4.5 Inhomogeneous Equations

The method of undetermined coefficients

$y'' + py' + qy = f \qquad (1)$

Theorem. Suppose y_p is a particular solution to the equation (1) and y_1, y_2 form FSS to the associated homogeneous equation

$y'' + py' + qy = 0 \qquad (2)$

Then the general solution to (1) is given by

$y = y_p + c_1 y_1 + c_2 y_2$

where c_1 and c_2 are arbitrary constants and y_p is a particular solution.

From before we know how to find y_1 and y_2.

We need to find y_p. Below is the table that explains what form of a trial solution to use for $y_p(t)$ for different right hand sides (forcing terms):

Forcing term $f(t)$	Trial solution $y_p(t)$
e^{rt}	ae^{rt}
$\cos wt$ or $\sin wt$	$a\cos wt + b\sin wt$
$P(t)$	$p(t)$
$P(t)e^{rt}$	$p(t)e^{rt}$
$P(t)\cos wt$ or $P(t)\sin wt$	$p(t)\cos wt + q(t)\sin wt$
$e^{rt}\cos wt$ or $e^{rt}\sin wt$	$e^{rt}(a\cos wt + b\sin wt)$
$e^{rt}P(t)\cos wt$ or $e^{rt}P(t)\sin wt$	$e^{rt}[p(t)\cos wt + q(t)\sin wt]$

Here a, b are constants and $p(t)$, $q(t)$ are polynomials that has to be found. $P(t)$ is a polynomial in the forcing term. In the same line $P(t)$, $p(t)$ and $q(t)$ must have the same degree. $p(t)$ and $q(t)$ are written in general form. For example, if $P(t) = t^3 - t - 2$ is a given polynomial of degree 3 then $p(t)$ has to be a polynomial of degree 3 in general form, i.e. $p(t) = at^3+bt^2+ct+d$.

Example If $f(t) = 3e^{3t}$ then $y_p(t) = ae^{3t}$

If $f(t) = t^2e^{3t}$ then $y_p = (at^2 + bt + c)e^{3t}$

If $f(t) = (2t+1)\cos 5t$ then $y_p(t) = (a_1 t + b_1)\cos 5t + (a_2 t + b_2)\sin 5t$

If $f(t) = te^{2t}\sin 7t$ then $y_p = e^{2t}((a_1 t + b_1)\cos 7t + (a_2 t + b_2)\sin 7t)$

Exceptional cases

If $f(t)$ has the same form as a solution of the homogeneous equation (2) then use the trial solution $ty_p(t)$. If it does not work, then try $t^2 y_p(t)$. And so on.

Example $y'' + 4y = \sin 3t$

Solution: The corresponding homogeneous equation is $y'' + 4y = 0$

Char. equation is $\lambda^2 + 4 = 0$, $\lambda = 2i$. FSS: $y_1 = \cos 2t$, $y_2 = \sin 2t$

$f(t)$ is not a solution of homogeneous equation.

The trial solution is $y_p = a\cos 3t + b\sin 3t$, $y_p' = -3a\sin 3t + 3b\cos 3t$, $y_p'' = -9a\cos 3t - 9b\sin 3t$

Put them into the original equation to get $y_p'' + 4y_p = -5a\cos 3t - 5b\sin 3t = \sin 3t$
$\Rightarrow \quad b = -\frac{1}{5}, a = 0.$

Hence $y_p = -\frac{1}{5}\sin 3t$

(Note also that $y_p'' = -9y_p$ by the property of the second derivative of sin and cos).

The general solution is $y(t) = c_1\cos 2t + c_2\sin 2t - \frac{1}{5}\sin 3t$

Example $y'' + 4t = \sin 2t$.

Solution: Now $f(t)$ is a solution of the homogeneous equation.

So we try $y_p = t(a\cos 2t + b\sin 2t) = at\cos 2t + bt\sin 2t$.

