5.5 Discontinuous Forcing Terms

Definition 1

$$H_c(t) = H(t-c) = \begin{cases} 0\,, & t < c \\ 1\,, & t \geq c \end{cases} \qquad \text{(Heaviside function)}$$

$$H_{ab}(t) = \begin{cases} 0\,, & t < a \\ 1\,, & a \leq t < b \\ 0\,, & t > b \end{cases} \qquad \text{(interval function)}$$

Remark 1 *Then* $H_{ab}(t) = H_a(t) - H_b(t) = H(t-a) - H(t-b)$ *(check this!).*

Example 1 *Describe the function* $g(t) = \begin{cases} 2t-1\,, & 0 \leq t < 3 \\ 5\,, & t \geq 3 \end{cases}$ *in terms of* H*-functions.*

Solution: $g(t) = (2t-1)H_{03}(t) + 5H_3(t) = (2t-1)(H(t) - H_3(t) + 5H_3(t)$

$= (2t-1)H(t) + (-2t+1+5)H_3(t) = (2t-1)H(t) - 2(t-3)H(t-3).$

Laplace transform of the Heaviside H-function

Proposition 1 (LT of the Heaviside function) $\mathcal{L}\{H_c(t)\}(s) = \dfrac{e^{-cs}}{s}.$

In particular, $\mathcal{L}\{H(t)\}(s) = \dfrac{1}{s}$, $\mathcal{L}\{H_{ab}(t)\}(s) = \mathcal{L}\{H_a(t)\}(s) - \mathcal{L}\{H_b(t)\}(s) = \dfrac{e^{-as} - e^{-bs}}{s}$.

Proposition 2 (The second translation formula)

$$\mathcal{L}\left[H(t-c)f(t-c)\right](s) = e^{-cs}F(s), \qquad \forall c > 0. \qquad \text{(second translation formula)}$$

Example: Find LT of $g(t)$ (see the previous example).

Solution: $L\left[g(t)\right](s) = L\left[(2t-1)H(t) - 2(t-3)H(t-3)\right](s)$

$= 2L\left[tH(t)\right] - L\left[H(t)\right] - 2L\left[(t-3)H(t-3)\right]$

By the (second translation formula) $\mathcal{L}\{g(t)\}(s) = \dfrac{2}{s^2} - \dfrac{1}{s} - 2\dfrac{e^{-3s}}{s^2}.$

Proposition 3 (Inverse of 2nd translation formula)

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\}(t) = H(t-c)f(t-c) \qquad \text{(Inverse of 2nd translation formula)}$$

Solving IVP with piecewise defined forcing functions

Example: Solve the IVP $x'' + 16x = f(t), \quad x(0) = 0, \quad x'(0) = 1$

where $f(t) = \begin{cases} \cos 4t\,, & 0 \le t < \pi \\ 0\,, & t \ge \pi \end{cases}$

Solution: $f(t) = \cos 4t \cdot H_{0\pi}(t) + 0 \cdot H_{\pi}(t) = (H(t) - H(t-\pi))\cos 4t.$

$L\left[x''\right] + 16L\left[x\right] = L\left[H(t)\cos 4t\right] - L\left[H(t-\pi)\cos 4t\right].$

$L\left[H(t)\cos 4t\right] = L\left[\cos 4t\right] = \dfrac{s}{s^2+16}.$

Due to periodicity $\cos 4t = \cos(4t - 4\pi) = \cos 4(t-\pi).$

$L\left[H(t-\pi)\cos 4t\right] = \left[H(t-\pi)\cos 4(t-\pi)\right] = e^{-\pi s}\dfrac{s}{s^2+16}.$

Hence $s^2X(s) - 1 + 16X(s) = \dfrac{s}{s^2+16} - \dfrac{s}{s^2+16}e^{-\pi s}$

$(s^2+16)X(s) = 1 + \dfrac{s}{s^2+16} - \dfrac{s}{s^2+16}e^{-\pi s}$

$X(s) = \dfrac{1}{s^2+16} + \dfrac{s}{(s^2+16)^2} - \dfrac{s}{(s^2+16)^2}e^{-\pi s}$

$x(t) = L^{-1}\left[\dfrac{1}{s^2+16}\right] + L^{-1}\left[\dfrac{s}{(s^2+16)^2}\right] - L^{-1}\left[\dfrac{s}{(s^2+16)^2}e^{-\pi s}\right]$

