5.2 Basic properties of LT

We will use lower-case letters for functions and upper-case letters for their LTs

$$\mathcal{L}\{y\} = Y, \qquad \mathcal{L}\{f\} = F, \qquad \mathcal{L}\{g\} = G, \qquad \text{and etc.}$$

We assume that all functions below and their derivatives are of exponential order.

LT of derivatives

Theorem 1 *Suppose y is a piecewise differentiable function. Then for large values of s*

$$\boxed{\mathcal{L}\{y'\}(s) = sY(s) - y(0)}$$

In general,

$$\boxed{\mathcal{L}\{y^{(k)}\}(s) = s^k Y(s) - s^{k-1}y(0) - \ldots - s^2 y^{(k-3)}(0) - sy^{(k-2)}(0) - y^{(k-1)}(0)}$$

Linearity of LT:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\}(s) = \alpha\mathcal{L}\{f(t)\}(s) + \beta\mathcal{L}\{g(t)\}(s)$$

Example 1 *Transform the IVP* $\qquad y^1 - 3y = e^{2t}, \qquad y(0) = 1$
and solve the resulting equation for LT.

Solution: LT of the left hand side is $L[y' - 3y] = [\text{by linearity}] = L[y'] - 3L[y]$
$= [\text{by Theorem}] = sY(s) - y(0) - 3Y(s) = (s-3)Y(s) - 1.$

LT of the right hand side is $\quad \mathcal{L}\{e^{2t}\}(s) = \dfrac{1}{s-2}$

Hence $(s-3)Y(s) - 1 = \dfrac{1}{s-2} \quad \Rightarrow \quad Y(s) = \dfrac{1}{s-3} + \dfrac{1}{(s-2)(s-3)} = \dfrac{s-1}{(s-2)(s-3)}$

Translation

Proposition 1 (The First Translation Formula)
Assume $f : (0, \infty) \to \mathbb{R}$ a piecewise continuous function of exponential order and $c \in \mathbb{R}$. Then

$$\boxed{\mathcal{L}\{e^{ct}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-c) = F(s-c).} \qquad \text{(First Translation Formula)}$$

Heaviside function (Unit step function): $\quad H(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t \geq 0 \end{cases}$

The Second Translation Formula:

$$\boxed{\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s)} \qquad a > 0$$

The derivative of LT

$$\boxed{\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)} \qquad \text{(derivative of Laplace Transform)}$$

In particular,

$$\boxed{\mathcal{L}\{t\, f(t)\}(s) = -F'(s)}$$

Example 2 *Find LT of t^2e^{5t}.*

Solution: $F(s) = \mathcal{L}\{e^{5t}\}(s) = \dfrac{1}{s-5} \quad \Rightarrow \quad L\left[t^2e^{5t}\right] = (-1)^2F''(s)$.

$F'(s) = -(s-5)^{-2}, \quad F''(s) = 2(s-5)^{-3} \quad \Rightarrow \quad L[t^2e^{5t}] = \dfrac{2}{(s-5)^3}$.

We also can apply the first translation formula:

$$\mathcal{L}\{t^2\}(s) = \frac{2}{s^3}, \quad \mathcal{L}\{t^2e^{5t}\} = \mathcal{L}\{e^{5t}t^2\}(s) = \mathcal{L}\{t^2\}(s-5) = \frac{2}{(s-5)^3}.$$

Example: Find LT of $g(t) = e^{-2t}\cos 4t$.

Solution: By the first translation formula

$$G(s) = \mathcal{L}\{e^{-2t}\cos 4t\}(s) = \mathcal{L}\{\cos 4t\}(s+2) = \frac{s+2}{(s+2)^2+16}.$$

- Exercise #4 page 201. Use the linearity of the Laplace transform to find the Laplace transform of the function

  $$y(t) = 3 - 5t - 11t^3.$$

  <u>`Solution`</u>: For $t > 0$ we have

  $$\mathcal{L}\{y(t)\}(s) = \mathcal{L}\{3-5t-11t^3\}(s) = 3\mathcal{L}\{1\}(s) - 5\mathcal{L}\{t\}(s) - 11\mathcal{L}\{t^3\}(s) = 3\frac{1}{2} - 5\frac{1!}{s^2} - 11\frac{3!}{s^4},$$

  where $s > 0$.

- Exercise #6 page 201. Use the linearity of the Laplace transform to find the Laplace transform of the function

  $$y(t) = 2\sin 3t + 3\cos 5t$$

  <u>`Solution`</u>: For $t > 0$ we have

  $$\mathcal{L}\{y(t)\}(s) = \mathcal{L}\{2\sin 3t + 3\cos 5t\}(s) = 2\mathcal{L}\{\sin 3t\}(s) + 3\mathcal{L}\{\cos 5t\}(s) = 2\frac{3}{s^2+9} + 3\frac{s}{s^2+25},$$

  where $s > 0$.

