4.1 Second Order DE's

$y'' = f(t, y, y')$

Linear 2nd order DE: $\quad y'' + p(t)y' + q(t)y = g(t) \qquad (1)$

homogeneous equation: $\quad y'' + p(t)y' + q(t)y = 0 \qquad (2)$

Theorem Suppose $p(t)$, $q(t)$, $g(t)$ are continuous on the interval (a, b) and t_0 is any point in (a, b). Then for any real numbers y_0 and y_1 there is one and only one solution $y(t)$ to the equation (1) on (a, b) that satisfies the IC $y(t_0) = y_0$, $y'(t_0) = y_1$ (two ICs (!))

Definition A **linear combination** of two functions u and v is any function of the form $w = Au + Bv$ where A and B are constants.

Definition 1 *Two functions u and v are said to be **LI (linearly independent)** on the interval (a, b) if neither is a constant multiple of the other on that interval. If one is a constant multiple of the other on (a, b) they are said to be **LD (linearly dependent)** there.*

Definition The **Wronskian** of two functions u and v is defined to be:

$$W(t) = \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} = u(t)v'(t) - u'(t)v(t) \quad (= uv' - u'v, \text{ compare to the product rule}).$$

Theorem Suppose that y_1 and y_2 are both solutions to the equation (2). Then their linear combination $y = c_1y_1 + c_2y_2$ is also a solution to (2) for any constants c_1 and c_2.

Theorem Suppose that y_1 and y_2 are LI solutions to the equation (2). Then its general solution is $y = c_1y_1 + c_2y_2$ where c_1 and c_2 are arbitrary constants.

Definition Two LI solutions of the equation (2) form a **FSS (fundamental set of solutions)**.

For two solutions we need to check their linear independence to see if they form a FSS.

The following theorem helps to find if two solutions are LI or not.

Theorem Suppose functions u and v are solutions to the equation (2) in the interval (a, b).

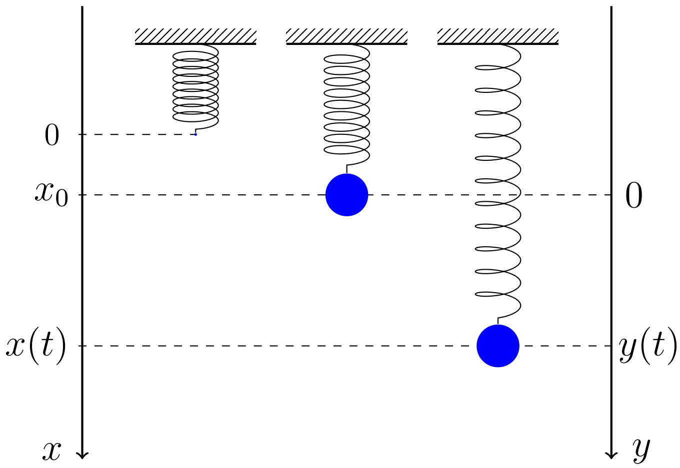
1. If $W(t_0) \neq 0$ for some $t_0 \in (a, b)$ then u and v are LI in (a, b).

2. If u and v are LI in (a, b) then $W(t) \neq 0$ for all $t \in (a, b)$.

IVP: The equation (1) together with the IC's $y(t_0) = y_0$, $y'(t_0) = y_1$.

Note: for a second order DE we need two IC's.

Example: (the vibrating spring, simple harmonic motion)



x_0 is the point of spring-mass equilibrium.

$x' = x'' = 0$ at x_0

Change of variables:

$$y = x - x_0$$

$$my'' = -ky + mg + D(y') + F(t) \qquad (3)$$

k is a constant, D is a damping force, F is an external force.

We assume that $D = -\mu y'$ and convert (3) into

$$my'' + \mu y' + ky = F(t)$$

If $\mu = 0$, $F(t) = 0$ then

$$y'' = -\frac{k}{m} y$$

Two LI solutions of the last equation are $y_1 = \cos w_0 t$ and $y_2 = \sin w_0 t$

where $w_0 = \sqrt{\frac{k}{m}}$ is the natural frequency.

So, the general solution is $y(t) = c_1 \cos w_0 t + c_2 \sin w_0 t$.

Assume that $\frac{k}{m} = 25$ and consider the IC's: $y(0) = 1$, $y'(0) = 10$.