$= \frac{1}{4}L^{-1}\left[\dfrac{4}{s^2+4^2}\right] + \frac{1}{8}L^{-1}\left[\dfrac{8s}{(s^2+4^2)^2}\right] - \frac{1}{8}L^{-1}\left[\dfrac{8s}{(s^2+4^2)e^{-\pi s}}\right]$

$= \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t - \frac{1}{8}(t-\pi)\sin 4(t-\pi)\cdot H(t-\pi).$

$$x(t) = \begin{cases} \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t\,, & 0 \le t < \pi \\ \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t - \frac{1}{8}t\sin 4t + \frac{\pi}{8}\sin 4t\,, & t \ge \pi \end{cases}$$

$$x(t) = \begin{cases} \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t\,, & 0 \le t < \pi \\ \frac{2+\pi}{8}\sin 4t\,, & t \ge \pi \end{cases}$$

- Exercise 2, page 214 . Use the (second translation formula) to find the Laplace transform of $H(t-1)e^{2(t-1)}$.

Solution: We have

$$c = 1, \qquad f(t-1) = e^{2(t-1)}, \qquad f(t) = e^{2t}, \qquad F(s) = \frac{1}{s-2},$$
$$\mathcal{L}\{H(t-1)e^{2(t-1)}\} = e^{-s}\frac{1}{s-2}.$$

- Exercise 4, page 214 . Use the (second translation formula) to find the Laplace transform of $H(t-\frac{\pi}{6})\sin(t-\frac{\pi}{6})$.
Solution: We have

$$c = \frac{\pi}{6}, \qquad f(t-\frac{\pi}{6}) = \sin(t-\frac{\pi}{6}), \qquad f(t) = \sin t, \qquad F(s) = \frac{1}{s^2+1},$$
$$\mathcal{L}\Big\{H\Big(t-\frac{\pi}{6}\Big)\sin\Big(t-\frac{\pi}{6}\Big)\Big\} = e^{-s\frac{\pi}{6}}\frac{1}{s^2+1}.$$

- Exercise 5, page 214 . Use the (second translation formula) to find the Laplace transform of $H(t-1)t^2$.
Solution: We have

$$\begin{aligned}
&c = 1,\\
&H(t-1)t^2 = H(t-1)\big((t-1)^2 + 2(t-1) + 1\big)\\
&\ = H(t-1)\cdot\underbrace{(t-1)^2}_{f_1(t)=t^2} + 2H(t-1)\cdot\underbrace{(t-1)}_{f_2(t)=2t} + H(t-1)\\
&\mathcal{L}\Big\{H(t-1)t^2\Big\} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right).
\end{aligned}$$

- Exercise 6, page 214 . Use the (second translation formula) to find the Laplace transform of $H(t-2)e^{-t}$.
Solution: We have

$$\begin{aligned}
&c = 2,\\
&H(t-2)e^{-t} = H(t-2)e^{-(t-2)}e^{-2} \qquad \Big(f(t) = e^{-t}, F(s) = \tfrac{1}{s+1}\Big)\\
&\mathcal{L}\Big\{H(t-2)e^{-t}\Big\} = e^{-2}e^{-2s}\frac{1}{s+1} = \frac{e^{-2(s+1)}}{s+1}.
\end{aligned}$$

- Exercise 7, page 214 . Use the (second translation formula) to find the Laplace transform of $H(t-\frac{\pi}{6})\sin(2t)$.

Solution: We have

$$
\begin{aligned}
&c=\frac{\pi}{6},\\
&H(t-\frac{\pi}{6})\sin(2t)=H(t-\frac{\pi}{6})\sin(2t)=H(t-\frac{\pi}{6})\sin\big(2(t-\frac{\pi}{6})+\frac{\pi}{3}\big)\\
&=H(t-\frac{\pi}{6})\Big(\sin\big(2(t-\frac{\pi}{6})\big)\underbrace{\cos(\pi/3)}_{1/2}+\cos\big(2(t-\frac{\pi}{6})\big)\underbrace{\sin(\pi/3)}_{\sqrt{3}/2}\Big)\\
&\qquad(\sin(a+b)=\sin a\cos b+\cos a\sin b)\\
&=\frac{1}{2}H(t-\frac{\pi}{6})\sin\big(2(t-\pi/6)\big)+\frac{\sqrt{3}}{2}H(t-\frac{\pi}{6})\cos\big(2(t-\pi/6)\big)\\
&\qquad(f_1(t)=\sin(2t),F_1(s)=\tfrac{2}{s^2+4},f_2(t)=\cos(2t),F_2(s)=\tfrac{s}{s^2+4})\\
&\mathcal{L}\Big\{H(t-2)e^{-t}\Big\}(s)=\frac{1}{2}e^{-s\pi/6}\frac{2}{s^2+4}+\frac{\sqrt{3}}{2}e^{-s\pi/6}\frac{s}{s^2+4}.
\end{aligned}
$$

