4.3 Second Order, Linear, Homogeneous Equations with Constant Coefficients

$$y'' + py' + qy = 0 \qquad (5)$$

where p and q are constants.

We try $y = e^{\lambda t}$ as a trial solution to (5).

Then (5) $\Leftrightarrow$ $(\lambda^2 + p\lambda + q)e^{\lambda t} = 0$ $\Leftrightarrow$ $\lambda^2 + p\lambda + q = 0 \qquad (6)$

(6) is called a characteristic equation for the DE (5)

Solutions of (6) are called eigenvalues or characteristic roots. They can be found by using the quadratic formula:

$\lambda = \frac{1}{2}\left(-p \pm \sqrt{D}\right)$ where $D = p^2 - 4q$ is the discriminant.

Case $D > 0$ (6) has two distinct real roots λ_1 and λ_2.

Then FSS: $y_1(t) = e^{\lambda_1 t}$, $y_2(t) = e^{\lambda_2 t}$ and the general solution to (5) is

$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$.

Case $D < 0$ (6) has two distinct complex roots (they must be complex conjugate numbers)

$\lambda = a + ib$ and $\overline{\lambda} = a - ib$

where $a = -\dfrac{p}{2}$, $b = \dfrac{\sqrt{-D}}{2}$ are real numbers (Note that $-D > 0$).

The number a is called the real part of the complex number λ and b is its imaginary part.

The general (real) solution is $y(t) = c_1 e^{at} \cos bt + c_2 e^{at} \sin bt$.

Functions $e^{at} \cos bt$ and $e^{at} \sin bt$ form a fundamental set of real-valued solutions.

We also can write $y(t) = e^{at}(c_1 \cos bt + c_2 \sin bt)$.

Case $D = 0$ (6) has one real root of multiplicity 2 (repeated root) λ

FSS is $e^{\lambda t}$, $te^{\lambda t}$.

The general solution is $y(t) = e^{\lambda t}(c_1 + c_2 t)$.

Example 1: $y'' + y' - 6y = 0$

Char. equation: $\lambda^2 + \lambda - 6 = 0 \quad \Leftrightarrow \quad (\lambda + 3)(\lambda - 2) = 0$

Real roots are $\lambda_1 = -3$ and $\lambda_2 = 2$.

FSS: e^{-3t} and e^{2t}.

General solution is $y(t) = c_1 e^{-3t} + c_2 e^{2t}$.

Example 2: $y'' - 2y' + 10y = 0$

Char. equation: $\lambda^2 - 2\lambda + 10 = 0$ Char. roots are $\lambda = 1 \pm \sqrt{1 - 10}$

$\lambda_1 = 1 + 3i,\ \lambda_2 = \overline{\lambda} = 1 - 3i$

Real-valued FSS is $e^t \cos 3t,\ e^t \sin 3t$ (Check that they are linearly independent)

General solution $y(t) = e^t(c_1 \cos 3t + c_2 \sin 3t)$

Example 3: $y'' - 4y' + 4y = 0$

Char. equation: $\lambda^2 - 4\lambda + 4 = 0, \quad (\lambda - 2)^2 = 0$

Char. root is $\lambda = 2$ (of multiplicity 2)

FSS: $e^{2t},\ te^{2t}$ (Check that they are linearly independent)

General solution is $y(t) = c_1 e^{2t} + c_2 te^2 t = e^{2t}(c_1 + c_2 t)$.

- Find the general solution
  - Exercise #3 page 145. $y'' + 5y' + 6y = 0$.
    Solution:
    $$\lambda^2 + 5\lambda + 6 = 0, \qquad \lambda_{1,2} = -3, -2, \qquad y(t) = C_1 e^{-3t} + C_2 e^{-2t}.$$
  - Exercise #4 page 145. $y'' + y' - 12y = 0$.
    Solution:
    $$\lambda^2 + \lambda - 12 = 0, \qquad \lambda_{1,2} = 3, -4, \qquad y(t) = C_1 e^{3t} + C_2 e^{-4t}.$$

    Exercise #11 page 145. $y'' + 4y' + 5y = 0$.
    Solution:
    $$\lambda^2 + 4\lambda + 5 = 0, \qquad \lambda_{1,2} = -2 \pm i, \qquad y(t) = e^{-2t}\big(C_1 \cos t + C_2 \sin t\big).$$

– Exercise #19 page 145. $4y'' + 4y' + y = 0$.
Solution:

$$4\lambda^2 + 4\lambda + 1 = 0, \qquad \lambda_{1,2} = -\frac{1}{2}, \qquad y(t) = C_1 e^{-\frac{t}{2}} + C_2 t e^{-\frac{t}{2}}.$$

