## 4.6 Variation of Parameters

To solve the equation

$$y'' + py' + qy = g(t) \tag{1}$$

we are looking for a particular solution in the form $y_p = v_1 y_1 + v_2 y_2$ where $y_1$ and $y_2$ form FSS to the associated homogeneous equation

$$y'' + py' + qy = 0 \tag{2}$$

To find $v_1$ and $v_2$ we use the system of two equations

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ v_1' y_1' + v_2' y_2' = g(t) \end{cases} \tag{3}$$

Solutions of the system (3) are $v_1' = -\dfrac{y_2 g}{W}$ and $v_2' = \dfrac{y_1 g}{W}$

where $W$ is the Wronskian of functions $y_1$ and $y_2$: $W = y_1 y_2' - y_2 y_1'$.

Then $v_1 = -\displaystyle\int \frac{y_2 g}{W}\, dt$ and $v_2 = \displaystyle\int \frac{y_1 g}{W}\, dt$

**Example** Solve $y'' + 9y = \tan 3t$.

*Solution:* The corresponding homogeneous equation is $y'' + 9y = 0$

Char. equation is $\lambda^2 + 9 = 0$, $\lambda = 3i$. FSS: $y_1 = \cos 3t$, $y_2 = \sin 3t$.

$W = 3\cos 3t \cos 3t + 3\sin 3t \sin 3t = 3$.

$$v_1 = -\int \frac{\sin 3t \tan 3t}{3}\, dt = -\frac{1}{3}\int \frac{\sin^2 3t}{\cos 3t}\, dt = -\frac{1}{3}\int \frac{1-\cos^2 3t}{\cos 3t}\, dt = -\frac{1}{3}\int (\cos 3t - \sec 3t)\ dt$$

$$v_1 = -\frac{1}{9}\left(\sin 3t - \ln|\sec 3t + \tan 3t|\right)$$

$$v_1 = \int \frac{\cos 3t \tan 3t}{3}\, dt = -\frac{1}{3}\int \sin 3t\, dt = -\frac{1}{9}\cos 3t$$

Hence $y_p = v_1 y_1 + v_2 y_2 = -\frac{1}{9}\left(\sin 3t - \ln|\sec 3t + \tan 3t|\right)\cos 3t - \frac{1}{9}\cos 3t \sin 3t$

$y_p = \frac{1}{9}\left(\ln|\sec 3t + \tan 3t| - 2\sin 3t\right)\cos 3t$

The general solution is $y(t) = c_1\cos 3t + c_2 \sin 3t + \frac{1}{9}\left(\ln|\sec 3t + \tan 3t| - 2\sin 3t\right)\cos 3t$

It also can be written as $y(t) = \frac{1}{9}\left(c_1 + \ln|\sec 3t + \tan 3t| - 2\sin 3t\right)\cos 3t + c_2 \sin 3t$

- Exercise #4 page 177. Find a particular solution for the following second-order differential equation:

$$x'' - 2x' - 3x = 4e^{3t}.$$

**Solution**: The characteristic equation gives the solution to the homogeneous equation:

$$r^2 - 2r - 3 = 0, \qquad r_{1,2} = \frac{2 \pm \sqrt{4+12}}{2} = -1, 3 \qquad \text{(characteristic equation)}$$
$$x_h(t) = c_1 e^{-t} + c_2 e^{3t}. \qquad \text{(general solution to the homogeneous equation)}$$

We seek a particular solution to the inhomogeneous equation of the form

$$x_p(t) = v_1 e^{-t} + v_2 e^{3t},$$
$$x_p'(t) = v_1' e^{-t} - v_1 e^{-t} + v_2' e^{3t} + 3v_2 e^{3t} = \underbrace{\left(v_1' e^{-t} + v_2' e^{3t}\right)}_{=0} - v_1 e^{-t} + 3v_2 e^{3t}$$
$$= -v_1 e^{-t} + 3v_2 e^{3t},$$
$$x_p''(t) = -v_1' e^{-t} + v_1 e^{-t} + 3v_2' e^{3t} + 9v_2 e^{3t}.$$

