5.3 Inverse LT

Definition 1 *If $\mathcal{L}\{f(t)\}(s) = F(s)$ and f is continuous of exponential order, then*

$$f = \mathcal{L}^{-1}(F) \qquad \text{(inverse Laplace transform)}$$

is the inverse LT (ILT).

Linearity of ILT: $\mathcal{L}^{-1}[aF + bG] = a\mathcal{L}^{-1}(F) + b\mathcal{L}^{-1}(G) = af + bg.$

A table of basic LTs and ILTs

$f(t)$, ILT	$L[f](s) = F(s)$
1	$\frac{1}{s}$, $s > 0$
t^n	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin at$	$\frac{a}{s^2 + a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2 + a^2}$, $s > 0$
e^{at}	$\frac{1}{s - a}$, $s > a$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$, $s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$e^c f(t)$	$F(s - c)$, $s > c$

Example: Compute ILT of (1) $\frac{1}{s^5}$, (2) $\frac{16}{s^2 + 64} + \frac{1}{s+2}$, (3) $\frac{3s - 2}{s^3(s^2 + 4)}$,

(4) $\frac{1}{s^2 - 2s + 17}$.

Solution: (1) $\mathcal{L}[t^4](s) = \frac{4!}{s^5} \quad \Rightarrow \quad t^4 = \mathcal{L}^{-1}\left[\frac{4!}{s^5}\right](t) = 4!\, \mathcal{L}^{-1}\left[\frac{1}{s^5}\right](t).$

Then $\mathcal{L}^{-1}\left[\frac{1}{s^5}\right](t) = \frac{t^4}{4!} = \frac{t^4}{24}$.

(2) By linearity $\mathcal{L}^{-1}\left[\frac{16}{s^2+64} + \frac{1}{s+2}\right](t) = \mathcal{L}^{-1}\left[\frac{2\cdot 8}{s^2+8^2}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t)$

$= 2\mathcal{L}^{-1}\left[\frac{8}{s^2+8^2}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t) = 2\sin 8t + e^{-2t}$.

(3) Partial fractions: $\frac{3s-2}{s^3(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+4}$

Then $3s-2 = As^2(s^2+4) + Bs(s^2+4) + C(s^2+4) + s^3(Ds+E)$

Set $s=0$ to get $C = -\frac{1}{2}$

We combine and equalize coefficients for

$s^4: \quad A + D = 0$
$s^3: \quad B + E = 0$
$s^2: \quad 4A + C = 0$
$s: \quad 4B = 3$

to get $A = \frac{1}{8}$, $B = \frac{3}{4}$, $D = -\frac{1}{8}$, $E = -\frac{3}{4}$. Therefore

$\mathcal{L}^{-1}\left[\frac{3s-2}{s^3(s^2+4)}\right](t) =$ [by linearity]

$= \frac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s}\right](t) + \frac{3}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^3}\right](t) - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right](t) - \frac{3}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2+2^2}\right](t)$

$= \frac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s}\right](t) + \frac{3}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{2}{s^3}\right](t) - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right](t) - \frac{3}{8}\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right](t)$

$= \frac{1}{8} + \frac{3}{4}t - \frac{1}{4}t^2 - \frac{1}{8}\cos 2t - \frac{3}{8}\sin 2t$.

(4) $\frac{1}{s^2-2s+17} = \frac{1}{(s-1)^2+4^2} = \frac{1}{2}\cdot\frac{2}{(s-1)^2+2^2}$

$\mathcal{L}^{-1}\left[\frac{1}{s^2-2s+17}\right](t) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s-1)^2+2^2}\right](t) = \frac{1}{2}e^t\sin 2t$.

- Exercise #4 page 208. Find the (inverse Laplace transform) of the function

$$\mathcal{L}^{-1}\left\{\frac{5s}{s^2+9}\right\} = 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = 5\cos 3t.$$

- Exercise #5 page 208. Find the (inverse Laplace transform) of the function

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 3t.$$

- Exercise #6 page 208. Find the (inverse Laplace transform) of the function

$$\mathcal{L}^{-1}\left\{\frac{2}{s^4}\right\} = \frac{2}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{3}t^3.$$

- Exercise #8 page 208. Find the (inverse Laplace transform) of the function

$$\mathcal{L}^{-1}\left\{\frac{2-5s}{s^2+9}\right\} = \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} - 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \frac{2}{3}\sin 3t - 5\cos 3t.$$

