4.4 Harmonic Motion

Harmonic Motion is described by the equation

$$x'' + 2cx' + w_0^2 x = f(t) \qquad (1)$$

where $c \geq 0,\ w_0 > 0$ are constants, c is damping constant, $f(t)$ is the forcing term.

Examples

Simple RLC circuit: $\dfrac{d^2 I}{dt^2} + \dfrac{R}{L} \cdot \dfrac{dI}{dt} + \dfrac{1}{LC} \cdot I = \dfrac{1}{L} \cdot \dfrac{dE}{dt}$

Motion of vibrating string: $y'' + \frac{\mu}{m} y' + \frac{k}{m} y = \frac{1}{m} F(t)$

Unforced harmonic motion $(f(t) = 0)$: $x'' + 2cx' + w_0^2 x = 0 \qquad (2)$

Simple harmonic motion (no damping, $c = 0$): $x'' + w_0^2 x = 0 \qquad (3)$

Solution to (3): Char. equation: $\lambda^2 + w_0^2 = 0, \quad \lambda = \pm i w_0$

Then the general solution is $x(t) = a \cos w_0 t + b \sin w_0 t$. w_0 is called natural frequency.

Switch to polar coordinates: $a = A\cos\phi,\ b = A \sin\phi \quad \left(\dfrac{b}{a} = \tan\phi, \quad a^2 + b^2 = A^2 \right)$.

Then the solution is $x(t) = A \cos(w_0 t - \phi)$

Now let's find the general solution to the equation (2).

Char. equation is $\lambda^2 + 2c\lambda + w_0^2 = 0$

$\lambda = -c \pm \sqrt{D}$, where $D = c^2 - w_0^2$ (discriminant)

Case 1 (underdamped case). $c < w_0 \quad \Rightarrow \quad D < 0$.

Complex roots are $\lambda = -c \pm iw$, where $w = \sqrt{w_0^2 - c^2}$

The general solution is $x(t) = e^{-ct}(c_1 \cos wt + c_2 \sin wt)$

Case 2 (overdamped case) $c > w_0 \quad \Rightarrow \quad D > 0$.

Distinct real roots: $\lambda_1 = -c - \sqrt{c^2 - w_0^2}$ and $\lambda_1 = -c + \sqrt{c^2 - w_0^2}$. Note that $\lambda_1 < \lambda_2 < 0$.

The general solution is $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

Case 3 (critically damped case). $c = w_0$

$\lambda = -c$ is a double (repeated) root.

The general solution is $x(t) = e^{-ct}(c_1 + c_2 t)$

Example A mass of 4kg is attached to a spring with a spring constant $k = 169$ kg/s^2. It is the stretched 10 cm from the spring-mass equilibrium and set to oscillating with an initial velocity of 130 cm/s. Let damping constant $\mu = 77.6$ kg/s. Find position of the spring at time t.

Here $m = 4,\ k = 169,\ \mu = 77.6$.

DE: $4x'' + 77.6x' + 169x = 0 \quad \Leftrightarrow \quad x'' + 19.4x' + 42.25x = 0$

Char equation: $\lambda^2 + 19.4\lambda + 42.25 = 0$

$\lambda = -9.7 \pm \sqrt{9.7^2 - 42.25} = -9.7 \pm \sqrt{94.09 - 42.25} = -9.7 \pm 7.2 \quad \Rightarrow \quad \lambda_1 = -16.9, \lambda_2 = -2.5$

It is an overdamped motion.

$x(t) = c_1 e^{-16.9t} + c_2 e^{-2.5t}$

ICs: $0.1 = x(0) = c_1 + c_2, \quad 1.3 = x'(0) = -16.9c_1 - 2.5c_2$

Exact values are $c_1 = -\frac{31}{288},\ c_2 = \frac{299}{1440}$

$x(t) = -\frac{31}{288}e^{-16.9t} + \frac{299}{1440}e^{-2.5t}$

- Exercise #3 page 163.

  (i) Use a computer or calculator to plot the graph of the given function

  $$y(t) = \cos 4t + \sqrt{3}\sin 4t,$$

  and

  (ii) place the solution in the form $y = A\cos(\omega t - \phi)$ and compare the graph of your answer with the plot found in part (i).