$y_p' = a\cos 2t + b\sin 2t + 2at\sin 2t + 2bt\cos 2t$

$y_p'' = -2a\sin 2t + 2b\cos 2t - 2a\sin 2t + 2b\cos 2t - 4at\cos 2t - 4bt\sin 2t = -4a\sin 2t + 4b\cos 2t - 4at\cos 2t - 4bt\sin 2t$

(Alternative way: Denote $y = a\cos 2t + b\sin 2t$. Then $y_p = ty$, $y_p' = y + ty'$, $y_p'' = 2y' + ty'' = 2y' - 4ty$. We have $y_p'' + 4y_p = 2y' - 4ty + 4ty = 2y'$).

$y_p'' + 4y = -4a\sin 2t + 4b\cos 2t = \sin 2t \quad \Rightarrow \quad a = -\frac{1}{4}, \; b = 0.$

$y_p = -\frac{1}{4} t \cos 2t$

(Check: $y_p' = -\frac{1}{4}\cos 2t + \frac{1}{2} t \sin 2t$, $y_p'' = \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2t + t\cos 2t = \sin 2t + t \cos 2t$,
$y_p'' + 4y_p = \sin 2t + t\cos 2t - t\cos t = \sin 2t$)

The general solution is $\quad y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{4} t \cos 2t$

Combination of forcing terms

Theorem Suppose $y_f(t)$ is a particular solution of the linear equation $y'' + py' + qy = f(t)$

and $y_g(t)$ is a particular solution of $y'' + py' + qy = g(t)$.

Then $y_p(t) = \alpha y_f(t) + \beta y_g(t)$ is a particular solution of the equation

$$y'' + py' + qy = \alpha f(t) + \beta g(t)$$

where α and β are constants.

Example Find the general solution to the equation

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$

Solution: Char. equation: $\lambda^2 - 2\lambda + 3 = 0$, $\lambda_1 = -1$, $\lambda_2 = 3 \quad \Rightarrow \quad y_1 = e^{-x}$, $y_2 = e^{3x}$

Neither $f(x) = 4x - 5$ nor $g(x) = 6xe^{2x}$ is a solution to the homogeneous equation.

We split the right hand side as $f(x) + g(x)$ where $f(x) = 4x - 5$ and $g(x) = 6xe^{2x}$

Consider the equation $y'' - 2y' - 3y = 4x - 5 = f(x)$

The trial solution is $y_f = ax + b$. Then $y_f' = a$, $y_f'' = 0$.

$y_f'' - 2y_f' - 3y_f = 0 - 2a - 3(ax + b) = 4x - 5 \quad \Rightarrow \quad -2a - 3b - 3ax = 4x - 5$

Then by combining like terms we get two equations with two unknowns
$-3a = 4$ and $-2a - 3b = -5$. Solutions are $a = -\frac{4}{3}$ and $b = \frac{23}{9}$.

Therefore $y_f(x) = -\frac{4}{3}x + \frac{23}{9}$.

Now consider the equation $y'' - 2y' - 3y = 6xe^{2x} = g(x)$

The trial solution is $y_g = (ax + b)e^{2x}$. Then $y_g' = ae^{2x} + 2(ax + b)e^{2x} = (2ax + a + 2b)e^{2x}$

and $y_g'' = 2ae^{2x} + 2(2ax + a + 2b)e^{2x} = (4ax + 4a + 4b)e^{2x} = 4(ax + a + b)e^{2x}$

$y_g'' - 2y_g' - 3y_g = 4(ax + a + b)e^{2x} - 2(2ax + a + 2b)e^{2x} - 3(ax + b)e^{2x} = (4ax + 4a + 4b - 4ax - 2a - 4b - 3ax - 3b)e^{2x} = (-3ax + 2a - 3b)e^{2x} = 6xe^{2x}$

or $-3ax + 2a - 3b = 6x$.

By combining like terms we get two equations with two unknowns $-3a = 6$ and $2a - 3b = 0$.

Solutions are $a = -2$ and $b = -\frac{4}{3}$.

