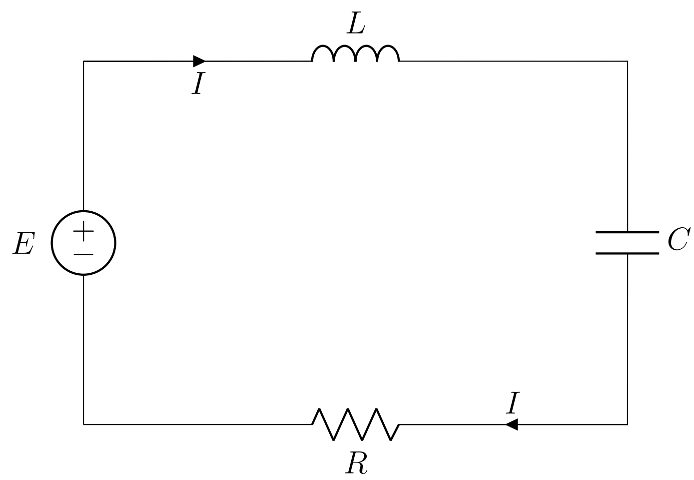
**3.4 Electrical Circuits:**




An RLC circuit.

E is the voltage source (voltage drop in V, volts)

R is the resistor (resistance in Ω, ohms)

C is the capacitor (capacitance in F, farads)

L is the inductor (inductance in H, henrys)

I is the current (in A, amperes)

**Component laws:**

1. Ohm's law: $E_R = RI$

2. Faraday's law: $E_L = L\dfrac{dI}{dt}$

3. Capacitance law: $E_C = \dfrac{Q}{C}$ $\quad Q$ is the charge on the capacitor (in C, coulombs)

4. Kirchhoff's voltage law: The sum of the voltage drops around any closed loop in a circuit must be zero.

5. Kirchoff's current law: The sum of currents flowing into any junction is zero.

From the law 4 we get $E_L + E_C + E_R - E = 0$ or

$$L\,\frac{dI}{dt} + \frac{Q}{C} + RI = E \qquad (1)$$

We use $I = \dfrac{dQ}{dt}$ and differentiate (1) to get

$$L\,\frac{d^2I}{dt^2} + \frac{1}{C}\,\frac{dQ}{dt} + R\,\frac{dI}{dt} = \frac{dE}{dt} \qquad (2)$$

or

$$L\,\frac{d^2I}{dt^2} + R\,\frac{dI}{dt} + \frac{1}{C}\,I = \frac{dE}{dt} \qquad (3)$$

If there is no capacitor then from (1) we obtain

$$L\,\frac{dI}{dt} + RI = E \qquad (4)$$

If there is no inductor then $RI + \dfrac{Q}{C} = E$ or

$$R\,\frac{dQ}{dt} + \frac{Q}{C} = E \qquad (5)$$

**Example 1:** $R = \frac{1}{2}\,\Omega$, $L =$ 1H, $E = 1\,\text{V}$ and there is no capacitor. If $I(0) = 0$ then find $I$.

Solution: The formula (4) gives the IVP $\dfrac{dI}{dt} + \frac{1}{2}I = 1, \quad I(0) = 0.$

It is a first order linear and separable equation. Its solution is $I(t) = 2(1 - e^{-t/2})$

**Example 2:** $R = 2\,\Omega$, $C = \frac{1}{5}$ F, $E = \cos t\,\text{V}$ and there is no inductor. Find $I$ if $I(0) = 0$.

Solution: The formula (5) gives the IVP $2\,\dfrac{dQ}{dt} + 5Q = \cos t, \quad I(0) = 0.$

It is a first order linear equation. The integrating factor is $u(t) = e^{\int \frac{5}{2}dt} = e^{\frac{5}{2}t}$

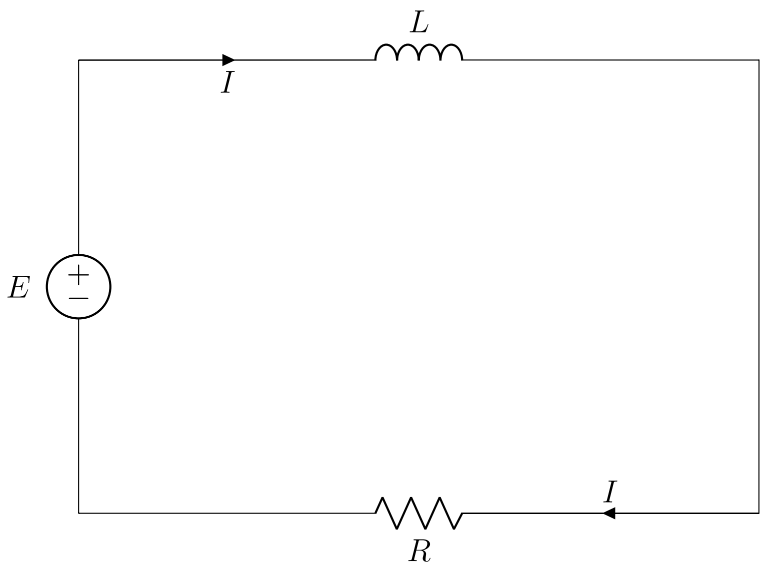
Then $\left(e^{\frac{5}{2}t}\,Q\right)' = \frac{1}{2}\,e^{\frac{5}{2}t}\,\cos t \;\Rightarrow\; e^{\frac{5}{2}t}\,Q = \frac{1}{2}\,e^{\frac{5}{2}t}(2\sin t + 5\cos t) + C \;\Rightarrow$