- Exercise #12 page 209. Find the (inverse Laplace transform) of the function

$$Y(s) = \frac{1}{(s-1)^6}.$$

<u>Solution</u>: We write

$$Y(s) = \frac{1}{(s-1)^6} \equiv F(s-1) = \mathcal{L}\{e^t f(t)\}$$

where

$$F(s) = \frac{1}{s^6}, \qquad f(t) = \frac{1}{5!}t^5,$$

hence

$$y(t) = \frac{1}{5!}t^5 e^t.$$

- Exercise #13 page 209. Find the (inverse Laplace transform) of the function

$$Y(s) = \frac{3}{(s+2)^2+25} \equiv \frac{3}{5}\frac{5}{(s+2)^2+25} = \mathcal{L}^{-1}\{\frac{3}{5}e^{-2t}\sin 5t\}.$$

- Exercise #14 page 209. Find the (inverse Laplace transform) of the function

$$Y(s) = \frac{4(s-1)}{(s-1)^2+4} \equiv 4\frac{s-1}{(s-1)^2+2^2}, \qquad y(t) = 4e^t\cos 2t.$$

- Exercise #15 page 209. Find the (inverse Laplace transform) of the function

$$\begin{aligned} Y(s) &= \frac{2s-3}{(s-1)^2+5} \equiv \frac{2(s-1)}{(s-1)^2+5} - \frac{1}{(s-1)^2+5} \\ &= 2\frac{s-1}{(s-1)^2+(\sqrt{5})^2} - \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{(s-1)^2+(\sqrt{5})^2} \\ &= 2\frac{s-1}{(s-1)^2+(\sqrt{5})^2} - \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{(s-1)^2+(\sqrt{5})^2}, \\ y(t) &= 2e^t\cos\sqrt{5}t - \frac{1}{\sqrt{5}}e^t\sin\sqrt{5}t. \end{aligned}$$

- Exercise #16 page 209. Find the (inverse Laplace transform) of the function

$$\begin{aligned} Y(s) &= \frac{s}{(s+2)^2+4} \equiv \frac{s+2}{(s+2)^2+2^2} - \frac{2}{(s+2)^2+2^2} \\ y(t) &= e^{-t}\cos 2t - e^{-t}\sin 2t. \end{aligned}$$

- Exercise #17 page 209. Find the (inverse Laplace transform) of the function

$$\begin{aligned} Y(s) &= \frac{3s+2}{s^2+4s+29} \equiv \frac{3s+2}{(s+2)^2+25} = \frac{3(s+2)-4}{(s+2)^2+5^2} \\ &= 3\frac{s+2}{(s+2)^2+5^2} - \frac{4}{5}\frac{5}{(s+2)^2+5^2} \\ y(t) &= 3e^{-2t}\cos 5t - \frac{4}{5}e^{-2t}\sin 5t. \end{aligned}$$

- Exercise #18 page 209. Find the (inverse Laplace transform) of the function

$$\begin{aligned} Y(s) &= \frac{5-2s}{s^2-2s+5} \equiv \frac{5-2s}{(s-1)^2+4} = \frac{-2(s-1)+3}{(s-1)^2+4} \\ &= -2\frac{s-1}{(s-1)^2+2^2} + \frac{3}{2}\frac{2}{(s-1)^2+2^2}, \\ y(t) &= -2e^{t}\cos 2t + \frac{3}{2}e^{t}\sin 2t. \end{aligned}$$

- Exercise #19 page 209. Perform the appropriate partial fraction decomposition, and then use the result to find the Laplace transform of the function

$$Y(s) = \frac{1}{(s+2)(s-1)}$$

Solution: Use partial fractions

$$\frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

to obtain

$$1 = A(s-1) + B(s+2)$$

which, by setting $s = 1$ and $s = -2$, gives

$$A = -\frac{1}{3}, \qquad B = \frac{1}{3},$$

and therefore

$$Y(s) = \frac{1}{(s+2)(s-1)} \equiv -\frac{1}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1}.$$

Finally,

$$y(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}.$$

- Exercise #20 page 209. Perform the appropriate partial fraction decomposition, and then use the result to find the Laplace transform of the function

$$Y(s) = \frac{1}{(s+3)(s-4)}$$

Solution: Use partial fractions

$$\frac{1}{(s+3)(s-4)} = \frac{A}{s+3} + \frac{B}{s-4} = \frac{A(s-4)+B(s+3)}{(s+3)(s-4)}$$

to obtain

$$1 = A(s-4) + B(s+3)$$

which, by setting $s = 4$ and $s = -3$, gives

$$A = -\frac{1}{7}, \qquad B = \frac{1}{7},$$

and therefore

$$Y(s) = \frac{1}{(s+3)(s-4)} \equiv -\frac{1}{7}\frac{1}{s+3} + \frac{1}{7}\frac{1}{s-4}.$$