Then $w_0 = 5$ and $y(t) = c_1 \cos 5t + c_2 \sin 5t$

$y'(t) = 5(-c_1 \sin 5t + c_2 \cos 5t)$

$y(0) = c_1 = 1, \quad y'(0) = 5c_2 = 10 \quad \Rightarrow \quad c_2 = 2.$

So, we get the solution $\quad y(t) = \cos 5t + 2\sin 5t.$

- Exercise #13 page 145. Show, by direct substitution, that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation

$$y'' - y' - 6y = 0, \qquad y_1(t) = e^{3t}, \quad y_2(t) = e^{-2t}.$$

Then verify, again by substitution, that any linear combination $C_1 y_1(t) + C_2 y_2(t)$ of the two given solutions is also a solution.

Solution: Let's first check linear independence:

$$W(t) := \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} \equiv \begin{vmatrix} e^{3t} & e^{-2t} \\ 3e^{3t} & -2e^{-2t} \end{vmatrix} = -5e^{t} \neq 0.$$

Also

$$\begin{aligned} &\big(C_1 e^{3t} + C_2 e^{-2t}\big)'' - \big(C_1 e^{3t} + C_2 e^{-2t}\big)' - 6\big(C_1 e^{3t} + C_2 e^{-2t}\big) \\ &= 9C_1 e^{3t} + 4C_2 e^{-2t} - 3C_1 e^{3t} + 2C_2 e^{-2t} - 6C_1 e^{3t} - 6C_2 e^{-2t} = 0. \end{aligned}$$

- Exercise #15 page 145. Show, by direct substitution, that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation

$$y'' - 2y' + 2y = 0, \qquad y_1(t) = e^{t}\cos t, \quad y_2(t) = e^{t}\sin t.$$

Then verify, again by substitution, that any linear combination $C_1 y_1(t) + C_2 y_2(t)$ of the two given solutions is also a solution.

Solution: Let's first check linear independence:

$$\begin{aligned} W(t) &:= \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} \equiv \begin{vmatrix} e^{t}\cos t & e^{t}\sin t \\ e^{t}(\cos t - \sin t) & e^{t}(\sin t + \cos t) \end{vmatrix} \\ &= e^{2t}\big(\sin t\cos t + \cos^2 t - \sin t\cos t + \sin^2 t\big) = e^{2t} \neq 0. \end{aligned}$$

Also

$$\begin{aligned} &\big(C_1 e^{t}\cos t + C_2 e^{t}\sin t\big)'' - 2\big(C_1 e^{t}\cos t + C_2 e^{t}\sin t\big)' + 2\big(C_1 e^{t}\cos t + C_2 e^{t}\sin t\big) \\ &= \big(C_1 e^{t}\cos t + C_2 e^{t}\sin t\big)'' - 2e^{t}\big(C_1\cos t + C_2\sin t - C_1\sin t + C_2\cos t\big) + 2\big(C_1 e^{t}\cos t + C_2 e^{t}\sin t\big) \\ &= e^{t}\big(\cancel{C_1\cos t} + \cancel{C_2\sin t} - \cancel{C_1\sin t} + C_2\cos t - \cancel{C_1\sin t} + C_2\cos t - \cancel{C_1\cos t} - \cancel{C_2\sin t}\big) \\ &\qquad - 2e^{t}\big(-\cancel{C_1\sin t} + C_2\cos t\big) = 0. \end{aligned}$$

- Exercise #16 page 145. Show, by direct substitution, that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation

$$y'' + 4y' + 4y = 0, \qquad y_1(t) = e^{-2t}, \quad y_2(t) = te^{-2t}.$$

Then verify, again by substitution, that any linear combination $C_1 y_1(t) + C_2 y_2(t)$ of the two given solutions is also a solution.

<u>`Solution`</u>: Let's first check linear independence:

$$W(t) := \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} \equiv \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}(1-2t) \end{vmatrix}$$
$$= e^{-4t}\big(1 - 2t + 2t\big) = e^{-4t} \neq 0.$$

Also

$$\big(C_1 e^{-2t} + C_2 te^{-2t}\big)'' + 4\big(C_1 e^{-2t} + C_2 te^{-2t}\big) + 4\big(C_1 e^{-2t} + C_2 te^{-2t}\big)$$
$$= \big(C_1 e^{-2t} + C_2 te^{-2t}\big)'' + 4e^{-2t}\big(-2C_1 + C_2 - 2tC_2\big) + 4\big(C_1 e^{-2t} + C_2 te^{-2t}\big)$$
$$= e^{-2t}\big(4C_1 - 2C_2 + 4tC_2 - 2C_2\big)'' + 4e^{-2t}\big(-C_1 + C_2 - C_2 t\big) = 0.$$

- Exercise #18, page 145. Use Definition 1 to explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the differential equation

$$y'' + 9y = 0, \qquad y_1(t) = \cos(3t), \quad y_2(t) = \sin(3t).$$

In addition, calculate the Wronskian and use it to explain the independence of the given solutions.

