5.7 Solution to General IVP. Convolutions

Definition 1 *The convolution of two piecewise continuous functions f and g is the function*

$$f * g = \int_0^t f(u)g(t-u)\,du. \qquad \text{(convolution)}$$

Note that $f * g$ is a function of the variable t, i.e. it is $(f * g)(t)$.

Properties of the convolution:

1. $\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s) = F \cdot G$ (if f and g are of exponential order)

2. $f * g = g * f$ (*comutativity*)

3. $f * (g + h) = f * g + f * h$ (*distributivity*)

4. $(f * g) * h = f * (g * h)$ (*associativity*)

5. $f * 0 = 0$ (*null ellement*)

6. $f * \delta = \delta * f = f$ (*identity element*)

7. $\mathcal{L}\{f * \delta\} = \mathcal{L}\{f\} = f$

8. $\mathcal{L}^{-1}\{F \cdot G\}(t) = f * g$

9. $\dfrac{d}{dt}(f * g) = f' * g + f(0)g(t)$

Example:

$$e^t * e^{-2t} = \int_0^t e^u e^{-2(t-u)} du = \int_0^t e^u e^{-2t} e^{2u} du = e^{-2t} \int_0^t e^{3u} du = e^{-2t} \cdot \frac{1}{3} e^{3u} \Big|_0^t = \frac{1}{3}[e^t - e^{-2t}]$$

Example: Solve IVP: $y'' + y = g(t), \quad y(0) = y'(0) = 0.$

Solution: Characteristic polynomial is $s^2 + 1$.

Applying LT to both sides we get

$(s^2+1)Y(s) = G(s), \quad Y(s) = \dfrac{1}{s^2+1} \cdot G(s)$

By the theorem from the section 5.6 $\dfrac{1}{s^2+1} = E(s)$. Hence $Y(s) = E(s) \cdot G(s)$.

$y(t) = \mathcal{L}^{-1}\left[E(s) \cdot G(s)\right] = e * g.$

$e(t)$ for the given IVP can be found as ILT of $\frac{1}{\text{characteristic polynomial}}$ or $\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t.$

Hence $y(t) = e * g = \displaystyle\int_0^t \sin u \cdot g(t-u)\,du = \int_0^t \sin(t-u) g(u)\,du.$

Theorem The solution to the IVP

$$ay'' + by' + cy = f(t), \quad y(0) = y_0,\ y'(0) = y_1$$

can be written as $y(t) = y_s(t) + y_i(t)$

where y_s is the solution to the IVP $ay_s'' + by_s' + cy_s = f(t), \quad y_s(0) = y_s'(0) = 0$

and y_i is the solution to the IVP $ay_i'' + 6y_i' + cy_i = 0, \quad y_i(0) = y_0,\ y_i'(0) = y_1$

If $e(t)$ is the unit response function for the DE then $y_s = e*f$ and $y_i = ay_0 e'(t) + (ay_1 + by_0)e(t)$.

Example: Compute the solution to the IVP $y'' + y = g(t), \quad y(0) = 1,\ y'(0) = -2.$

Solution: We found before that $e(t) = \sin t$. Then $y_s(t) = e * g = \displaystyle\int_0^t \sin(t-u) g(u) du.$

$y_i(t) = y_0 e'(t) + y' e(t) = \cos t - 2\sin t \Rightarrow y(t) = y_s(t) + y_i(t) = \cos t - 2\sin t + \displaystyle\int_0^t \sin(t-u) g(u)\,du.$

Example: Solve the IVP $y'' + 4y' + 13y = g(t), \quad y(0) = -5,\ y'(0) = 2.$

Solution: Characteristic polynomial is $P(s) = s^2 + 4s + 13 = (s+2)^2 + 9.$

$E(s) = \dfrac{1}{(s+2)^2 + 3^2}, \quad e(t) = \mathcal{L}^{-1}\left[E(s)\right](t) = \frac{1}{3}e^{-2t}\sin 3t, \quad e'(t) = \frac{1}{3}e^{-2t}(3\cos 3t - 2\sin 3t)$

Then $y(t) = e * g(t) + y_0 e'(t) + (y' + 4y_0)e(t)$

$= \dfrac{1}{3}\displaystyle\int_0^t e^{-2(t-u)} \sin 3(t-u) g(u)\,du - \dfrac{5}{3}e^{-2t}(3\cos 3t - 2\sin 3t) - 18 \cdot \dfrac{1}{3}e^{-2t}\sin 3t$

$y(t) = \dfrac{1}{3}\displaystyle\int_0^t e^{-2(t-u)} \sin(t-u) g(u)\,du - \dfrac{1}{3}e^{-2t}\left(8\sin 3t + 15\cos 3t\right)$

In particular if $g(t) = \tan(t^3)$ then

$$y(t) = \frac{1}{3}\int_0^t e^{-2(t-u)} \sin(t-u)\tan(u^3)\,du - \frac{1}{3}e^{-2t}\,(8\sin 3t + 15\cos 3t).$$

