5.4 Using LT to solve IVPs

Example: Solve the IVP $y'' - y = e^{2t}, \quad y(0) = 0, \quad y'(0) = 1$

Solution: LT of the left hand side is

$\mathcal{L}\{y'' - y\} = \mathcal{L}\{y''\} - \mathcal{L}\{y\} = s^2Y - sy(0) - y'(0) - Y = (s^2 - 1)Y - 1.$

LT of the right hand side is $\mathcal{L}\{e^{2t}\} = \dfrac{1}{s-2}$. Therefore

$$(s^2 - 1)Y = 1 + \frac{1}{s-2} = \frac{s-2+1}{s-2} = \frac{s-1}{s-2}$$

$$Y = \frac{1}{(s+1)(s-2)} = \tfrac{1}{3}\left(\frac{1}{s-2} - \frac{1}{s+1}\right)$$

Then $y(t) = \mathcal{L}^{-1}(Y) = \tfrac{1}{3}\mathcal{L}^{-1}\left[\dfrac{1}{s-2}\right] - \tfrac{1}{3}\mathcal{L}^{-1}\left[\dfrac{1}{s+1}\right] = \tfrac{1}{3}\left(e^{2t} - e^{-t}\right).$

Example: Solve the IVP $y'' - 6y' + 9y = t^2e^{3t}, \quad y(0) = 2, \quad y'(0) = 6.$

Solution: $\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2e^3t\}$

$$s^2Y - sy(0) - y'(0) - 6\left(sY - y(0)\right) + 9Y = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y = 2s - 6 + \frac{2}{(s-3)^3}$$

$$(s-3)^2y = 2(s-3) + \frac{2}{(s-3)^3}$$

$$(s-3)^2y = 2(s-3) + \frac{2}{(s-3)^3}$$

$$Y(s) = \frac{2}{s-3} + \frac{2}{(s-3)^5}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\{\frac{1}{s-3}\} + \tfrac{2}{4!}\mathcal{L}^{-1}\{\frac{4!}{(s-3)^5}\} = 2e^{3t} + \tfrac{1}{12}t^4e^{3t}$$

Example: Solve the IVP $y'' + 4y' + 6y = 1 + e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$

Solution: $\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{1\} + \mathcal{L}\{e^{-t}\}$

$$s^2Y + 4sY + 6Y = \tfrac{1}{s} + \frac{1}{s+1}$$

$$(s^2 + 4s + 6)Y = \frac{2s+1}{s(s+1)}$$

$$Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)}$$

Partial fraction decomposition gives:

$$Y(s) = \tfrac{1}{6} \cdot \frac{1}{s} + \tfrac{1}{3} \cdot \frac{1}{s+1} - \frac{\frac{1}{2}s + \frac{5}{3}}{s^2+4s+6}.$$

$$\frac{\frac{1}{2}s + \frac{5}{3}}{s^2+4s+6} = \frac{\frac{1}{2}(s+2) + \frac{2}{3}}{(s+2)^2+2} = \tfrac{1}{2} \cdot \frac{s+2}{(s+2)^2+2} + \tfrac{2}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{(s+2)^2+2}$$

$$y(t) = \tfrac{1}{6}\mathcal{L}^{-1}\{\frac{1}{s}\} + \tfrac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \tfrac{1}{2}\mathcal{L}^{-1}\{\frac{s+2}{(s+2)^2+2}\} - \tfrac{\sqrt{2}}{3}\mathcal{L}^{-1}\{\frac{\sqrt{2}}{(s+2)^2+2}\}$$

$$= \tfrac{1}{6} + \tfrac{1}{3}e^{-t} - \tfrac{1}{2}e^{-2t}\cos\sqrt{2}t - \tfrac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t.$$

(by using the first translation formula or from the table).

Example: Solve the IVP $x'' + 16x = \cos 4t, \quad x(0) = 0, \quad x'(0) = 1.$

Solution: $\mathcal{L}\{x''\} + 16\mathcal{L}\{x\} = \mathcal{L}\{\cos 4t\}.$

$$s^2X(s) - x'(0) + 16X(s) = \frac{s}{s^2+16^2}.$$

$$(s^2+16)X = 1 + \frac{s}{s^2+16} \quad \Rightarrow \quad X = \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2}.$$

Using the derivative of LT $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)$ we find

$$\mathcal{L}\{t\sin 4t\} = -F'(s) \text{ where } F(s) = \mathcal{L}\{\sin 4t\} = \frac{4}{s^2+16} = 4(s^2+16)^{-1}.$$