Hence $y_g = -\left(2x + \frac{4}{3}\right)e^{2x}$

Then $y_p = y_f + y_g = -\frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}$

The general solution is $y = c_1 e^{-x} + c_2 e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}$

- Exercise #2 page 172. Find a particular solution for the following differential equation:

$$y'' + 6y' + 8y = -3e^{-t}.$$

Solution: First let us find the solution to the homogeneous equation:

$$\begin{aligned}
&y_h'' + 6y_h' + 8y_h = 0, \\
&r^2 + 6r + 8 = 0, && \text{(characteristic equation)} \\
&r_{1,2} = \frac{-6 \pm \sqrt{36 - 4 \times 8}}{2} = \frac{-6 \pm 2}{2} = -4, -2, \\
&y_h(t) = C_1 e^{-4t} + C_2 e^{-2t},
\end{aligned}$$

and note that the fundamental set of solutions $\{e^{-4t}, e^{-2t}\}$ does not contain e^{-t}.
For a particular solution we seek a solution of the form:

$$\begin{aligned}
&y_p(t) = ae^{-t}, \\
&y_p' = -ae^{-t}, \qquad y_p'' = ae^{-t}, \\
&e^{-t}\big(a - 6a + 8a\big) = -3e^{-t}, \qquad a = -1, \\
&y_p(t) = -e^{-t}.
\end{aligned}$$

- Exercise #3 page 172. Find a particular solution for the following differential equation:

$$y'' + 2y' + 5y = 12e^{-t}.$$

Solution: First let us find the solution to the homogeneous equation:

$$\begin{aligned}
&y_h'' + 2y_h' + 5y_h = 0, \\
&r^2 + 2r + 5 = 0, && \text{(characteristic equation)} \\
&r_{1,2} = \frac{-2 \pm \sqrt{4 - 4 \times 5}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i, \\
&y_h(t) = e^{-t}\big(C_1 \cos 2t + C_2 \sin 2t\big),
\end{aligned}$$

and note that the fundamental set of solutions $\{e^{-t}\cos 2t, e^{-t}\sin 2t\}$ does not contain e^{-t}. For a particular solution we seek a solution of the form:

$$\begin{aligned}
&y_p(t) = ae^{-t}, \\
&y_p' = -ae^{-t}, \qquad y_p'' = ae^{-t}, \\
&e^{-t}\big(a - 2a + 5a\big) = 12e^{-t}, \qquad a = 3, \\
&y_p(t) = 3e^{-t}.
\end{aligned}$$

- Exercise #4 page 172. Find a particular solution for the following differential equation:

$$y'' + 3y' - 18y = 18e^{2t}.$$

Solution: First let us find the solution to the homogeneous equation:

$$\begin{aligned}
&y_h'' + 3y_h' - 18y_h = 0, \\
&r^2 + 3r - 18 = 0, && \text{(characteristic equation)} \\
&r_{1,2} = \frac{-3 \pm \sqrt{9 + 4 \times 18}}{2} = \frac{-3 \pm 9}{2} = -6, 3, \\
&y_h(t) = C_1 e^{-6t} + C_2 e^{3t},
\end{aligned}$$

and note that the fundamental set of solutions $\{e^{-6t}, e^{3t}\}$ does not contain e^{2t}. For a particular solution we seek a solution of the form:

$$\begin{aligned}
&y_p(t) = ae^{2t}, \\
&y_p' = 2ae^{2t}, \qquad y_p'' = 4ae^{2t}, \\
&e^{-t}\big(4a + 3 \times 2a - 18a\big) = 18e^{2t}, \qquad a = -\frac{9}{4}, \\
&y_p(t) = -\frac{9}{4}e^{2t}.
\end{aligned}$$

- Exercise #6 page 172. Use the form $y_p = a\cos\omega t = b\sin\omega t$ to help find a particular solution for the following differential equation:

$$y'' + 9y = \sin 2t.$$

`Solution`: First let us find the solution to the homogeneous equation:

$$\begin{aligned} &y_h'' + 9y_h = 0, \\ &r^2 + 9 = 0, && \text{(characteristic equation)} \\ &r_{1,2} = \pm 3i, \\ &y_h(t) = C_1 \cos 3t + C_2 \sin 3t, \end{aligned}$$

and note that the fundamental set of solutions $\{\cos 3t, \sin 3t\}$ does not contain $\sin 2t$. For a particular solution we seek a solution of the form:

$$\begin{aligned} &y_p(t) = a\cos 2t + b \sin 2t, \\ &y_p' = -2a \sin 2t + 2b \cos 2t, \qquad y_p'' = -4a\cos 2t - 4b \sin 2t, \\ &\cos 2t\big(-4a + 9a\big) + \sin 2t(-4b + 9b) = \sin 2t, \qquad a = 0, \quad b = \frac{1}{5}, \\ &y_p(t) = \frac{1}{5} \sin 2t. \end{aligned}$$

See also Exercise 11 for the "complex method".