Finally,

$$y(t) = -\frac{1}{7}e^{-3t} + \frac{1}{7}e^{4t}.$$

- Exercise #21 page 209. Perform the appropriate partial fraction decomposition, and then use the result to find the Laplace transform of the function

$$Y(s) = \frac{2s-1}{(s+1)(s-2)}$$

Solution: Use partial fractions

$$\frac{2s-1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} = \frac{A(s-2)+B(s+1)}{(s+1)(s-2)}$$

to obtain

$$2s-1 = A(s-2) + B(s+1)$$

which, by setting $s = 2$ and $s = -1$, gives

$$A = 1, \qquad B = 1,$$

and therefore

$$Y(s) = \frac{2s-1}{(s+1)(s-2)} \equiv \frac{1}{s+1} + \frac{1}{s-2}.$$

Finally,

$$y(t) = e^{-t} + e^{2t}.$$

- Exercise #22 page 209. Perform the appropriate partial fraction decomposition, and then use the result to find the Laplace transform of the function

$$Y(s) = \frac{2s-2}{(s-4)(s+2)}$$

<u>**Solution**</u>: Use partial fractions

$$\frac{2s-2}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2} = \frac{A(s+2)+B(s-4)}{(s-4)(s+2)}$$

to obtain

$$2s-2 = A(s+2)+B(s-4)$$

which, by setting $s=-2$ and $s=4$, gives

$$A = 1, \qquad B = 1,$$

and therefore

$$Y(s) = \frac{1}{s-4} + \frac{1}{s+2}.$$

Finally,

$$y(t) = e^{4t} + e^{-2t}.$$

- Exercise #23 page 209. Perform the appropriate partial fraction decomposition, and then use the result to find the Laplace transform of the function

$$Y(s) = \frac{7s+13}{s^2+2s-3}$$

<u>**Solution**</u>: Since $s^2+2s-3 = (s+3)(s-1)$, we use partial fractions as follows

$$\frac{7s+13}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1} = \frac{A(s-1)+B(s+3)}{(s+3)(s-1)}$$

to obtain

$$7s+13 = A(s-1)+B(s+3)$$

which, by setting $s=1$ and $s=-3$, gives

$$A = 2, \qquad B = 5,$$

and therefore

$$Y(s) = 2\frac{1}{s+3} + 5\frac{1}{s-1}.$$

Finally,

$$y(t) = 2e^{-3t} + 5e^{t}.$$

- Exercise #24 page 209. Perform the appropriate partial fraction decomposition, and then use the result to find the Laplace transform of the function

$$Y(s) = \frac{7-s}{s^2+s-2}$$

Solution: Since $s^2+s-2 = (s+2)(s-1)$, we use partial fractions as follows

$$\frac{7-s}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

to obtain

$$7-s = A(s-1)+B(s+2)$$

which, by setting $s=1$ and $s=-2$, gives

$$A=-3, \qquad B=2,$$

and therefore

$$Y(s) = -3\frac{1}{s+2} + 2\frac{1}{s-1}.$$

Finally,

$$y(t) = -3e^{-2t} + 2e^{t}.$$

- Exercise #25 page 209. Perform the appropriate partial fraction decomposition, and then use the result to find the Laplace transform of the function

$$Y(s) = \frac{13s-5}{2s^2-s}$$

Solution: Since $2s^2-s = s(2s-1)$, we use partial fractions as follows

$$\frac{13s-5}{s(2s-1)} = \frac{A}{s} + \frac{B}{2s-1} = \frac{A(2s-1)+Bs}{s(2s-1)}$$

to obtain

$$13s-5 = A(2s-1)+Bs$$

which, by setting $s=\frac{1}{2}$ and $s=0$, gives

$$A=5, \qquad B=3,$$

and therefore

$$Y(s) = 5\frac{1}{s} + 3\frac{1}{2s-1} = 5\frac{1}{s} + \frac{3}{2}\frac{1}{s-\frac{1}{2}}.$$