Then $F'(s) = -4(s^2+16)^{-2} \cdot 2s = -\dfrac{8s}{(s^2+16)^2}.$

Hence $\mathcal{L}\{t\sin 4t\} = \dfrac{8s}{(s^2+16)^2}.$ Then $\mathcal{L}^{-1}\{\dfrac{8s}{(s^2+16)^2}\} = t\sin 4t.$

So we have: $x(t) = \mathcal{L}^{-1}\{\dfrac{1}{s^2+16}\} + \mathcal{L}^{-1}\{\dfrac{s}{(s^2+16)^2}\} = \tfrac{1}{4}\mathcal{L}^{-1}\{\dfrac{4}{s^2+4^2}\} + \tfrac{1}{8}\mathcal{L}^{-1}\{\dfrac{8s}{(s^2+16)^2}\}$
$= \tfrac{1}{4}\sin 4t + \tfrac{1}{8}t\sin 8t.$

- Exercise 0. Use the Laplace transform to solve the third-order initial value problem

$$y''' - y = \frac{t^2}{2}, \qquad y(0) = 0, y'(0) = 0, y''(0) = -1.$$

Solution: For $t > 0$ we have

$$\big(s^3Y - s^2\cancel{y(0)} - s\cancel{y'(0)} - \underbrace{y''(0)}_{=-1}\big) - Y = \frac{1}{2}\frac{2!}{s^3},$$

$$(s^3-1)Y + 1 = \frac{1}{s^3}, \qquad Y(s) = -\frac{1}{s^3},$$

$$y(t) = -\frac{1}{2}t^2.$$

Check the solution!

- Exercise #2 page 214. Use the Laplace transform to solve the first-order initial value problem

$$y' + 9y = e^{-t}, \qquad y(0) = 0.$$

Solution: For $t > 0$ we have

$$\big(sY - \cancel{y(0)}\big) + 9Y = \frac{1}{s+1},$$

$$(s+9)Y = \frac{1}{s+1}, \qquad Y(s) = \frac{1}{(s+1)(s+9)} = \frac{1}{8}\frac{1}{s+1} - \frac{1}{8}\frac{1}{s+9},$$

$$y(t) = \frac{1}{8}e^{-t} - \frac{1}{8}e^{-9t}.$$

- Exercise #6 page 214. Use the Laplace transform to solve the first-order initial value problem

$$y' + 8y = t^2, \qquad y(0) = -1.$$

Solution: For $t > 0$ we have

$$\big(sY - \underbrace{y(0)}_{=-1}\big) + 8Y = \frac{2}{s^3},$$

$$(s+8)Y = -1 + \frac{2}{s^3},$$

$$Y(s) = -\frac{1}{s+8} + \frac{2}{s^3(s+8)} = -\frac{1}{s+8} + \Big(-\frac{1}{256}\frac{1}{s+8} + \frac{1}{256}\frac{1}{s} - \frac{1}{32}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s^3}\Big)$$

$$= -\frac{257}{256}\frac{1}{s+8} + \frac{1}{256}\frac{1}{s} - \frac{1}{32}\frac{1}{s^2} + \frac{1}{4}\frac{2!}{s^3}\frac{1}{2}$$

Therefore

$$y(t) = -\frac{257}{256}e^{-8t} + \frac{1}{256} - \frac{1}{32}t + \frac{1}{8}t^2.$$

Here we used

$$\frac{2}{s^3(s+8)} = \frac{A}{s+8} + \frac{B}{s} + \frac{C}{s^2} + \frac{D}{s^3} = \frac{As^3 + (s+8)(Bs^2 + Cs + D)}{s^3(s+8)},$$

where

$$A = -\frac{1}{256} \qquad \text{(s=-8)}$$

$$D = \frac{1}{4} \qquad \text{(s=0)}$$

$$2 = -\frac{1}{256} + 9(B + C + \frac{1}{4}), \qquad 2 = \frac{1}{256} + 7(B - C + \frac{1}{4}), \qquad \text{(s=1,s=-1)}$$

$$B = \frac{1}{256}, \quad , C = -\frac{1}{32}.$$

- Exercise #8 page 214. Use the Laplace transform to solve the first-order initial value problem