- Exercise #8 page 156. The equation

$$6y'' + 5y' - 6y = 0$$

has **distinct, real**, characteristic roots. Find the general solution.
Solution:

$$\begin{aligned} &6\lambda^2 + 5\lambda - 6 = 0, \qquad \lambda_{1,2} = \frac{-5 \pm \sqrt{25 + 4 \cdot 36}}{12} = \frac{-5 \pm 13}{12} = -\frac{3}{2}, \frac{2}{3}, \\ &\{e^{-\frac{3}{2}t}, e^{\frac{2}{3}t}\} && \text{(fundamental set of solutions)} \\ &y(t) = C_1 e^{-\frac{3}{2}t} + C_2 e^{\frac{2}{3}t}. && \text{(general solution)} \end{aligned}$$

- Exercise #12 page 156. The equation

$$y'' + 2y' + 17y = 0$$

has **complex** characteristic roots. Find the general solution.
Solution:

$$\begin{aligned} &\lambda^2 + 2\lambda + 17 = 0, \qquad \lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 4 \cdot 17}}{2} = -1 \pm 4i, \\ &\{e^{-t}\cos 4t, e^{-t}\sin 4t\} && \text{(fundamental set of solutions)} \\ &y(t) = C_1 e^{-t}\cos 4t + C_2 e^{-t}\sin 4t. && \text{(general solution)} \end{aligned}$$

- Exercise #24 page 156. The equation

$$y'' + 8y' + 16y = 0$$

has **repeated, real**, characteristic roots. Find the general solution.
Solution:

$$\begin{aligned} &\lambda^2 + 8\lambda + 16 = 0, \qquad \lambda_{1,2} = \frac{-8 \pm \sqrt{64 - 4 \cdot 16}}{2} = -4, \\ &\{e^{-4t}, te^{-t}\} && \text{(fundamental set of solutions)} \\ &y(t) = C_1 e^{-4t} + C_2 t e^{-4t}. && \text{(general solution)} \end{aligned}$$

- Exercise #29 page 156. Find the solution of the

$$y'' + 10y' + 25y = 0, \qquad y(0) = 2, \quad y'(0) = -1$$

initial value problem.
<u>`Solution`</u>: The characteristic polynomial is

$$\lambda^2 + 10\lambda + 25 = 0, \qquad \lambda_{1,2} = -5, \qquad \text{(repeated roots)}$$
$$y(t) = C_1 e^{-5t} + C_2 t e^{-5t}, \quad y'(t) = -5C_1 e^{-5t} + C_2 e^{-5t} - 5C_2 t e^{-5t},$$
$$2 \equiv y(0) = C_1, \quad -1 \equiv y'(0) = -5C_1 + C_2, \qquad \text{(initial conditions)}$$
$$C_1 = 2, C_2 = 9,$$
$$y(t) = 2e^{-5t} + 9te^{-5t}.$$

- Exercise #30 page 156. Find the solution of the

$$y'' - 2y' - 3y = 0, \qquad y(0) = 2, \quad y'(0) = -3$$

initial value problem.
<u>`Solution`</u>:

$$\lambda^2 - 2\lambda - 3 = 0, \qquad \lambda_{1,2} = \frac{2 \pm \sqrt{4 + 4 \cdot 3}}{2} = -1, 3 \qquad \text{(real, distinct)}$$
$$y(t) = C_1 e^{-t} + C_2 e^{3t}, \quad y'(t) = -C_1 e^{-t} + 3C_2 e^{3t},$$
$$2 \equiv y(0) = C_1 + C_2, \quad -3 \equiv y'(0) = -C_1 + 3C_2, \qquad \text{(initial conditions)}$$
$$C_1 = \frac{9}{4}, \quad C_2 = -\frac{1}{4},$$
$$y(t) = \frac{9}{4} e^{-t} - \frac{1}{4} e^{3t},$$

- Exercise #31 page 156. Find the solution of the

$$y'' + 2y' + 3y = 0, \qquad y(0) = 1, \quad y'(0) = 0$$

initial value problem.
<u>`Solution`</u>: The characteristic polynomial is

$$\lambda^2 + 2\lambda + 3 = 0, \qquad \lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 12}}{2} = -1 \pm i\sqrt{2}, \qquad \text{(complex roots)}$$
$$y(t) = C_1 e^{-t} \cos \sqrt{2}t + C_2 e^{-t} \sin \sqrt{2}t,$$
$$y'(t) = -C_1 e^{-t} \cos \sqrt{2}t - C_1 \sqrt{2} e^{-t} \sin \sqrt{2}t - C_2 e^{-t} \sin \sqrt{2}t + C_2 \sqrt{2} e^{-t} \cos \sqrt{2}t,$$
$$1 \equiv y(0) = C_1, \qquad 0 \equiv y'(0) = -C_1 + C_2\sqrt{2}, \qquad \text{(initial conditions)}$$
$$C_1 = 1, \qquad C_2 = \frac{\sqrt{2}}{2}$$
$$y(t) = e^{-t} \cos \sqrt{2}t + \frac{\sqrt{2}}{2} e^{-t} \sin \sqrt{2}t.$$