- Exercise #8 page 172. Use the form $y_p = a\cos\omega t = b \sin\omega t$ to help find a particular solution for the following differential equation:

$$y'' + 7y' + 10y = -4\sin 3t.$$

`Solution`: First let us find the solution to the homogeneous equation:

$$\begin{aligned} &y_h'' + 7y_h' + 10y_h = 0, \\ &r^2 + 7r + 10 = 0, && \text{(characteristic equation)} \\ &r_{1,2} = -5, -2, \\ &y_h(t) = C_1 e^{-5t} + C_2 e^{-2t}, \end{aligned}$$

and note that the fundamental set of solutions $\{e^{-5t}, e^{-2t}\}$ does not contain $\sin 3t$. For a particular solution we seek a solution of the form:

$$\begin{aligned} &y_p(t) = a\cos 3t + b\sin 3t, \\ &y_p' = -3a\sin 3t + 3b\cos 3t, \qquad y_p'' = -9a\cos 3t - 9b\sin 3t, \\ &\cos 3t\big(-9a + 7\cdot 3b + 10a\big) + \sin 3t(-9b + 7\cdot(-3a) + 10b) \\ &\qquad \equiv \cos 3t\big(a + 21b\big) + \sin 3t(-21a + b) = \sin 2t, \\ &\qquad a + 21b = 0, \quad -21a + b = -4, \\ &a = \frac{42}{221}, \quad -\frac{2}{221}, \\ &y_p(t) = \frac{42}{221}\cos 3t - \frac{2}{221}\sin 3t. \end{aligned}$$

See also Exercise 13 for the "complex method".

- Exercise #10 page 172. Use the complex method to find a particular solution for the following differential equation:

$$y'' + 4y = \cos 3t.$$

<u>**Solution**</u>: Since $\sin 3t$ is the real part of e^{i3t}, we look for a solution to

$$z'' + 4z = e^{3it} \tag{***}$$

where $z(t) = x(t) + iy(t)$. The particular solution to equation (***) is

$$z_p(t) = ae^{3it}, \quad z_p'(t) = 3iae^{3it}, \quad z_p'' = -9ae^{3it},$$
$$e^{3it}(-9a + 4a) = e^{3it}, \qquad a = -\frac{1}{5}, \qquad z_p(t) = -\frac{1}{5}\cos 3t - \frac{1}{5}\sin 3t,$$
$$y_p(t) = -\frac{1}{5}\cos 3t.$$

(see Exercise #5 page 172)

- Exercise #11 page 172. Use the complex method to find a particular solution for the following differential equation:

$$y'' + 9y = \sin 2t.$$

<u>**Solution**</u>: Since $\sin 2t$ is the imaginary part of e^{i2t}, we look for a solution to

$$z'' + 9z = e^{2it} \tag{*}$$

where $z(t) = x(t) + iy(t)$. The particular solution to equation (*) is

$$z_p(t) = ae^{2it}, \quad z_p'(t) = 2iae^{it}, \quad z_p'' = -4ae^{it},$$
$$e^{2it}(-4a + 9a) = e^{2it}, \qquad a = \frac{1}{5}, \qquad z_p(t) = \frac{1}{5}\cos 2t + \frac{1}{5}\sin 2t,$$
$$y_p(t) = \frac{1}{5}\sin 2t.$$

(see Exercise #6 page 172)