Finally,

$$y(t) = 5e^{t} + \frac{3}{2}e^{t/2}.$$

- Exercise #27 page 209. Perform the appropriate partial fraction decomposition, and then use the result to find the Laplace transform of the function

$$Y(s) = \frac{7s^2 + 3s + 16}{(s+1)(s^2+4)}$$

<u>Solution</u>: We use partial fractions as follows

$$\frac{7s^2 + 3s + 16}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs + C}{s^2+4} = \frac{A(s^2+4) + (Bs+C)(s+1)}{(s+1)(s^2+4)}$$

to obtain

$$7s^2 + 3s + 16 = A(s^2+4) + (Bs+C)(s+1)$$

which, by setting $s = -1$ gives

$$A = 4,$$

while with $s = 2i$, gives

$$\underbrace{-28 + 6i + 16}_{=6i-12} = \underbrace{(2iB + C)(2i+1)}_{-4B+C+i(2C+2B)},$$
$$6 = 2(B+C), \qquad -12 = -4B + C,$$
$$B = 3, \qquad C = 0,$$

and therefore

$$Y(s) = 4\frac{1}{s+1} + 3\frac{s}{s^2+2^2}.$$

Finally,

$$y(t) = 4e^{-t} + 3\cos 2t.$$

- Exercise #28 page 209. Perform the appropriate partial fraction decomposition, and then use the result to find the Laplace transform of the function

$$Y(s) = \frac{3s^2 + s + 1}{(s-2)(s^2+1)}$$

<u>Solution</u>: We use partial fractions as follows

$$\frac{3s^2 + s + 1}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs + C}{s^2+1} = \frac{A(s^2+4) + (Bs+C)(s+1)}{(s-2)(s^2+1)}$$

to obtain

$$3s^2 + s + 1 = A(s^2+1) + (Bs+C)(s-2)$$

which, by setting $s = 2$ gives

$$A = 3,$$

while with $s = i$, gives

$$\underbrace{-3 + i + 1}_{=i-2} = \underbrace{(iB + C)(i - 2)}_{i(-2B+C)-B-2C},$$
$$1 = -2B + C, \qquad -2 = -B - 2C,$$
$$B = 0, \qquad C = 1,$$

and therefore

$$Y(s) = 3\frac{1}{s-2} + \frac{1}{s^2+1}.$$

Finally,

$$y(t) = 3e^{2t} + \sin t.$$

- Exercise #29 page 209. Perform the appropriate partial fraction decomposition, and then use the result to find the Laplace transform of the function

$$Y(s) = \frac{2s^2 + 9s + 11}{(s+1)(s^2 + 4s + 5)}$$

Solution: We use partial fractions as follows

$$\frac{2s^2 + 9s + 11}{(s+1)(s^2 + 4s + 5)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 4s + 5} = \frac{A(s^2 + 4s + 5) + (Bs + C)(s+1)}{(s+1)\big((s+2)^2 + 1\big)}$$

to obtain

$$2s^2 + 9s + 11 = A(s^2 + 4s + 5) + (Bs + C)(s + 1)$$

which, by setting $s = -1$ gives

$$A = 2,$$

while with $s = 0$, $s = 1$, gives

$$B = 0, \quad C = 1$$

and therefore

$$Y(s) = 2\frac{1}{s+1} + \frac{1}{(s+2)^2 + 1}.$$

Finally,

$$y(t) = 2e^{-t} + e^{-2t}\sin t.$$

- Exercise #32 page 209. Perform the appropriate partial fraction decomposition, and then use the result to find the Laplace transform of the function

$$Y(s) = \frac{1}{(s-1)^2(s^2 + 4)}$$

Solution: We use partial fractions as follows

$$\frac{1}{(s-1)^2(s^2+4)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+4} = \frac{\big(A(s-1)+B\big)(s^2+4) + (Cs+D)(s-1)^2}{(s-1)^2(s^2+4)}$$

to obtain

$$1 = \big(A(s-1)+B\big)(s^2+4) + (Cs+D)(s-1)^2$$

which, by setting $s=1$ gives

$$B = \frac{1}{5},$$

while $s = 2i$, gives

$$\begin{aligned} 1 &= (2iC+D)(2i-1)^2 \equiv (2iC+D)(-4+4i+1)(2iC+D)(-4i-3) \\ &= i(-6C-4D) + 8C - 3D, \end{aligned}$$

i.e,

$$C = \frac{2}{25}, \qquad D = -\frac{3}{25}.$$

Then $s = 0$ yields

$$\begin{aligned} &1 = 4(-A+B) + D \equiv -4A + \frac{4}{5} - \frac{3}{25}, \\ &A = -\frac{2}{25} \end{aligned}$$

and therefore

$$Y(s) = -\frac{2}{25}\frac{1}{s-1} + \frac{1}{5}\frac{1}{(s-1)^2} + \frac{2}{25}\frac{s}{s^2+4} - \frac{3}{2\cdot 25}\frac{2}{s^2+4}.$$

Finally,

$$y(t) = -\frac{2}{25}e^t + \frac{1}{5}te^t + \frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t.$$