- Exercise 8, page 214 . Use the (second translation formula) to find the Laplace transform of $H(t-\frac{\pi}{2})\cos 3t$.
  Solution: We have

$$
\begin{aligned}
&H\Big(t-\frac{\pi}{2}\Big)\cos 3t=H\Big(t-\frac{\pi}{2}\Big)\cos\big(3(t-\pi/2)+3\pi/2\big)\\
&=H\Big(t-\frac{\pi}{2}\Big)\Big(\cos 3(t-\pi/2)\underbrace{\cancel{\cos(3\pi/2)}}_{=0}-\sin 3(t-\pi/2)\underbrace{\sin(3\pi/2)}_{=1}\Big)\\
&\qquad(\cos(a+b)=\cos a\cos b-\sin a\sin b)\\
&=-H\Big(t-\frac{\pi}{2}\Big)\sin 3(t-\pi/2)\\
&c=\frac{\pi}{2},\qquad f(t-\frac{\pi}{2})=\sin 3(t-\frac{\pi}{2}),\qquad f(t)=\sin 3t,\qquad F(s)=\frac{s}{s^2+9},\\
&\mathcal{L}\Big\{\cdots\Big\}=e^{-s\frac{\pi}{2}}\frac{3}{s^2+9}.
\end{aligned}
$$

- Exercise 10, page 214 . Use the (Heaviside function) to redefine the piecewise function

$$
f(t)=\begin{cases}0, & t<0\\ 3, & t\geq 0\end{cases}
$$

  and then use the (second translation formula) to find its Laplace transform.
  Solution: We have

$$
f(t)=\begin{cases}0, & t<0\\ 3, & t\geq 0\end{cases}=3H_0(t),
$$

$$
\mathcal{L}\{f(t)\}(s)=3\mathcal{L}\{H_0(t)\}(s)=\frac{3}{s}.
$$

- Exercise 12, page 214 . Use the (Heaviside function) to redefine the piecewise function

$$
f(t)=\begin{cases}0, & t<0\\ t, & t\in[0,3)\\ 3, & t\geq 3\end{cases}
$$

and then use the (second translation formula) to find its Laplace transform.
Solution: We have

$$f(t) = tH_{03}(t) + 3H_3(t) = tH_0(t) - tH_3(t) + 3H_3(t) = tH_0(t) + (-t+3)H_3(t),$$
$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{tH_0(t) + (-t+3)H_3(t)\}(s)$$
$$= \underbrace{\mathcal{L}\{tH(t)\}(s)}_{c=0} - \underbrace{\mathcal{L}\{(t-3)H(t-3)\}(s)}_{c=3, f(t)=t, F(s)=\frac{1}{s^2}} = \frac{1}{s^2} - \frac{e^{-3s}}{s^2}.$$

- Exercise 13, page 214 . Use the (Heaviside function) to redefine the piecewise function

$$f(t) = \begin{cases} 0, & t < 0 \\ t^2, & t \in [0,2) \\ 4, & t \geq 2 \end{cases}$$

and then use the (second translation formula) to find its Laplace transform.
Solution: We have

$$\begin{aligned} f(t) &= t^2H_{02}(t) + 4H_2(t) = t^2H_0(t) - (t^2-4)H_2(t) \\ &= t^2H(t) - \underbrace{(t^2-4t+4)}_{=(t-2)^2}H(t-2) + \underbrace{(-4t+8)}_{=-4(t-2)}H(t-2) \\ &= t^2H(t) - (t-2)^2H(t-2) - 4(t-2)H(t-2), \\ \mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{t^2H(t) - (t-2)^2H(t-2) - 4(t-2)H(t-2)\}(s) \\ &= \underbrace{\mathcal{L}\{t^2H(t)\}(s)}_{c=0, f(t)=t} - \underbrace{\mathcal{L}\{(t-2)^2H(t-2)\}(s)}_{c=2, f(t)=t^2} - 4\underbrace{\mathcal{L}\{(t-2)H(t-2)\}(s)}_{c=2, f(t)=t} \\ &= \frac{2}{s^3} - e^{-2s}\frac{2}{s^3} - 4e^{-2s}\frac{1}{s^2}. \end{aligned}$$