  <u>`Solution`</u>: Since $a = 1, b = \sqrt{3}$ and the natural frequency $\omega_0 = 4$, we obtain

  $$A = \sqrt{a^2 + b^2} \equiv 2, \qquad \text{(amplitude)}$$

  $$\phi = \arctan\frac{b}{a} \equiv \frac{\pi}{3} \qquad \text{(phase)}$$

  and therefore

  $$y(t) = A\cos(\omega_0 t - \phi) \equiv 2\cos(4t - \frac{\pi}{3}).$$

```
t=linspace(0,2*pi,200);
y1=cos(4*t)+sqrt(3)*sin(4*t);
y2=2*cos(4*t-pi/3);
plot(t,y1,'r.-.',t,y2,'b.-.')
```

- Exercise #4 page 163.

  (i) Use a computer or calculator to plot the graph of the given function

  $$y(t) = -\sqrt{3}\cos 2t + \sin 2t,$$

  and

  (ii) place the solution in the form $y = A\cos(\omega t - \phi)$ and compare the graph of your answer with the plot found in part (i).

<u>Solution</u>: Since $a = -\sqrt{3}, b = 1$ and the natural frequency $\omega_0 = 2$, we obtain

$$A = \sqrt{a^2 + b^2} \equiv 2, \qquad \text{(amplitude)}$$

$$\phi = \arctan\left(\frac{b}{a}\right) + \pi \equiv -\frac{\pi}{6} + \pi = \frac{5\pi}{6} \qquad \text{(since } a < 0, b > 0\text{)} \qquad \text{(phase)}$$

and therefore

$$y(t) = A\cos(\omega_0 t - \phi) \equiv 2\cos(2t - \frac{5\pi}{6}).$$

```
t=linspace(0,2*pi,200);
z1=-sqrt(3)*cos(2*t)+ sin(2*t);
z2 = 2*cos(2*t-5*pi/6);
plot(t,z1,'r-',t,z2,'b.-.')
```

- Exercise #5 page 163.

  (i) Use a computer or calculator to plot the graph of the given function

  $$y(t) = 0.2\cos 2.5t - 0.1\sin 2.5t,$$

  and

  (ii) place the solution in the form $y = A\cos(\omega t - \phi)$ and compare the graph of your answer with the plot found in part (i).

<u>Solution</u>: Since $a = 0.2, b = -0.1$ and the natural frequency $\omega_0 = 2.5$, we obtain

$$A = \sqrt{a^2 + b^2} \equiv \frac{\sqrt{5}}{10}, \qquad \text{(amplitude)}$$

$$\phi = \arctan\left(\frac{b}{a}\right) \equiv -\frac{\pi}{6} \qquad \text{(since } a > 0\text{)} \qquad \text{(phase)}$$

and therefore

$$y(t) = A\cos(\omega_0 t - \phi) \equiv \frac{\sqrt{5}}{10}\cos(2.5t + \frac{\pi}{6}).$$

```
t=linspace(0,2*pi,200);
x1=0.2*cos(2.5*t)-0.1*sin(2.5*t);
x2=sqrt(5)/10*cos(2.5*t+pi/6);
plot(t,x1,'r.-.',t,x2,'b.-.')
```

- Exercise #7 page 163. Place the function

$$y = e^{-t/2}\big(\cos 5t + \sin 5t\big)$$

in the form $Ae^{-ct}(\cos\omega_0 t - \phi)$. Then, on one plot place the graphs of

$$Ae^{-ct}(\cos\omega_0 t - \phi), \qquad Ae^{-ct}, \qquad -Ae^{-ct}.$$

For the last two use a different style and/or color than for the first.
Solution:

- Exercise #8 page 163. Place the function

$$y = e^{-t/4}\big(\sqrt{3}\cos 4t - \sin 4t\big)$$

in the form $Ae^{-ct}(\cos\omega_0 t - \phi)$. Then, on one plot place the graphs of

$$Ae^{-ct}(\cos\omega_0 t - \phi), \qquad Ae^{-ct}, \qquad -Ae^{-ct}.$$

For the last two use a different style and/or color than for the first.
Solution: Here $a = \sqrt{3}, b = -1$, and the natural frequency $\omega_0 = 4$, hence

$$\begin{aligned}
&A = \sqrt{a^2+b^2} \equiv 2, \\
&Ae^{-ct} \equiv 2e^{-t/4}, && \text{(damped amplitude)} \\
&\phi = \arctan\frac{b}{a} \equiv -\frac{\pi}{6}, && \text{(phase)} \\
&y(t) = 2e^{-t/4}\cos(4t + \pi/6).
\end{aligned}$$

```
t=linspace(0,2*pi,200);
y1 = 2*exp(-t/4).* cos( 4*t + pi/6 ) ;
y2 = 2*exp(-t/4);
y3 = - 2*exp(-t/4);
plot(t,y1,'r.-.',t,y2,'b.-.',t,y3,'g.-.')
```