- Exercise #12 page 172. Use the complex method to find a particular solution for the following differential equation:

$$y'' + 7y' + 6y = 3\sin 2t.$$

<u>**Solution**</u>: Since $\sin 2t$ is the imaginary part of e^{i2t}, we look for a solution to

$$z'' + 7y' + 6z = e^{2it} \tag{$\star$}$$

where $z(t) = x(t) + iy(t)$. The particular solution to equation ($\star$) is

$$\begin{aligned}
&z_p(t) = ae^{2it}, \quad z_p'(t) = 2iae^{it}, \quad z_p'' = -4ae^{it}, \\
&e^{2it}(-4a + 7\cdot 2ia + 6a) = 3e^{2it}, \\
&\quad a = \frac{3}{2+14i} = \frac{3}{2(1+7i)} = \frac{3(1-7i)}{2(1+49)} = \frac{3}{100}(1-7i), \\
&z_p(t) = \frac{3}{100}(1-7i)e^{2it} = \frac{3}{100}(1-7i)\big(\cos 2t + i\sin 2t\big) \\
&\qquad = \frac{3}{100}\cos 2t + \frac{21}{100}\sin 2t + i\big(\frac{3}{100}\sin 2t - \frac{21}{100}\cos 2t\big), \\
&y_p(t) = \frac{3}{100}\sin 2t - \frac{21}{100}\cos 2t.
\end{aligned}$$

(see Exercise #7 page 172)

- Exercise #13 page 172. Use the complex method to find a particular solution for the following differential equation:

$$y'' + 7y' + 10y = -4\sin 3t.$$

<u>`Solution`</u>: Since $\sin 3t$ is the imaginary part of e^{i3t}, we look for a solution to

$$z'' + 7z' + 10z = -4e^{3t}. \tag{**}$$

where $z(t) = x(t) + iy(t)$. The particular solution to equation (**) is

$$\begin{aligned}
&z_p(t) = ae^{3it}, \quad z_p'(t) = 3iae^{3it}, \quad z_p'' = -9ae^{3it}, \\
&e^{3it}(-9a + 7\cdot 3ia + 10a) = -4e^{3it}, \\
&a + 21ia = -4, \quad a = \frac{-4}{1+21i} = \frac{-4(1-21i)}{(1+21i)(1-21i)} = \frac{-4+84i}{442} = -\frac{2}{221} + i\frac{42}{221}, \\
&z_p(t) = \big(-\frac{2}{221} + i\frac{42}{221}\big)\cdot\big(\cos 3t + i\sin 3t\big) \\
&\qquad = -\frac{2}{221}\cos 3t - \frac{42}{221}\sin 3t + i\Big(-\frac{2}{221}\sin 3t + \frac{42}{221}\cos 3t\Big), \\
&y_p(t) = \frac{42}{221}\cos 3t - \frac{2}{221}\sin 3t.
\end{aligned}$$

(see Exercise #8 page 172)

- Exercise #14 page 172. Find a particular solution to the following differential equation

$$y'' + 5y' + 4y = 2 + 3t.$$

<u>`Solution`</u>: First let us find the solution to the homogeneous equation:

$$\begin{aligned}
&y_h'' + 5y_h' + 4y_h = 0, \\
&r^2 + 5r + 4 = 0, \qquad \text{(characteristic equation)} \\
&r_{1,2} = \frac{-5 \pm \sqrt{25 - 4\times 4}}{2} = \frac{-5\pm 3}{2} = -4, -1, \\
&y_h(t) = C_1e^{-4t} + C_2e^{-t},
\end{aligned}$$

and note that the fundamental set of solutions $\{e^{-4t}, e^{-2t}\}$ does not contain $at+b$.
For a particular solution we seek a solution of the form:

$$y_p(t) = at+b, \quad y_p' = a, \quad , y_p'' = 0,$$
$$0 + 5a + 4(at+b) = 2+3t, \qquad 5a+4b = 2, 4a = 3, \qquad a = \frac{3}{4}, b = -\frac{7}{16},$$
$$y_p(t) = \frac{3}{4}t - \frac{7}{16}.$$