- Exercise 14, page 214 . Use the (Heaviside function) to redefine the piecewise function

$$f(t) = \begin{cases} 3, & t \in [0,1) \\ 2, & t \in [1,2) \\ 1, & t \in [2,3) \\ 0, & \text{otherwise} \end{cases}$$

and then use the (second translation formula) to find its Laplace transform.
Solution: We have

$$\begin{aligned} f(t) &= 3H_{0,1}(t) + 2H_{1,2}(t) + 1 \cdot H_{2,3}(t) = 3H(t) + (-3+2)H_1(t) + (2-1)\cdot H_2(t) - H_3(t) \\ &= 3H(t) - H_1(t) - H_2(t) - H_3(t) \\ &= 3H(t) - H(t-1) - H(t-2) - H(t-3), \\ \mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{3H(t) - H(t-1) - H(t-2) - H(t-3)\}(s) \\ &= 3\underbrace{\mathcal{L}\{1\cdot H(t)\}(s)}_{c=0, f(t)=1} - \underbrace{\mathcal{L}\{1\cdot H(t-1)\}(s)}_{c=1, f(t)=1} - \underbrace{\mathcal{L}\{1\cdot H(t-2)\}(s)}_{c=2, f(t)=1} - \underbrace{\mathcal{L}\{1\cdot H(t-3)\}(s)}_{c=3, f(t)=1} \\ &= 3\frac{1}{s} - e^{-s}\frac{1}{s} - e^{-2s}\frac{1}{s} - e^{-3s}\frac{1}{s}. \end{aligned}$$

- Exercise 15, page 214 . Use the (Heaviside function) to redefine the piecewise function

$$f(t) = \begin{cases} t, & t \in [0,1) \\ 2-t, & t \in [1,3) \\ t-4, & t \in [3,4) \\ 0, & \text{otherwise} \end{cases}$$

and then use the (second translation formula) to find its Laplace transform.
<u>Solution</u>: We have

$$\begin{aligned} f(t) &= tH_{0,1}(t) + (2-t)H_{1,3}(t) + (t-4)H_{3,4}(t) \\ &= tH(t) + (-t+2-t)H_1(t) + (-2+t+t-4)H_3(t) - (t-4)H_4(t) \\ &= tH(t) - 2(t-1)H(t-1) + 2(t-3)H(t-3) - (t-4)H(t-4), \\ \mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{tH(t) - 2(t-1)H(t-1) + 2(t-3)H(t-3) - (t-4)H(t-4)\}(s) \\ &= \underbrace{\mathcal{L}\{t \cdot H(t)\}(s)}_{c=0, f(t)=t} - 2\underbrace{\mathcal{L}\{(t-1) \cdot H(t-1)\}(s)}_{c=1, f(t)=t} \\ &\quad + 2\underbrace{\mathcal{L}\{(t-3) \cdot H(t-3)\}(s)}_{c=3, f(t)=t} - \underbrace{\mathcal{L}\{(t-4) \cdot H(t-4)\}(s)}_{c=4, f(t)=t} \\ &= \frac{1}{s^2} - 2\frac{e^{-s}}{s^2} + 2\frac{e^{-3s}}{s^2} - \frac{e^{-4s}}{s^2} = \frac{1}{s^2}\left(1 - 2e^{-s} + 2e^{-3s} - e^{-4s}\right). \end{aligned}$$

- Exercise 16, page 214 . Find the inverse Laplace transform of the function

$$Y(s) = \frac{e^{-2s}}{s+3}.$$

Create a piecewise definition for your solution that doesn't use the (Heaviside function).
<u>Solution</u>: Using the (second translation formula), we have

$$Y(s) = \frac{e^{-2s}}{s+3} = e^{-2s} \underbrace{\mathcal{L}\{e^{-3t}\}(s)}_{=F(s), f(t)=e^{-3t}, c=2} = \mathcal{L}\{H(t-2)e^{-3(t-2)}\}(s).$$

$$y(t) = H(t-2)e^{-3(t-2)} = \begin{cases} 0, & t < 2 \\ e^{-3(t-2)}, & t \geq 2 \end{cases}$$