Review

5.6 δ-function

e is the unit impulse response function

$e(t)$ is the solution to the IVP: $ae'' + be' + ce = \delta(t) \quad e(0) = e'(0) = 0.$

$$\mathcal{L}\{\delta\}(s) = 1.$$

$\Rightarrow \quad \mathcal{L}\{e\}(s) = E(s) = \dfrac{1}{P(s)},$ where $P(s) = as^2 + bs + c.$

5.7 Convolutions

IVP: $\quad ay'' + by' + cy = f(t) \quad y(0) = y_0, \; y'(0) = y_1$

The solution is $\; y = y_s + y_i.$

There the state-free solution y_s is the solution to the IVP:

$ay_s'' + by_s' + cy_s = f(t) \quad y_s(0) = y_s'(0) = 0.$

In the Laplace space: $Y_s(s) = \dfrac{F(s)}{P(s)} = \dfrac{1}{P(s)} \cdot F(s) = E(s)F(s)$

$y_s(t) = e * f.$

The input-free solution y_i is the solution to the IVP:

$ay_i'' + by_i' + cy_i = 0 \quad y_i(0) = y_0, \; y_i'(0) = y_1.$

$Y_i(s) = E(s)\left(y_0(as + b) + ay_1\right) = E(s)\left(ay_0 s + (ay_1 + by_0)\right)$

$y_i(t) = ay_0 e'(t) + (ay_1 + by_0)e(t).$

Therefore, $\; y(t) = e * f(t) + ay_0 e'(t) + (ay_1 + by_0)e(t).$

In sections up to 5.5 we could find LT of the right hand side $f(t)$.

In the section 5.7 the right hand side is an arbitrary function $f(t)$ and we do not know its LT.

- Exercise 4, page 241. Use Definition 1 to calculate the convolution of

$$f(t) = e^{3t}, \qquad g(t) = e^{-2t}$$

Solution: Using (convolution) we have

$$(f*g)(t) = \int_0^t f(u)g(t-u)\,du = \int_0^t e^{3u}e^{-2(t-u)}\,du = e^{-2t}\int_0^t e^{5u}\,du = e^{-2t}\frac{1}{5}e^{5u}\Big|_{u=0}^{u=t}$$
$$= \frac{1}{5}e^{-2t}\big(e^{5t}-1\big) = \frac{1}{5}\big(e^{3t}-e^{-2t}\big)$$

- Exercise 5, page 241. Use Definition 1 to calculate the convolution of

$$f(t) = e^{t}, \qquad g(t) = e^{2t}$$

Solution: Using (convolution) we have

$$(f*g)(t) = \int_0^t f(u)g(t-u)\,du = \int_0^t e^{u}e^{2(t-u)}\,du = e^{2t}\int_0^t e^{-u}\,du = -e^{2t}e^{-u}\Big|_{u=0}^{u=t}$$
$$= -e^{2t}\big(e^{-t}-1\big) = e^{2t}-e^{t}.$$

- Exercise 7, page 241. Use Definition 1 to calculate the convolution of

$$f(t) = t, \qquad g(t) = 3-t$$

Solution: Using (convolution) we have

$$(f*g)(t) = \int_0^t f(u)g(t-u)\,du = \int_0^t u\big(3-(t-u)\big)\,du = \int_0^t \big((3-t)u+u^2\big)\,du$$
$$= \Big((3-t)\frac{u^2}{2}+\frac{u^3}{3}\Big)\Big|_{u=0}^{u=t} = \frac{3}{2}t^2-\frac{1}{6}t^3.$$

- Exercise 9, page 241. Use Definition 1 to calculate the convolution of

$$f(t) = t^2, \qquad g(t) = e^{-t}$$

Solution: Using (convolution) we have

$$(f*g)(t) = \int_0^t f(u)g(t-u)\,du = \int_0^t u^2e^{-(t-u)}\,du = e^{-t}\int_0^t u^2\underbrace{e^{u}}_{(e^u)'}\,du$$
$$= e^{-t}\Big(u^2e^u\Big|_{u=0}^{u=t} - \int_0^t 2u\underbrace{e^{u}}_{(e^u)'}\,du\Big) = e^{-t}\Big[t^2e^t-2\Big(ue^u\Big|_{u=0}^{u=t}-\int_0^t e^u\,du\Big)\Big]$$
$$= e^{-t}\big[t^2e^t-2\big(te^t-(e^t-1)\big)\big] = e^{-t}(t^2e^t-2te^t+2e^t-2)$$
$$= t^2-2t+2-2e^{-t}.$$

- Exercise 11, page 241.

(i) Use Definition 1 to calculate the convolution of

$$f(t) = \sin t, \qquad g(t) = t$$

(ii) Use any techniques, other than $\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s)$, to find the Laplace transform of the convolution found in part (i).

(iii) Compute $F(s) = \mathcal{L}\{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$, and the product $F(s)G(s)$. Compare this result with that found in part (ii).