- Exercise #16 page 172. Find a particular solution to the following differential equation

$$y'' + 5y' + 6y = 4 - t^2.$$

Solution:
For a particular solution we seek a solution of the form:

$$y_p(t) = at^2 + bt + c, \quad y_p' = 2at + b, \quad , y_p'' = 2a,$$
$$2a + 5(2at+b) + 6(at^2+bt+c) = 4 - t^2,$$
$$6at^2 + t(10a+6b) + (2a+5b+6c) = -t^2 + 4$$
$$a = -\frac{1}{6}, \qquad b = \frac{5}{18}, \qquad c = \frac{53}{108},$$
$$y_p(t) = -\frac{1}{6}t^2 + \frac{5}{18}t + \frac{53}{108}.$$

- Exercise #24 page 172. The forcing therm is also a solution of the associated homogeneous equation. Find a particular solution to the following differential equation

$$y'' - 3y' - 10y = 3e^{-2t}.$$

Solution: First let us find the solution to the homogeneous equation:

$$y_h'' - 3y_h' - 10y_h = 0,$$
$$r^2 - 3r - 10 = 0, \qquad \text{(characteristic equation)}$$
$$r_{1,2} = \frac{3 \pm \sqrt{9 + 4 \times 10}}{2} = \frac{3 \pm 7}{2} = -2, 5,$$
$$y_h(t) = C_1 e^{-2t} + C_2 e^{5t},$$

and note that the fundamental set of solutions $\{e^{-2t}, e^{5t}\}$ does contain e^{-2t}.
For a particular solution we then seek a solution of the form:

$$y_p(t) = Ate^{-2t}, \quad y_p' = Ae^{-2t}(1-2t), \quad y_p'' = Ae^{-2t}(-2+4t-2) = Ae^{-2t}(-4+4t),$$
$$Ae^{-2t}[(-4+4t) - 3(1-2t) - 10t] = 3^{-2t}, \qquad Ae^{-2t} \cdot (-7) = 3^{-2t}$$
$$A = -\frac{3}{7},$$
$$y_p(t) = -\frac{3}{7}te^{-2t}.$$

Verify your solution!

- Exercise #26 page 172. The forcing therm is also a solution of the associated homogeneous equation. Find a particular solution to the following differential equation

$$y'' + 4y = 4\cos 2t.$$

<u>Solution</u>: First let us find the solution to the homogeneous equation:

$$\begin{aligned} & y_h'' + 4y_h = 0, \\ & r^2 + 4 = 0, && \text{(characteristic equation)} \\ & r_{1,2} = \pm 2i, \\ & y_h(t) = C_1 \cos 2t + C_2 \sin 2t, \end{aligned}$$

and note that the fundamental set of solutions $\{\cos 2t, \sin 2t\}$ does contain $\cos 2t$.
For a particular solution we then seek a solution of the form:

$$\begin{aligned} y_p(t) &= a\,t\cos 2t + b\,t\sin 2t, \\ y_p' &= a\cos 2t - 2a\,t\sin 2t + b\sin 2t + 2b\,t\cos 2t \\ &= \cos 2t(a + 2b\,t) + \sin 2t(-2a\,t + b), \\ y_p'' &= -2\sin 2t\cdot(a + 2b\,t) + 2b\cos 2t + 2\cos 2t(-2a\,t + b) - 2a\sin 2t \\ &= \sin 2t\cdot(-2a - 4b\,t - 2a) + 2\cos 2t(b - 2a\,t + b) \\ &= -4\sin 2t\cdot(a + b\,t) + 4\cos 2t(b - a\,t), \end{aligned}$$

$$\begin{aligned} & -4\sin 2t\cdot(a + \cancel{b\,t}) + 4\cos 2t(b - \cancel{a\,t}) + 4\cdot\left(\cancel{a\,t\cos 2t} + \cancel{b\,t\sin 2t}\right) \equiv 4\cos 2t, \\ & -4a\sin 2t + 4b\cos 2t \equiv 4\cos 2t, \\ & a = 0, b = 1, \\ & y_p(t) = t\sin 2t. \end{aligned}$$

Verify your solution!
Remark that the solution to the (forced) differential equation

$$y'' + 4y = 4\cos 2t$$

is

$$y(t) \equiv y_h(t) + y_p(t) = C_1\cos 2t + C_2\sin 2t + t\sin 2t.$$