- Exercise 18, page 214 . Find the inverse Laplace transform of the function

$$Y(s) = \frac{1-e^{-s}}{s^2}.$$

Create a piecewise definition for your solution that doesn't use the (Heaviside function).
<u>Solution</u>: Using the (second translation formula), we have

$$Y(s) = \frac{1-e^{-s}}{s^2} = \frac{1}{s^2} + \frac{e^{-s}}{s^2} = \mathcal{L}\{t\} - \underbrace{e^{-s}F(s)}_{c=1, f(t)=t} = \mathcal{L}\{t - H(t-1)(t-1)\}(s).$$

$$y(t) = t - H(t-1)(t-1) = \begin{cases} t, & t \in [0,1) \\ t-(t-1), & t \geq 1 \end{cases} = \begin{cases} t, & t \in [0,1) \\ 1, & t \in [1,\infty) \end{cases}$$

- Exercise 19, page 214 . Find the inverse Laplace transform of the function

$$Y(s) = \frac{2+e^{-2s}}{s^3}.$$

Create a piecewise definition for your solution that doesn't use the (Heaviside function).
Solution: Using the (second translation formula), we have

$$Y(s) = \frac{2+e^{-2s}}{s^3} = \frac{2}{s^3} + \frac{e^{-2s}}{s^3} = \mathcal{L}\{t^2\} - \frac{1}{2}\underbrace{e^{-2s}F(s)}_{c=2, f(t)=t^2} = \mathcal{L}\{t^2 - \frac{1}{2}H(t-2)(t-2)^2\}(s).$$

$$y(t) = t^2 - \frac{1}{2}H(t-2)(t-2)^2 = \begin{cases} t^2, & t \in [0,2) \\ t^2 - \frac{1}{2}(t-2)^2, & t \geq 2 \end{cases} = \begin{cases} t, & t \in [0,2) \\ 1, & t \in [2,\infty) \end{cases}$$

- Exercise 20, page 214 . Find the inverse Laplace transform of the function

$$Y(s) = \frac{e^{-s}}{s^2+4}.$$

Create a piecewise definition for your solution that doesn't use the (Heaviside function).
Solution: Using the (second translation formula), we have

$$Y(s) = \frac{e^{-s}}{s^2+4} = \frac{1}{2}e^{-s}\frac{2}{s^2+4} = \frac{1}{2}\underbrace{e^{-s}\mathcal{L}\{\sin(2t)\}}_{c=1, f(t)=\sin(2t)} = \frac{1}{2}\mathcal{L}\{H(t-1)\sin\big(2(t-1)\big)\}(s).$$

$$y(t) = \frac{1}{2}H(t-1)\sin\big(2(t-1)\big) = \begin{cases} 0, & t \in [0,1) \\ \frac{1}{2}\sin 2(t-1), & t \in [1,\infty) \end{cases}$$

- Exercise 21, page 214 . Find the inverse Laplace transform of the function

$$Y(s) = \frac{se^{-3s}}{s^2+4}.$$

Create a piecewise definition for your solution that doesn't use the (Heaviside function).
Solution: Using the (second translation formula), we have

$$Y(s) = \frac{se^{-3s}}{s^2+4} = e^{-3s}\frac{s}{s^2+4} = \underbrace{e^{-3s}\mathcal{L}\{\cos(2t)\}}_{c=3} = \mathcal{L}\{H(t-3)\cos 2(t-3)\}(s).$$

$$y(t) = H(t-3)\cos 2(t-3) = \begin{cases} 0, & t \in [0,3) \\ \cos 2(t-3), & t \in [3,\infty) \end{cases}$$

- Exercise 22, page 214 . Find the inverse Laplace transform of the function

$$Y(s) = \frac{1-e^{-s}}{s(s+2)}.$$

Create a piecewise definition for your solution that doesn't use the (Heaviside function).
Solution: Using the (second translation formula), we have

$$
\begin{aligned}
Y(s) &= \frac{1-e^{-s}}{s(s+2)} = \frac{1}{s(s+2)} - e^{-s}\frac{1}{s(s+2)} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2} - e^{-s}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}\right) \\
&= \mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}e^{-2t}\right\} - \underbrace{e^{-s}\mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}e^{-2t}\right\}}_{c=1} = \mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}e^{-2t}\right\} - \mathcal{L}\left\{H(t-1)\left(\frac{1}{2} - \frac{1}{2}e^{-2t}\right)\right\} \\
&= \mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}e^{-2t} - H(t-1)\left(\frac{1}{2} - \frac{1}{2}e^{-2t}\right)\right\}. \\
y(t) &= \big(1 - H(t-1)\big)\left(\frac{1}{2} - \frac{1}{2}e^{-2t}\right) = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2t}, & t \in [0,1) \\ 0, & t \in [1,\infty) \end{cases}
\end{aligned}
$$