Substituting these into the inhomogeneous equation we obtain

$$\Big(-v_1' e^{-t} + \cancel{v_1 e^{-t}} + 3v_2' e^{3t} + \cancel{9v_2 e^{3t}}\Big) - 2\Big(-\cancel{v_1 e^{-t}} + \cancel{3v_2 e^{3t}}\Big) - 3\Big(\cancel{v_1 e^{-t}} + \cancel{v_2 e^{3t}}\Big) = 4e^{3t},$$
$$-v_1' e^{-t} + 3v_2' e^{3t} = 4e^{3t},$$

hence

$$v_1' e^{-t} + v_2' e^{3t} = 0, \qquad (v_1' y_1 + v_2' y_2 = 0)$$
$$-v_1' e^{-t} + 3v_2' e^{3t} = 4e^{3t}, \qquad (v_1' y_1' + v_2' y_2' = RHS)$$

which gives

$$v_2' = 1, \qquad v_1' = -e^{4t},$$
$$v_1 = -\frac{1}{4}e^{4t}, \qquad v_2 = t$$

and finally

$$x_p(t) = -\frac{1}{4}e^{4t}e^{-t} + te^{3t}. \qquad \text{(a particular solution)}$$

- Exercise #6 page 177. Find a particular solution for the following second-order differential equation:

$$x'' - 4x' + 4x = e^{2t}.$$

**Solution**: The characteristic equation gives the solution to the homogeneous equation:

$$r^2 - 4r + 4 = 0, \qquad r_{1,2} = \frac{4 \pm \sqrt{16-16}}{2} = 2 \qquad \text{(characteristic equation)}$$
$$x_h(t) = c_1 e^{2t} + c_2 t e^{2t}. \qquad \text{(general solution to the homogeneous equation)}$$

We seek a particular solution to the inhomogeneous equation of the form

$$x_p(t) = v_1 e^{2t} + v_2 t e^{2t},$$

where $v_1, v_2$ are to be found from the system

$$v_1' e^{2t} + v_2' \cdot (te^{2t}) = 0, \qquad (v_1' y_1 + v_2' y_2 = 0)$$
$$v_1' 2e^{2t} + v_2' \cdot (e^{2t} + t2e^{2t}) = e^{2t}, \qquad (v_1' y_1' + v_2' y_2' = RHS)$$

hence

$$v_1' + tv_2' = 0,$$
$$2v_1' + v_2'(1+2t) = 1,$$

which gives

$$v_2' = 1, \qquad v_1' = -t,$$
$$v_1 = -\frac{t^2}{2}, \qquad v_2 = t$$

and finally

$$x_p(t) = -\frac{t^2}{2} e^{2t} + t \cdot te^{2t} \equiv \frac{t^2}{2} e^{2t}. \qquad \text{(a particular solution)}$$

- Exercise #8 page 177. Find a particular solution for the following second-order differential equation:

$$x'' + x = \frac{1}{\cos^2 t}.$$

**Solution**: The characteristic equation gives the solution to the homogeneous equation:

$$r^2 + 1 = 0, \qquad r_{1,2} = \pm i \qquad \text{(characteristic equation)}$$
$$x_h(t) = c_1 \cos t + c_2 \sin t. \qquad \text{(general solution to the homogeneous equation)}$$

We seek a particular solution to the inhomogeneous equation of the form

$$x_p(t) = v_1 \cos t + v_2 \sin t,$$

where $v_1, v_2$ are to be found from the system

$$v_1' \cos t + v_2' \sin t = 0, \qquad (v_1'y_1 + v_2'y_2 = 0)$$
$$v_1'(-\sin t) + v_2'(\cos t) = \frac{1}{\cos^2 t}, \qquad (v_1'y_1' + v_2'y_2' = RHS)$$

hence

$$v_1' \cos t + v_2' \sin t = 0, \qquad (\cdot \sin t)$$
$$-v_1' \sin t + v_2' \cos t = \frac{1}{\cos^2 t}, \qquad (\cdot \cos t)$$

which gives

$$v_2' = \frac{1}{\cos t}, \qquad v_1' = -\frac{\sin t}{\cos^2 t} \equiv \left(\frac{1}{\cos t}\right)',$$
$$v_1 = \frac{1}{\cos t}, \qquad v_2 = \ln\left(\frac{1+\sin t}{\cos t}\right),$$

and finally

$$x_p(t) = v_1 \cos t + v_2 \sin t \equiv \frac{1}{\cos t}\cos t + \sin t \ln\left(\frac{1+\sin t}{\cos t}\right)$$
$$= 1 + \sin t \ln\left(\frac{1+\sin t}{\cos t}\right). \qquad \text{(a particular solution)}$$

- Exercise #9 page 177. Find a particular solution for the following second-order differential equation:

$$x'' + x = \sin^2 t.$$

__Solution__: The characteristic equation gives the solution to the homogeneous equation:

$$r^2 + 1 = 0, \qquad r_{1,2} = \pm i \qquad \text{(characteristic equation)}$$
$$x_h(t) = c_1 \cos t + c_2 \sin t. \qquad \text{(general solution to the homogeneous equation)}$$