- Exercise #9 page 163. Place the function

$$y = e^{-0.1t}\big(0.2\cos 2t + 0.1\sin 2t\big)$$

in the form $Ae^{-ct}(\cos\omega_0 t - \phi)$. Then, on one plot place the graphs of

$$Ae^{-ct}(\cos\omega_0 t - \phi), \qquad Ae^{-ct}, \qquad -Ae^{-ct}.$$

For the last two use a different style and/or color than for the first.
<u>Solution</u>: Here $a = 0.2, b = 0.1$, and the natural frequency $\omega_0 = 2$, hence

$$A = \sqrt{a^2 + b^2} \equiv \frac{\sqrt{5}}{10},$$

$$Ae^{-ct} \equiv \frac{\sqrt{5}}{10}e^{-0.1t}, \qquad \text{(damped amplitude)}$$

$$\phi = \arctan\frac{b}{a} \equiv \arctan 0.5, \qquad \text{(phase)}$$

$$y(t) = \frac{\sqrt{5}}{10}e^{-0.1t}\cos(2t - \arctan 0.5).$$

```
t=linspace(0,2*pi,200);
y1 = sqrt(5)/10*exp(-0.1*t).* cos( 2*t - atan(0.5) ) ;
y2 = sqrt(5)/10*exp(-0.1*t);
y3 = - sqrt(5)/10*exp(-0.1*t);
plot(t,y1,'r.-.',t,y2,'b.-.',t,y3,'g.-.')
```

- Exercise #11 page 163. A $m = 0.2$-kg mass is attached to a spring having a spring constant $\kappa = 5$ kg/s^2. The system is displaced 0.5 m from its equilibrium position and released from rest. If there is no damping present ($\mu = 0$), find the amplitude A, frequency ω_0, and phase ϕ of the resulting motion.
<u>Solution</u>: With the given data, the vibrating spring equation

$$my'' + \mu y' + \kappa y = F(t), \qquad y'' + \frac{\mu}{m}y' + \frac{\kappa}{m}y = \frac{F(t)}{m},$$

writes as

$$y'' + 25y = 0,$$

with the initial conditions

$$y(0) = 0.5, \qquad y'(0) = 0.$$

Then the general solution is

$$y(t) = C_1 \cos 5t + C_2 \sin 5t,$$
$$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t,$$

and from the initial conditions we obtain

$$C_1 = 0.5, C_2 = 0.$$

Finally

$$y(t) = 0.5 \cos 5t,$$
$$A = 0.5, \qquad \text{(amplitude)}$$
$$\omega_0 = 5, \qquad \text{(natural frequency)}$$
$$\phi = 0. \qquad \text{(phase)}$$

- Exercise #12 page 163. A $m = 0.1$-kg mass is attached to a spring having a spring constant $\kappa = 3.6$ kg/s^2. The system is allowed to come to rest.

  Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4m/s. If there is no damping present ($\mu = 0$), find the amplitude A, frequency ω_0, and phase ϕ of the resulting motion. Plot the solution.
  <u>`Solution`</u>: With the given data, the vibrating spring equation

$$my'' + \mu y' + \kappa y = F(t), \qquad y'' + \frac{\mu}{m} y' + \frac{\kappa}{m} y = \frac{F(t)}{m},$$

writes as

$$y'' + 36y = 0,$$

with the initial conditions

$$y(0) = 0, \qquad y'(0) = 0.4.$$

Then the general solution is

$$y(t) = C_1 \cos 6t + C_2 \sin 6t,$$
$$y'(t) = -6C_1 \sin 5t + 6C_2 \cos 5t,$$

and from the initial conditions we obtain

$$C_1 = 0, C_2 = \frac{1}{15}.$$

Finally

$$y(t) = \frac{1}{15} \sin 6t,$$
$$A = 1/15, \qquad \text{(amplitude)}$$
$$\omega_0 = 6, \qquad \text{(natural frequency)}$$
$$\phi = 0. \qquad \text{(phase)}$$

```
t=linspace(0,2*pi,200) ;
y =1/15*sin( 6*t ) ;
plot(t,y,'.-.')
```