- Exercise 23, page 214 . Find the inverse Laplace transform of the function

$$Y(s) = \frac{e^{-2s}}{s^2-2s-3}.$$

Create a piecewise definition for your solution that doesn't use the (Heaviside function).
Solution: Using the (second translation formula), we have

$$
\begin{aligned}
Y(s) &= \frac{e^{-2s}}{s^2-2s-3} = e^{-2s}\left(\frac{1}{4}\frac{1}{s-3} - \frac{1}{4}\frac{1}{s+1}\right) = \frac{1}{4}e^{-2s}\frac{1}{s-3} - \frac{1}{4}e^{-2s}\frac{1}{s+1}\Big) \\
&= \frac{1}{4}e^{-2s}\mathcal{L}\{e^{3t}\}(s) - \frac{1}{4}e^{-2s}\mathcal{L}\{e^{-4}\}(s) \\
&= \frac{1}{4}\mathcal{L}\{H(t-2)e^{3(t-2)}\}(s) - \frac{1}{4}\mathcal{L}\{H(t-2)e^{-4(t-2)}\}(s) \\
&= \frac{1}{4}\mathcal{L}\{H(t-2)\big(e^{3(t-2)} - e^{-4(t-2)}\big)\}(s) \\
y(t) &= \frac{1}{4}H(t-2)\big(e^{3(t-2)} - e^{-4(t-2)}\big) = \begin{cases} 0, & t \in [0,2) \\ \frac{1}{4}\big(e^{3(t-2)} - e^{-4(t-2)}\big), & t \geq 2 \end{cases}
\end{aligned}
$$

- Exercise 24, page 214 . Find the inverse Laplace transform of the function

$$Y(s) = \frac{e^{-s}}{s(s-2)^2}.$$

Create a piecewise definition for your solution that doesn't use the (Heaviside function).
Solution: Using the (second translation formula), we have

$$
\begin{aligned}
Y(s) &= \frac{e^{-s}}{s(s-2)^2} = e^{-s}\left(\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{1}{s-2} + \frac{1}{2}\frac{1}{(s-2)^2}\right) \\
&= \frac{1}{4}e^{-s}\mathcal{L}\{1\}(s) - \frac{1}{4}e^{-s}\mathcal{L}\{e^{2t}\}(s) + \frac{1}{2}e^{-s}\mathcal{L}\{te^{2t}\}(s) \\
&= \mathcal{L}\left\{\frac{1}{4}H(t-1) - \frac{1}{4}H(t-1)e^{2(t-1)} + \frac{1}{2}H(t-1)(t-1)e^{2(t-1)}\right\}(s)
\end{aligned}
$$

where we used partial fractions

$$\frac{1}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2} = \frac{A(s-2)^2 + s\big(B(s-2)s + D\big)}{s(s-2)^2},$$

$$1 = 2C, \qquad C = 1/2, \qquad (s=2)$$

$$1 = 4A, \qquad A = 1/4, \qquad (s=0)$$

$$1 = A + (-B + C) = \frac{1}{4} + (-B + \frac{1}{2}) = \frac{3}{4} - B, \qquad B = -\frac{1}{4}. \qquad (s=1)$$

- Exercise 25, page 214 . Find the inverse Laplace transform of the function

$$Y(s) = \frac{2 - e^{-2s}}{s^2 + 2s + 2}.$$

Create a piecewise definition for your solution that doesn't use the (Heaviside function).
Solution: Using the (second translation formula), we have

$$\begin{aligned}
Y(s) &= \frac{2 - e^{-2s}}{s^2 + 2s + 2} = \frac{2}{(s+1)^2 + 1} - e^{-2s}\frac{1}{(s+1)^2 + 1} \\
&= 2\mathcal{L}\{e^{-t}\sin t\} - e^{-2s}\mathcal{L}\{e^{-t}\sin t\} \\
&= 2\mathcal{L}\{e^{-t}\sin t\} - \mathcal{L}\{H(t-2)e^{-(t-2)}\sin(t-2)\} \\
y(t) &= 2e^{-t}\sin t - H(t-2)e^{-(t-2)}\sin(t-2) \\
&= \begin{cases} 2e^{-t}\sin t, & t \in [0,2) \\ 2e^{-t}\sin t - e^{-(t-2)}\sin(t-2), & t \in [2,\infty) \end{cases}
\end{aligned}$$