5.1 Laplace Transform

LT transforms a DE into an algebraic equation

Definition 1 (Laplace transform) *Let $f : (0, \infty) \to \mathbb{R}$. The Laplace transform of f is a function*

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-st}dt, \quad \text{for } s > 0. \qquad \text{(Laplace transform)}$$

Note that variables swapped $t \Leftrightarrow s$.

The integral in the definition is improper. So use a limit to evaluate it.

Example 1 *Let $f(t) = e^{at}$. Then*

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{e^{at}\}(s) = F(s) = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{(a-s)t}dt = \lim_{A\to\infty} \int_0^A e^{(a-s)t}dt$$

$$= \lim_{A\to\infty} \frac{1}{a-s} e^{(a-s)t}\Big|_0^A = \frac{1}{a-s} \lim_{A\to\infty} [e^{(a-s)A} - 1] = \frac{1}{a-s}(0-1) = \frac{1}{s-a}.$$

We assumed that $a - s < 0$ (or $a < s$) so that $\lim_{A\to\infty} e^{(a-s)A} = 0$. Therefore

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \text{for } s \in (a, \infty). \tag{1}$$

Example 2 $f(t) = 1 = e^{0\cdot t}$ $(a = 0)$.

$$\mathcal{L}\{1\}(s) = \mathcal{L}\{e^{at}\}(s) = \frac{1}{s}, \qquad s \in (0, \infty). \tag{2}$$

Example 3 $f(t) = t$.

$$\mathcal{L}\{t\}(s) = \int_0^\infty te^{-st}dt = \lim_{A\to\infty} \int_0^A te^{-st}dt \qquad \text{by parts}: \begin{cases} u = t & dv = e^{-st}dt \\ du = dt & v = -\frac{1}{s}e^{-st} \end{cases} \tag{3}$$

$$\mathcal{L}\{t\}(s) = \lim_{A\to\infty} \left[-\frac{1}{s}te^{-st}\Big|_0^A + \frac{1}{s}\int_0^A e^{-st}dt \right] = \lim_{A\to\infty} \left[-\frac{1}{2}Ae^{-sA} + 0 - \frac{1}{s^2}e^{-st}\Big|_0^A \right]$$

We assume that $s > 0$ so that $\lim_{A\to\infty} e^{-sA} = 0$. Then

$$\mathcal{L}\{t\}(s) = -\frac{1}{s^2} \lim_{A\to\infty} \left[e^{-sA} - 1\right] = \frac{1}{s^2}. \tag{4}$$

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{\sin at\}(s) = \frac{a}{s^2+a^2}, \quad \mathcal{L}\{\cos at\}(s) = \frac{s}{s^2+a^2}$$

(Laplace transform) of piecewise continuous functions

Example 4 $g(t) = \begin{cases} 1\,, & \textit{for } 0 \le t \le 1 \\ 0\,, & \textit{for } t > 1 \end{cases}$

$$\mathcal{L}\{g(t)\}(s) = \int_0^1 1 \cdot e^{-st} dt + \int_1^\infty 0 \cdot e^{-st} dt = \int_0^1 1 \cdot e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^1 = -\frac{1}{s}\left(e^{-s} - 1\right)$$
$$= \frac{1 - e^{-s}}{s} \tag{5}$$

(Laplace transform) of piecewise differentiable functions

Example 5 $f(t) = \begin{cases} 1 - t, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^1 (1-t)e^{-st} dt + \int_1^\infty 0 \cdot e^{-st} dt = \int_0^1 (1-t)e^{-st} dt$$

$$= \int_0^1 e^{-st} dt - \int_0^1 te^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^1 - \left[-\frac{1}{s} te^{-st} - \frac{1}{s^2} e^{-st}\right]_0^1 = -\frac{1}{s} e^{-s} + \frac{1}{s} + \frac{1}{s} e^{-s} + \frac{1}{s^2} e^{-s} - \frac{1}{s^2}$$

$$= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2} e^{-s} = \frac{e^{-s} + s - 1}{s^2}.$$

Definition 2 *A function $f(t)$ is of an exponential order if there are constants C and a such that*

$$|f(t)| \le C\, e^{at} \qquad \textit{for all } t > 0.$$

Theorem 1 *Suppose $f(t)$ is a piecewise continuous function defined on $[0, \infty)$, which is of exponential order. Then the* (Laplace transform) $\mathcal{L}\{f\}(s)$ *exists for large values of s.*

- Exercise #7 page 196. Use the Definition 1 of the Laplace transform to find the Laplace transform of each of the following functions defined for $t > 0$.

$$f(t) = te^{2t}.$$

Solution: For $t > 0$ we have

$$\mathcal{L}\{te^{2t}\}(s) = \int_0^\infty te^{2t}e^{-st}\,dt = \int_0^\infty te^{(2-s)t}\,dt$$
$$= \int_0^\infty t\frac{\left(e^{(2-s)t}\right)'}{2-s}\,dt = t\frac{e^{(2-s)t}}{2-s}\Big|_0^\infty - \int_0^\infty \frac{e^{(2-s)t}}{2-s}\,dt = t\frac{e^{(2-s)t}}{2-s}\Bigg|_{t=0}^{t=\infty} - \int_0^\infty \frac{\left(e^{(2-s)t}\right)'}{(2-s)^2}\,dt$$
$$= t\frac{e^{(2-s)t}}{2-s}\Bigg|_{t=0}^{t=\infty} - \frac{1}{(2-s)^2}e^{(2-s)t}\Bigg|_{t=0}^{t=\infty}$$
$$= \begin{cases} \text{undefined} & \text{if } s < 2 \\ \displaystyle\int_0^\infty t dt = \frac{1}{s^2} & \text{if } s = 2 \\ \frac{1}{(2-s)^2} & \text{if } s > 2. \end{cases}$$