- Exercise #22 page 202. Transform the given initial value problem into an algebraic equation involving $\mathcal{L}(y) \equiv Y$. Solve the resulting equation for the Laplace transform of y.

$$y'' + y = \sin 4t, \qquad y(0) = 0, \quad y'(0) = 1.$$

<u>Solution</u>: Applying the Laplace transform to the differential equation we obtain

$$s^2Y - sy(0) - y'(0) + Y = \frac{4}{s^2+16}, \qquad (1+s^2)Y = 1 + \frac{4}{s^2+16},$$

$$Y(s) = \frac{1}{s^2+1} + 4\frac{1}{(s^2+16)(s^2+1)} = \frac{1}{s^2+1} + \frac{4}{15}\Big(\frac{1}{s^2+1} - \frac{1}{s^2+16}\Big)$$

$$= \frac{19}{15}\frac{1}{s^2+1} - \frac{4}{15}\frac{1}{s^2+16} = \mathcal{L}\Big\{\frac{19}{15}\sin t - \frac{1}{15}\sin 4t\Big\}(s).$$

- Exercise #27 page 201. Use the (First Translation Formula) to find the Laplace transform of the function

$$y(t) = e^{2t}\cos 2t$$

<u>Solution</u>: For $t > 0$ we have, using (First Translation Formula) with $c = 2$, and $f(t) = \cos 2t$ we have

$$F(s) = \frac{s}{s^2+4}, \qquad Y(s) = F(s-c) \equiv \frac{s-2}{(s-2)^2+4}.$$

- Exercise #28 page 202. Use the (First Translation Formula) to find the Laplace transform of the function

$$y(t) = e^{-2t}(2t+3)$$

<u>Solution</u>: For $t > 0$ we have, using (First Translation Formula) with $c = -2$, and $f(t) = 2t + 3$ we have

$$F(s) = 2\frac{1!}{s^2} + 3\frac{1}{2} \equiv \frac{2+3s}{s^2}, \qquad Y(s) = F(s-c) \equiv \frac{2+3(s+2)}{(s+2)^2} = \frac{3s+8}{(s+2)^2}.$$

- Exercise #30 page 202. Use the (derivative of Laplace Transform) formula to find the Laplace transform of the function

$$y(t) = t\sin 3t$$

<u>Solution</u>: For $t > 0$ we have, using (derivative of Laplace Transform) formula with $n = 1$, and $f(t) = \sin 3t$ we have

$$F(s) = \frac{3}{s^2+9}, \qquad Y(s) = (-1)^n F^{(n)}(s) \equiv -\Big(\frac{3}{s^2+9}\Big)' = \frac{3\cdot 2s}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}.$$

- Exercise #31 page 202. Use the (derivative of Laplace Transform) formula to find the Laplace transform of the function

$$y(t) = te^{-t}$$

Solution: For $t > 0$ we have, using (derivative of Laplace Transform) formula with $n = 1$, and $f(t) = e^{-t}$ we have

$$F(s) = \frac{1}{s+1},$$
$$Y(s) = (-1)^n F^{(n)}(s) \equiv -\Big(\frac{1}{s+1}\Big)' = \frac{1}{(s+1)^2}.$$

- Exercise #32 page 202. Use the (derivative of Laplace Transform) formula to find the Laplace transform of the function

$$y(t) = t^2 \cos 2t$$

Solution: For $t > 0$ we have, using (derivative of Laplace Transform) formula with $n = 2$, and $f(t) = \cos 2t$ we have

$$F(s) = \frac{s}{s^2+4},$$
$$Y(s) = (-1)^n F^{(n)}(s) \equiv \Big(\frac{s}{s^2+4}\Big)'' = \Big(\frac{s^2+4-2s^2}{(s^2+4)^2}\Big)' = \Big(\frac{4-s^2}{(s^2+4)^2}\Big)'$$
$$= \frac{-2s\cdot(s^2+4)^2 - (4-s^2)\cdot 2\cdot 2s\cdot(s^2+4)}{(s^2+4)^3} = \frac{-2s\cdot(s^2+4) - (4-s^2)\cdot 2\cdot 2s}{(s^2+4)^2}$$
$$= \frac{-2s^3 - 8s - 16s + 4s^3}{(s^2+4)^2} = \frac{2s^3 - 24s}{(s^2+4)^2}.$$

- Exercise #33 page 202. Use the (derivative of Laplace Transform) formula to find the Laplace transform of the function

$$y(t) = t^2 e^{2t}$$

Solution: For $t > 0$ we have, using (derivative of Laplace Transform) formula with $n = 2$, and $f(t) = e^{2t}$ we have

$$F(s) = \frac{1}{s-2},$$
$$Y(s) = (-1)^n F^{(n)}(s) \equiv \Big(\frac{1}{s-2}\Big)'' = \frac{2}{(s-2)^3}.$$

- Exercise #34 page 202. Transform the initial value problem

$$y' + 2y = t\sin t, \qquad y(0) = 1$$

into an algebraic equation involving $Y = \mathcal{L}\{y\}$. Solve the resulting equation for the Laplace transform of y.

Solution: For $t > 0$, using (derivative of Laplace Transform) with $f(t) = \sin t$, $F(s) = \frac{1}{s^2+1}$, $F'(s) = -\frac{2s}{(s^2+1)^2}$, and $n = 1$ we have

$$\begin{aligned}
\Big(sY - y(0)\Big) + 2Y &= \mathcal{L}\{t \sin t\}(s), \\
(s+2)Y - 1 &= (-1)^1 F'(s) \equiv \frac{2s}{(s^2+1)^2}, \\
(s+2)Y &= 1 + \frac{2s}{(s^2+1)^2}, \\
Y &= \frac{1}{s+2} + \frac{2s}{(s+2)(s^2+1)^2}
\end{aligned}$$