$$y' - 2y = e^{-t}\cos t, \qquad y(0) = -2.$$

Solution: For $t > 0$ we have

$$\begin{aligned}
\big(sY - \underbrace{y(0)}_{=-2}\big) - 2Y &= \mathcal{L}\{e^{at}\cos\omega t\} \overset{a=-1,\omega=1}{=\!=\!=} \frac{s-a}{(s-a)^2+\omega^2} \equiv \frac{s+1}{(s+1)^2+1},\\
(s-2)Y &= -2 + \frac{s+1}{(s+1)^2+1},\\
Y(s) &= -\frac{2}{s-2} + \frac{s+1}{(s-2)\big((s+1)^2+1\big)} = -\frac{2}{s-2} + \left[\frac{3}{10}\frac{1}{s-2} + \frac{-\frac{3}{10}s - \frac{1}{5}}{(s+1)^2+1}\right]\\
&= -\frac{17}{10}\frac{1}{s-2} + \Big(-\frac{3}{10}(s+1) + \frac{1}{10}\Big)\frac{1}{(s+1)^2+1}\\
&= -\frac{17}{10}\frac{1}{s-2} - \frac{3}{10}\frac{s+1}{(s+1)^2+1} + \frac{1}{10}\frac{1}{(s+1)^2+1},\\
y(t) &= -\frac{17}{10}e^{2t} - \frac{3}{10}e^{-t}\cos t + \frac{1}{10}e^{-t}\sin t.
\end{aligned}$$

Here we used

$$\frac{s+1}{(s-2)\big((s+1)^2+1\big)} =$$

- Exercise #9 page 214. Use the Laplace transform to solve the first-order initial value problem

$$y' + y = te^{t}, \qquad y(0) = -2.$$

<u>`Solution`</u>: For $t > 0$ we have

$$\big(sY - \underbrace{y(0)}_{=-2}\big) + Y = \mathcal{L}\{e^{at}f(t)\} \xlongequal{a=1, f(t)=t, F(s)=\frac{1}{s^2}} F(s-a) \equiv \frac{1}{(s-1)^2},$$

$$(s+1)Y = -2 + \frac{1}{(s-1)^2},$$

$$Y(s) = -\frac{2}{s+1} + \frac{1}{(s+1)(s-1)^2} = -\frac{2}{s+1} + \left[\frac{\frac{1}{4}}{s+1} + \frac{-\frac{1}{4}}{s-1} + \frac{\frac{1}{2}}{(s-1)^2}\right]$$

$$= -\frac{7}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2},$$

$$y(t) = -\frac{7}{4}e^{-t} - \frac{1}{4}e^{t} + \frac{1}{2}te^{t}.$$

Here we used

$$\frac{1}{(s+1)(s-1)^2} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$= \frac{A(s-1)^2 + (s+1)\big(B(s-1)+C\big)}{(s-1)^2},$$

$$C = \frac{1}{2}, \qquad (s=1)$$

$$A = \frac{1}{4}, \qquad (s=-1)$$

$$1 = \frac{1}{4} - B + \frac{1}{2}, \qquad B = -\frac{1}{4}. \qquad (s=0)$$

- Exercise #10 page 214. Use the Laplace transform to solve the first-order initial value problem

$$y' - 4y = e^{-2t}t^2, \qquad y(0) = 1.$$

<u>`Solution`</u>: For $t > 0$ we have

$$\big(sY - \underbrace{y(0)}_{=1}\big) - 4Y = \mathcal{L}\{e^{at}f(t)\} \xlongequal{a=-2, f(t)=t^2, F(s)=\frac{2}{s^3}} F(s-a) \equiv \frac{2}{(s+2)^3},$$

$$(s-4)Y = 1 + \frac{2}{(s+1)^3},$$

$$Y(s) = \frac{1}{s-4} + \frac{2}{(s-4)(s+1)^3} = \frac{1}{s-4} + \left[\frac{1}{108}\frac{1}{s-4} - \frac{1}{108}\frac{1}{s+2} - \frac{1}{18}\frac{1}{(s+2)^2} - \frac{1}{3}\frac{1}{(s+2)^3}\right]$$

$$= \frac{109}{108}\frac{1}{s-4} - \frac{1}{108}\frac{1}{s+2} - \frac{1}{18}\frac{1}{(s+2)^2} - \frac{1}{6}\frac{2!}{(s+2)^3}$$

$$y(t) = \frac{109}{108}e^{4t} - \frac{1}{108}e^{-2t} - \frac{1}{18}te^{-2t} - \frac{1}{6}t^2e^{-2t}.$$

Here we used

$$\frac{2}{(s-4)(s+1)^3} = \frac{A}{s-4} - \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$
$$= \frac{A(s+2)^3 + (s-4)\big(B(s+2)^2 + C(s+2) + D\big)}{(s-4)(s+2)^3},$$
$$A = \frac{1}{108}, \qquad (s=4)$$
$$D = -\frac{1}{3}, \qquad (s=-2)$$
$$2 = 8\frac{1}{108} - 4\big(4B + 2C - \frac{1}{3}\big), \qquad 2 = \frac{1}{108} - 5\big(B + C - \frac{1}{3}\big), \qquad (s=0, s=-1)$$
$$B = -\frac{1}{108}, \qquad C = -\frac{1}{18}.$$