- Exercise #8 page 196. Use the Definition 1 of the (Laplace transform) to find the Laplace transform of each of the following functions defined for $t > 0$.

$$f(t) = te^{-3t}.$$

Solution: For $t > 0$ we have

$$\mathcal{L}\{te^{-3t}\}(s) = \int_0^\infty te^{-3t}e^{-st}\,dt = \int_0^\infty te^{-(3+s)t}\,dt$$
$$= \int_0^\infty t\frac{\left(e^{-(3+s)t}\right)'}{-(3+s)}\,dt = \cancel{t\frac{e^{-(3+s)t}}{-(3+s)}\Big|_0^\infty} + \int_0^\infty \frac{e^{-(3+s)t}}{3+s}\,dt = -\frac{\left(e^{-(3+s)t}\right)'}{(3+s)^2}\Bigg|_0^\infty = \frac{1}{(3+s)^2}.$$

Later on we shall see another way of solving this, using

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s),$$

with $f(t) = e^{-3t}$ and $n = 1$, which by (1) yields $F(s) = \frac{1}{s+3}$, $F'(s) = -\frac{1}{(s+3)^2}$ and therefore

$$\mathcal{L}\{te^{-3t}\}(s) = \frac{1}{(3+s)^2}.$$

- Exercise #11 page 196. Use the Definition 1 of the (Laplace transform) and mathematical induction to show that the Laplace transform of function $f(t) = t^n$ is

$$F_n(s) := \mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}.$$

Solution: Indeed

$$F_n(s) := \mathcal{L}\{t^n\}(s) = \int_0^\infty t^n e^{-st}\,dt = \int_0^\infty \left(\frac{e^{-st}}{-s}\right)' t^n\,dt = \cancel{\frac{e^{-st}}{-s}t^n\Bigg|_{t=0}^{t=\infty}} - \int_0^\infty \frac{e^{-st}}{-s}\cdot nt^{n-1}\,dt$$
$$= \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt \equiv \frac{n}{s}F_{n-1}(s) = \frac{n}{s}\frac{n-1}{s}F_{n-2}(s) = \frac{n(n-1)(n-2)}{s^3}F_{n-3}(s)$$
$$= \cdots = \frac{n(n-1)\cdots 1}{s^n}F_0(s) \equiv \frac{n(n-1)\cdots 1}{s^n}\underbrace{\mathcal{L}\{1\}(s)}_{=1/s} = \frac{n!}{s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}.$$

- Exercise #13 and 14 page 196. Use the Definition 1 of the Laplace transform to show that the (Laplace transform) of functions

$$\mathcal{L}\{e^{at}\cos\omega t\}(s) = \frac{s-a}{(s-a)^2+\omega^2}, \qquad \mathcal{L}\{e^{at}\sin\omega t\}(s) = \frac{\omega}{(s-a)^2+\omega^2}.$$

Solution: Indeed, for all $s \geq a$, using twice integration by parts, we have

$$\begin{aligned}
\mathcal{L}_1 := \mathcal{L}\{e^{at}\sin\omega t\}(s) &= \int_0^\infty e^{at}\sin(\omega t)\,e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\sin(\omega t)\,dt \\
&= \int_0^\infty \Big(\frac{e^{(a-s)t}}{a-s}\Big)'\sin(\omega t)\,dt = \cancel{\frac{e^{(a-s)t}}{a-s}\sin(\omega t)\Big|_{t=0}^{t=\infty}} - \int_0^\infty \frac{e^{(a-s)t}}{a-s}\omega\cos(\omega t)\,dt \\
&= -\frac{\omega}{a-s}\int_0^\infty e^{(a-s)t}\cos(\omega t)\,dt = -\frac{\omega}{a-s}\int_0^\infty \Big(\frac{e^{(a-s)t}}{a-s}\Big)'\cos(\omega t)\,dt \\
&= -\frac{\omega}{a-s}\left(\frac{e^{(a-s)t}}{a-s}\cos(\omega t)\Big|_{t=0}^{t=\infty} - \int_0^\infty \frac{e^{(a-s)t}}{a-s}(-\omega)\sin(\omega t)\,dt\right) \\
&= -\frac{\omega}{a-s}\left(-\frac{1}{a-s} + \frac{\omega}{a-s}\int_0^\infty e^{(a-s)t}\sin(\omega t)\,dt\right) \\
&= \frac{\omega}{(a-s)^2} - \frac{\omega^2}{(a-s)^2}\underbrace{\int_0^\infty e^{(a-s)t}\sin(\omega t)\,dt}_{=\mathcal{L}_1} \\
&= \frac{\omega}{(a-s)^2} - \frac{\omega^2}{(a-s)^2}\mathcal{L}_1.
\end{aligned}$$

Therefore

$$\frac{\omega}{(a-s)^2} = \Big(1+\frac{\omega^2}{(a-s)^2}\Big)\mathcal{L}_1,$$

hence

$$\mathcal{L}_1 = \frac{\omega}{(s-a)^2+\omega^2}$$