$Q(t) = \sin t + \frac{5}{2}\cos t + C_1 e^{-\frac{5}{2}t} \;\Rightarrow\; I(t) = \dfrac{dQ}{dt} = \cos t - \frac{5}{2}\sin t + Ce^{-\frac{5}{2}t}$

$I(0) = 1 + C = 0 \;\Rightarrow\; C = -1$

$I(t) = \cos t - \frac{5}{2}\sin t - e^{-\frac{5}{2}t}$

- Exercise #10, page 131. An inductor ($1H$) and resistor ($0.1\Omega$) are joined in series with an electromotive force (emf) $E = E(t)$ as in the Figure.




If there is no current in the circuit at time $t = 0$, find the ensuing current in the circuit at time $t$ for the emf $E(t) = 4\cos(3t)\,V$.

<u>`Solution`</u>: $L = 1H$ (Henry), $R = 0.1\Omega$,

$$E = E_R + E_L + \cancel{E_C} = RI + L\frac{dI}{dt} + \cancel{\frac{1}{C}Q},$$

hence

$$4\cos(3t) = 0.1I + 1\frac{dI}{dt},$$
$$I' = -0.1I + 4\cos(3t), \qquad (\cdot e^{0.1t})$$
$$(Ie^{0.1t})' = 4\cos(3t)\cdot e^{0.1t}, \qquad Ie^{0.1t} = 4\int \cos(3t)\cdot e^{0.1t}dt, \qquad I(t) = 4e^{-0.1t}\int \cos(3t)\cdot e^{0.1t}dt.$$

Using integration by parts twice

$$I(t) = \frac{4}{901}\big(10\cos(3t) + 300\sin(3t)\big) + Ce^{-0.1t},$$

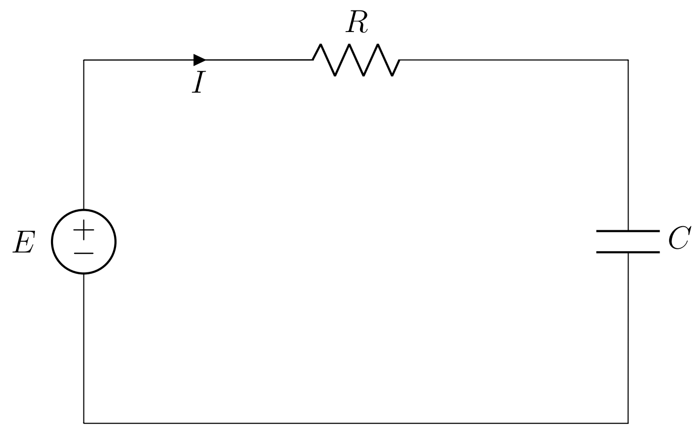
and using the initial condition $I(0) = 0$ we obtain

$$C = -\frac{40}{901},$$

hence

$$I(t) = \frac{4}{901}\big(10\cos(3t) + 300\sin(3t)\big) - \frac{40}{901}e^{-0.1t}.$$

- Exercise #15, page 131. A resistor ($20\Omega$) and capacitor ($1F$) are linked in series with an electromotive force (emf) $E = E(t)$ in $RC$ circuit.




If the emf is given as $E(t) = 10e^{-0.05t}$ and the current is zero at time $t = 0$, find the maximum charge on the capacitor and the time that it will occur.

<u>`Solution`</u>: $R = 20\Omega, C = 1F, Q(0) = 0,$

$$E = E_R + E_C \equiv RI + \frac{1}{C}Q = RQ' + \frac{1}{C}Q,$$
$$10e^{-\frac{t}{100}} = 20Q' + Q, \qquad Q' = -\frac{1}{20}Q + \frac{1}{2}e^{-\frac{t}{100}}$$

and using as integrating factor $u(t) = e^{\frac{t}{20}}$ we obtain

$$\left(e^{\frac{t}{20}}Q\right) = \frac{1}{2}e^{\frac{t}{25}},$$
$$e^{\frac{t}{20}}Q(t) = \frac{25}{2}e^{\frac{t}{25}} + C,$$
$$Q(t) = 12.5e^{\frac{t}{25}-\frac{t}{20}} + Ce^{-\frac{t}{20}} \equiv 12.5e^{-0.01t} + Ce^{-\frac{t}{20}}.$$

Using the initial condition $Q(0) = 0 \equiv 12.5 + C$ we obtain $C = -12.5$, and therefore

$$Q(t) = 12.5e^{-0.01t}\left(e^{-0.04t} - 1\right).$$

To find the maximum charge, we compute

$$Q'(t) = e^{-0.05t}\left(625 \times 10^{-4} - 125 \times 10^{-4}e^{0.04t}\right),$$

hence the maximum charge is when $Q'(T) = 0$, i.e.,

$$T = 25\ln(5),$$
$$Q(25\ln(5)) \approx 6.6874.$$