- Exercise #14 page 163. Consider the undamped oscillator

$$mx'' + \kappa x = 0, \qquad x(0) = x_0, \quad , x'(0) = v_0.$$

Show that the amplitude of the resulting motion is $\sqrt{x_0^2 + mv_0^2/\kappa}$.
Solution: The equation of the vibrating spring writes

$$x'' + \frac{\kappa}{m}x = 0,$$
$$x(t) = a\cos\left(\sqrt{\frac{\kappa}{m}}t\right) + b\sin\left(\sqrt{\frac{\kappa}{m}}t\right), \qquad \text{(general solution)}$$
$$x'(t) = -a\sqrt{\frac{\kappa}{m}}\sin\left(\sqrt{\frac{\kappa}{m}}t\right) + b\sqrt{\frac{\kappa}{m}}\cos\left(\sqrt{\frac{\kappa}{m}}t\right),$$

where

$$x_0 = a, v_0 = b\sqrt{\frac{\kappa}{m}}.$$

Therefore the solution is

$$x(t) = x_0\cos\left(\sqrt{\frac{\kappa}{m}}t\right) + \frac{v_0}{\sqrt{\frac{\kappa}{m}}}\sin\left(\sqrt{\frac{\kappa}{m}}t\right),$$

and the amplitude is

$$A = \sqrt{a^2 + b^2} = \sqrt{x_0^2 + \frac{v_0^2}{\kappa/m}}.$$

- Exercise #15 page 163. A $2\,\mu F$ capacitor ($1\mu F = 1 \times 10^{-6}F$) is charged to 20 V and then connected across a $6\mu H$ inductor ($1\mu H = 1 \times 10^{-6}H$), forming an LC circuit.

  (a) Find the initial charge on the capacitor.

  (b) At the time of connection, the initial current is zero $Q'(0) = 0$. Assuming no resistance, find the amplitude, frequency, and phase of the charge. Plot the graph of the current versus time. (Use the equations

$$RI + L\frac{dI}{dt} + \frac{1}{C}Q = E(t), \qquad I = \frac{dQ}{dt}$$

  form Section 3.4.)

Solution: Here $R = 0, C = 2 \times 10^{-6}, L = 6 \times 10^{-6}, E(t) = 20.$

  (a) Initially we have for the capacitor

$$\frac{1}{C}Q(0) = E(0), \qquad Q(0) = C \cdot E(0) \equiv 2 \times 10^{-6} \cdot 20 = 4 \times 10^{-5} \text{ coulombs}.$$

(b) Then the equation writes

$$
\begin{aligned}
&6 \times 10^{-6} Q'' + \frac{1}{2 \times 10^{-6}} Q = 0, \\
&Q'' + \frac{1}{12 \times 10^{-12}} Q = 0, \\
&Q(t) = a \cos\left(\frac{10^6}{\sqrt{12}} t\right) + b \sin\left(\frac{10^6}{\sqrt{12}} t\right) && \text{(general `charge' solution)} \\
&Q'(t) = -a \frac{10^6}{\sqrt{12}} \sin\left(\frac{10^6}{\sqrt{12}} t\right) + b \frac{10^6}{\sqrt{12}} \cos\left(\frac{10^6}{\sqrt{12}} t\right). && \text{(general `current' solution)}
\end{aligned}
$$

Using the initial conditions $Q(0) = 4 \times 10^{-5}$ and $I(0) = 0 \equiv Q'(0)$ we obtain

$$a = 4 \times 10^{-5}, \quad b = 0,$$

therefore

$$
\begin{aligned}
&Q(t) = 4 \times 10^{-5} \cos\left(\frac{10^6}{\sqrt{12}} t\right), && \text{(charge)} \\
&A = 4 \times 10^{-5} && \text{(amplitude)} \\
&\omega_0 = \frac{10^6}{\sqrt{12}} \approx 2.88675 \times 10^5 && \text{(frequency)} \\
&\phi = 0. && \text{(phase)}
\end{aligned}
$$