- Exercise #20 page 214. Use the Laplace transform to solve the second-order initial value problem

$$y'' - 2y' - 3y = e^{4t}, \qquad y(0) = 1, y'(0) = -1.$$

Solution: For $t > 0$ we have

$$\big(s^2Y - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{-1}\big) - 2\big(sY - \underbrace{y(0)}_{1}\big) - 3Y = \frac{1}{s-4},$$
$$\underbrace{(s^2 - 2s - 3)}_{(s+1)(s-3)}Y = s - 3 + \frac{1}{s-4}$$
$$Y = \frac{1}{s+1} + \frac{1}{(s+1)(s-3)(s-4)} = \frac{1}{s+1} + \left(\frac{1}{20}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-3} + \frac{1}{5}\frac{1}{s-4}\right)$$
$$= \frac{21}{20}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-3} + \frac{1}{5}\frac{1}{s-4},$$
$$y(t) = \frac{21}{20}e^{-t} - \frac{1}{4}\,e^{3t} + \frac{1}{5}e^{4t}.$$

$$\frac{1}{(s+1)(s-3)(s-4)} = \frac{A}{s+1} + \frac{B}{s-3} + \frac{C}{s-4}$$
$$= \frac{A(s-3)(s-4) + B(s+1)(s-4) + C(s+1)(s-3)}{(s+1)(s-3)(s-4)},$$
$$B = -\frac{1}{4}, \qquad (s=3)$$
$$C = \frac{1}{5}, \qquad (s=4)$$
$$A = \frac{1}{20}. \qquad (s=-1)$$

- Exercise #26 page 214. Use the Laplace transform to solve the second-order initial value problem

$$y'' + 9y = e^{-t}\sin t, \qquad y(0) = -1, y'(0) = 1.$$

Solution: For $t > 0$ we have

$$\big(s^2Y - s\underbrace{y(0)}_{-1} - \underbrace{y'(0)}_{1}\big) + 9Y = \mathcal{L}\{e^{ct}f(t)\} \xlongequal{c=-1, f(t)=\sin t, F(s)=\frac{1}{s^2+1}} F(s-c)$$
$$= \frac{1}{(s+1)^2+1},$$

hence

$$(s^2+9)Y = -s + 1 + \frac{1}{(s+1)^2+1}$$
$$Y = -\frac{s-1}{s^2+9} + \frac{1}{\big((s+1)^2+1\big)(s^2+9)}$$
$$y(t) = -\frac{87}{85}\cos 3t + \frac{26}{85}\sin 3t + \frac{2}{85}e^{-t}\cos t + \frac{9}{85}e^{-t}\sin t.$$

$$\frac{1}{\big((s+1)^2+1\big)(s^2+9)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{\big((s+1)^2+1\big)}$$
$$= \frac{(As+B)\big((s+1)^2+1\big) + (Cs+D)(s^2+9)}{(s^2+9)\big((s+1)^2+1\big)},$$

hence

$$1 = (3iA+B)\big((3i+1)^2+1\big) \equiv (3iA+B)\big(-7+6i\big) \quad (s=3i)$$
$$= i(-21A+6B) + (-18A-7B),$$

$$A = -\frac{2}{85}, \qquad B = -\frac{7}{85},$$

$$1 = (Ci - C + D)\big((i-1)^2+9\big) \equiv (Ci - C + D)\big(9-2i\big) \quad (s = i-1)$$
$$= i(9C + 2C - 2D) + 2C - 9C + 9D \equiv i(11C - 2D) - 7C + 9D,$$

$$C = \frac{2}{85}, \qquad D = \frac{11}{85},$$

so

$$Y(s) = -\frac{s-1}{s^2+9} + \frac{-\frac{2}{85}s - \frac{7}{85}}{s^2+9} + \frac{\frac{2}{85}s + \frac{11}{85}}{\big((s+1)^2+1\big)}$$
$$= \frac{-\frac{87}{85}s + \frac{78}{85}}{s^2+9} + \frac{\frac{2}{85}(s+1) - \frac{2}{85} + \frac{11}{85}}{\big((s+1)^2+1\big)}$$
$$= -\frac{87}{85}\frac{s}{s^2+9} + \frac{78}{3\cdot 85}\frac{3}{s^2+9} + \frac{2}{85}\frac{s+1}{\big((s+1)^2+1\big)} + \frac{9}{85}\frac{1}{\big((s+1)^2+1\big)}$$