<u>`Solution`</u>:

$$W(t) := \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} \equiv \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3\big(\cos^2(3t) + \sin^2(3t)\big) = 3 \neq 0.$$

- Exercise #19, page 145. Use Definition 1 to explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the differential equation

$$y'' + 4y' + 13y = 0, \qquad y_1(t) = e^{-2t}\cos(3t), \quad y_2(t) = e^{-2t}\sin(3t).$$

In addition, calculate the Wronskian and use it to explain the independence of the given solutions.

<u>`Solution`</u>:

$$W(t) := \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} \equiv \begin{vmatrix} e^{-2t}\cos(3t) & e^{-2t}\sin(3t) \\ e^{-2t}(-2\cos(3t) - 3\sin(3t)) & e^{-2t}\big(-2\sin(3t) + 3\cos(3t)\big) \end{vmatrix}$$
$$= e^{-4t}\big(\cancel{-2\sin(3t)\cos(3t)} + 3\cos^2(3t) + \cancel{2\sin(3t)\cos(3t)} + 3\sin^2(3t)\big) = 3e^{-4t} \neq 0.$$

- Exercise #20 page 145. Use Definition 1 to explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the differential equation

$$y'' + 6y' + 9y = 0, \qquad y_1(t) = e^{-3t}, \quad y_2(t) = te^{-3t}.$$

In addition, calculate the Wronskian and use it to explain the independence of the given solutions.

<u>`Solution`</u>:

$$W(t) := \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} \equiv \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & e^{-3t}(1-3t) \end{vmatrix} = e^{-6t}\big(1 - 3t_3t\big) = e^{-6t} \neq 0.$$

- Exercise #26 page 146. Unfortunately, Theorem 1.23 does not show how to find two independent solutions. However, there is a technique that can be used to find a second solution when one is known

  (a) Show that $y_1(t) = t^2$ is a solution of

$$t^2y'' + ty' - 4y = 0. \tag{1.32}$$

  (b) Let $y_2(t) = vy_1(t) = vt^2$, where v is a yet to be determined function of t. Note that if $\frac{y_2}{y_1} = v$ and v is non-constant, then y_1 and y_2 are independent. Show that the substitution $y_2 = vt^2$ reduces equation (1.32) to the separable equation

$$5v' + tv'' = 0. \tag{1.33}$$

  Solve equation (1.33) for v, form the solution $y_2 = vt^2$, and then state the general solution of equation (1.32).

<u>`Solution`</u>:

(a) Indeed

$$t^2y_1'' + ty_1' - 4y_1 \equiv t^2 \cdot 2 + t \cdot 2t - 4t^2 = 0.$$

(b) Substitute $y_2 = vt^2$ into equation (1.32) to obtain

$$\begin{aligned}
&t^2(vt^2)'' + t(vt^2)' - 4vt^2 = 0, \\
&t^2(v't^2 + 2vt)' + t(v't^2 + 2vt) - 4vt^2 = 0, \\
&t^2(v''t^2 + 2tv' + 2v't + \cancel{2v}) + t(v't^2 + \cancel{2vt}) - \cancel{4vt^2} = 0, \\
&v''t^4 + 5t^3v' = 0, \qquad v''t + 5v' = 0.
\end{aligned}$$

Denote $x(t) := v'(t)$, hence

$$\begin{aligned}
&tx' = -5x, \qquad x' = -\frac{5}{t}x, \qquad x(t) = t^{-5} \equiv v'. \\
&v(t) = t^{-4},
\end{aligned}$$

hence $y_2(t) = t^{-4}t^2 \equiv t^{-2}$, which also a solution, linearly independent from t^2.

- Exercise #28 page 146. Use the technique shown in Exercise 26 to find the general solution of the second-order equation

$$t^2y'' + ty' - y = 0, \qquad y_1(t) = t.$$

<u>**Solution**</u>: Substitute $vy_1(t) \equiv vt$ into the equation and obtain

$$t^2y'' + ty' - y \equiv t^2(vt)'' + t(vt)' - tv = t^2(v''t + 2v') + t(v't + v) - tv$$
$$= t^3v'' + 3t^2v' + \cancel{tv} - \cancel{tv} = t^2(tv$$

hence

$$0 = tv'' + 3v' = tx' + 3x,$$
$$x(t) = \frac{1}{t^3}, \qquad v'(t) = \frac{1}{t^3}, \qquad v(t) = \frac{1}{t^2},$$

and

$$y_2(t) = \frac{1}{t^2}t \equiv \frac{1}{t}$$

is the other linearly independent solution.