We seek a particular solution to the inhomogeneous equation of the form

$$x_p(t) = v_1 \cos t + v_2 \sin t,$$

where $v_1, v_2$ are to be found from the system

$$v_1' \cos t + v_2' \sin t = 0, \qquad (v_1'y_1 + v_2'y_2 = 0)$$
$$v_1'(-\sin t) + v_2'(\cos t) = \sin^2 t, \qquad (v_1'y_1' + v_2'y_2' = RHS)$$

hence

$$v_1' \cos t + v_2' \sin t = 0, \qquad (\cdot \sin t)$$
$$-v_1' \sin t + v_2' \cos t = \sin^2 t, \qquad (\cdot \cos t)$$

which gives

$$\begin{aligned} v_2' &= \cos t \sin^2 t \equiv \left(\frac{\sin^3 t}{3}\right)', \\ v_1' &= -\sin^3 t = -\sin t(1-\cos^2 t) = (\cos t)' - \left(\frac{\cos^3 t}{3}\right)', \\ v_1 &= \cos t - \frac{\cos^3 t}{3}, \qquad v_2 = \frac{\sin^3 t}{3}, \end{aligned}$$

and finally

$$\begin{aligned} x_p(t) &= v_1 \cos t + v_2 \sin t \equiv \left(\cos t - \frac{\cos^3 t}{3}\right)\cos t + \sin t \frac{\sin^3 t}{3} \\ &= \cos^2 t - \frac{\cos^4 t}{3} + \frac{\sin^4 t}{3} \equiv \cos^2 t - \frac{\cos^2 t}{3}(1-\sin^2 t) + \frac{\sin^2 t}{3}(1-\cos^2 t) \\ &= \frac{1}{3}\sin^2 t + \frac{2}{3}\cos^2 t \\ &= \frac{1}{2} + \frac{1}{6}\cos 2t. \qquad \text{(a particular solution)} \end{aligned}$$

Check your solution:

$$\begin{aligned} x_p(t) &= \cos^2 t - \frac{\cos^4 t}{3} + \frac{\sin^4 t}{3}, \\ x_p' &= -2\cos t \sin t - \frac{4}{3}\cos^3 t \sin t + \frac{4}{3}\sin^3 t \cos t \equiv -\sin 2t + \frac{4}{3}\sin t \cos t = -\frac{1}{3}\sin 2t, \\ x_p'' &= -\frac{2}{3}\cos 2t, \end{aligned}$$

which yields

$$\begin{aligned} LHS &\equiv -\frac{2}{3}\cos 2t + \cos^2 t - \frac{\cos^4 t}{3} + \frac{\sin^4 t}{3} \\ &= -\frac{2}{3}\left(\cos^2 t - \sin^2 t\right) + \cos^2 t - \frac{\cos^2 t}{3}(1-\sin^2 t) + \frac{\sin^2 t}{3}(1-\cos^2 t) \\ &= \sin^2 t \equiv RHS \end{aligned}$$

- Exercise #10 page 177. Find a particular solution for the following second-order differential equation:

$$y'' + 2y' + y = t^5 e^{-t}.$$

**Solution**: The characteristic equation gives the solution to the homogeneous equation:

$$\begin{aligned} &r^2 + 2r + 1 = 0, \qquad r_{1,2} = -1 && \text{(characteristic equation)} \\ &x_h(t) = c_1 e^{-t} + c_2 t e^{-t}. && \text{(general solution to the homogeneous equation)} \end{aligned}$$

We seek a particular solution to the inhomogeneous equation of the form

$$x_p(t) = v_1 e^{-t} + v_2 t e^{-t},$$

where $v_1, v_2$ are to be found from the system

$$\begin{aligned} v_1' e^{-t} + v_2' t e^{-t} &= 0, & (v_1' y_1 + v_2' y_2 = 0) \\ v_1'(-e^{-t}) + v_2'(e^{-t} - t e^{-t}) &= t^5 e^{-t}, & (v_1' y_1' + v_2' y_2' = RHS) \end{aligned}$$

hence

$$\begin{aligned} v_1' + t v_2' &= 0, & (\cdot \sin t) \\ -v_1' + v_2'(1-t) &= t^5, & (\cdot \cos t) \end{aligned}$$

which gives

$$\begin{aligned} v_2' &= t^5, \qquad v_1' = -t^6, \\ v_1 &= -\frac{t^7}{7}, \qquad v_2 = \frac{t^6}{6}, \end{aligned}$$

and finally

$$x_p(t) = v_1 e^{-t} + v_2 t e^{-t} \equiv -\frac{t^7}{7} e^{-t} + \frac{t^6}{6} t e^{-t} = \frac{t^7}{42} e^{-t}. \qquad \text{(a particular solution)}$$