(\cos(t)\, e^{\sin t}\big)dt = \int \sin(t)\big(e^{\sin t}\big)' dt\\
&= \sin(t)e^{\sin t} - \int \cos(t)e^{\sin t}dt = \sin(t)e^{\sin t} - \int \big(e^{\sin t}\big)' dt\\
&= \big(\sin(t) - 1\big)e^{\sin t} + C.
\end{aligned}$$

Therefore

$$e^{\sin t}\, x = \big(\sin(t) - 1\big)e^{\sin t} + C$$

and the general solution is

$$x(t) = \sin(t) - 1 + Ce^{-\sin t}.$$

Using the initial condition $x(0) = 1$ we obtain

$$1 = -1 + C, \qquad C = 2,$$

and the solution to the initial value problem is

$$x(t) = \sin(t) - 1 + 2e^{-\sin t}.$$

(II) "variation of parameters" The solution to the homogeneous equation

$$z' = -z\cos(t)$$

is $x_h(t) = e^{\int a(t)dt} = e^{\int -\cos(t)dt} = e^{-\sin(t)}$.
Looking for a solution $x(t) = v(t)x_h(t) \equiv v(t)e^{-\sin(t)}$, substituting into the equation we have

$$\begin{aligned}
&v'e^{-\sin t} + \cancel{v(-\cos t)e^{-\sin t}} = \cancel{-v(t)e^{-\sin(t)} \cdot \cos(t)} + \sin(t)\cos(t),\\
&v' = e^{\sin t}\sin(t)\cos(t), \qquad v(t) = \int e^{\sin t}\sin(t)\cos(t)\,dt = I,\\
&v(t) = \big(\sin(t) - 1\big)e^{\sin t} + C
\end{aligned}$$

and finally

$$x(t) = \sin(t) - 1 + Ce^{-\sin t},$$

where as before $C = 2$.

- Exercise #1, page 55. Find the general solution of the first-order, linear equation:

$$y' + y = 2.$$

Solution:

$$\begin{aligned}
&y' = \underbrace{(-1)}_{a(t)} \cdot y + 2,\\
&u(t) = e^{\int -a(t)dt} \equiv e^{\int dt} = e^t, && \text{(integrating factor)}\\
&y'e^t + ye^t = 2e^t,\\
&\big(ye^t\big)' = 2e^t,\\
&ye^t = 2e^t + C,\\
&y(t) = 2 + Ce^{-t}.
\end{aligned}$$

- Exercise #3, page 55. Find the general solution of the first-order, linear equation:

$$y' + \frac{2}{x}y = \frac{\cos x}{x^2}.$$

Solution:

$$\begin{aligned}
&y' = \underbrace{-\frac{2}{x}}_{a(x)}\,y + \frac{\cos x}{x^2},\\
&u(x) = e^{\int -a(x)dx} \equiv e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2, && \text{(integrating factor)}\\
&x^2y' + x^2\frac{2}{x}y = \frac{\cos x}{x^2} \cdot x^2, \qquad x^2y' + 2xy = \cos x,\\
&\big(x^2y\big)' = \cos x,\\
&x^2y = \sin x + C, \qquad y(x) = \frac{\sin x + C}{x^2}.
\end{aligned}$$

- Exercise #4, page 55. Find the general solution of the first-order, linear equation:

$$y' + 2ty = 5t.$$

Solution:

$$\begin{aligned}
y' &= \underbrace{-2t}_{a(t)}\, y + 5t \\
u(t) &= e^{\int -a(t)dt} \equiv e^{\int 2tdt} = e^{t^2} && \text{(integrating factor)} \\
e^{t^2} y' + 2te^{t^2} y &= 5te^{t^2} \\
\left(e^{t^2} y\right)' &= 5te^{t^2} \\
e^{t^2} y &= 5 \int te^{t^2} dt \equiv \frac{5}{2} e^{t^2} + C, \\
y &= \frac{5}{2} + Ce^{-t^2}.
\end{aligned}$$

- Exercise #21, page 55. Find the solution to the initial value problem:

$$(1+t)x' + x = \cos t, \qquad x\left(-\frac{\pi}{2}\right) = 0.$$

Solution:

$$\begin{aligned}
x' &= \underbrace{-\frac{1}{1+t}}_{a(t)} x + \frac{\cos t}{1+t}, \\
u(t) &= e^{\int -a(t)dt} \equiv e^{\int \frac{1}{1+t} dt} = e^{\ln(1+t)} = 1+t, && \text{(integrating factor)} \\
(1+t)x' + x &= \cos t, \\
\big((1+t)x\big)' &= \cos t, && \text{(integrate)} \\
(1+t)x &= \sin t + C, \\
x(t) &= \frac{\sin t + C}{1+t}, && \text{(general solution)} \\
0 &= \frac{C-1}{1-\frac{\pi}{2}}, \qquad C = 1, && \text{(initial condition)} \\
x(t) &= \frac{\sin(t)+1}{1+t}.
\end{aligned}$$

- Exercise #16, page 55. Find the solution to the initial value problem:

$$(1+t^2)y' + 4ty = (1+t^2)^{-2}, \qquad y(1) = 0.$$

Solution:

$$
\begin{aligned}
&y' = \underbrace{-\frac{4t}{1+t^2}}_{a(t)}\,y + \frac{1}{(1+t^2)^3}, \\
&u(t) = e^{\int -a(t)dt} \equiv e^{\int \frac{4t}{1+t^2}dt} = e^{2\ln(1+t^2)} = (1+t^2)^2, && \text{(integrating factor)} \\
&(1+t^2)^2 y' = -4t(1+t^2)y + \frac{1}{1+t^2}, \\
&\big((1+t^2)^2 y\big)' = \frac{1}{1+t^2}, && \text{(integrate)} \\
&(1+t^2)^2 y = \arctan(t) + C, \\
&y(t) = \frac{\arctan(t)+C}{(1+t^2)^2}, && \text{(general solution)} \\
&0 = \frac{\frac{\pi}{4}+C}{4}, \qquad C = -\frac{\pi}{4}, && \text{(initial condition)} \\
&y(t) = \frac{\arctan(t)-\frac{\pi}{4}}{(1+t^2)^2}.
\end{aligned}
$$