<u>Solution</u>:

(i) Using (convolution) we have

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du = \int_0^t (\sin u)\,(t-u)\,du = t\int_0^t \underbrace{\sin u}_{-(\cos u)'}\,du - \int_0^t u\,\underbrace{\sin u}_{-(\cos u)'}\,du$$

$$= -t\big(\cancel{\cos t} - 1\big) + (\cancel{t\cos t} - 0) - \int_0^t \cos u\,du = t - \sin t$$

(ii)

$$\mathcal{L}\{t - \sin t\}(s) = \frac{1}{s^2} - \frac{1}{s^2+1} = \frac{1}{s^2(s^2+1)}$$

(iii)

$$F(s) = \frac{1}{s^2+1},$$

$$G(s) = \frac{1}{s^2},$$

$$F(s)G(s) = \frac{1}{s^2(s^2+1)}.$$

- Exercise 12, page 241.

(i) Use Definition 1 to calculate the convolution of

$$f(t) = \cos t, \qquad g(t) = t^2$$

(ii) Use any techniques, other than $\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s)$, to find the Laplace transform of the convolution found in part (i).

(iii) Compute $F(s) = \mathcal{L}\{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$, and the product $F(s)G(s)$. Compare this result with that found in part (ii).

<u>Solution</u>:

(i) Using (convolution) we have

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du = \int_0^t (\underbrace{\cos u}_{(\sin u)'})\,(t-u)^2\,du$$

$$= \cancel{(\sin u)\,(t-u)^2\Big|_{u=0}^{u=t}} - 2\int_0^t \underbrace{\sin u}_{-(\cos u)'}\,(t-u)\,du$$

$$= 2(\cos u)(t-u)\Big|_{u=0}^{u=t} - 2\int_0^t \cos u\,du = 2t - 2\sin t$$

(ii)

$$\mathcal{L}\{2t - 2\sin t\}(s) = 2\frac{1}{s^2} - 2\frac{1}{s^2+1} = \frac{2}{s^2(s^2+1)}$$

(iii)

$$\begin{aligned} F(s) &= \frac{s}{s^2+1}, \\ G(s) &= \frac{2}{s^3}, \\ F(s)G(s) &= \frac{2s}{s^3(s^2+1)} \equiv \frac{2}{s^2(s^2+1)}. \end{aligned}$$

- Exercise 13, page 241.

(i) Use Definition 1 to calculate the convolution of

$$f(t) = e^{-2t}, \qquad g(t) = t$$

(ii) Use any techniques, other than $\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s)$, to find the Laplace transform of the convolution found in part (i).

(iii) Compute $F(s) = \mathcal{L}\{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$, and the product $F(s)G(s)$. Compare this result with that found in part (ii).

<u>`Solution`</u>:

(i) Using (convolution) we have

$$\begin{aligned} (f * g)(t) &= \int_0^t f(u)g(t-u)\,du = \int_0^t \left(e^{-2u}\right)(t-u)\,du \\ &= t\int_0^t \underbrace{e^{-2u}}_{-\frac{1}{2}(e^{-2u})'}\,du - \int_0^t u\underbrace{e^{-2u}}_{-\frac{1}{2}(e^{-2u})'}\,du \\ &= -\frac{1}{2}t\left(e^{-2u}\right)\Big|_{u=0}^{u=t} + \frac{1}{2}\left(ue^{-2u}\Big|_{u=0}^{u=t} - \int_0^t e^{-2u}\,du\right) \\ &= -\frac{1}{2}t\left(e^{-2u}\right)\Big|_{u=0}^{u=t} + \frac{1}{2}ue^{-2u}\Big|_{u=0}^{u=t} + \frac{1}{4}e^{-2u}\Big|_{u=0}^{u=t} = \left(\frac{1}{4} - \frac{1}{2}t\right)\left(e^{-2u}\right)\Big|_{u=0}^{u=t} + \frac{1}{2}ue^{-2u}\Big|_{u=0}^{u=t} \\ &= \left(\frac{1}{4} - \cancel{\frac{1}{2}t}\right)e^{-2t} - \left(\frac{1}{4} - \frac{1}{2}t\right) + \cancel{\frac{1}{2}te^{-2t}} = \frac{t}{2} + \frac{1}{4}\left(e^{-2t} - 1\right). \end{aligned}$$

(ii)

$$\mathcal{L}\left\{\frac{t}{2} + \frac{1}{4}\left(e^{-2t} - 1\right)\right\}(s) = \frac{1}{2s^2} - \frac{1}{4s} + \frac{1}{4}\frac{1}{s+2} = \frac{1}{s^2(s+2)}.$$

(iii)

$$\begin{aligned} F(s) &= \frac{1}{s+2}, \\ G(s) &= \frac{1}{s^2}, \\ F(s)G(s) &= \frac{1}{s^2(s+2)}. \end{aligned}$$

- Exercise 14, page 241.

  (i) Use Definition 1 to calculate the convolution of

  $$f(t) = e^{3t}, \qquad g(t) = t^2$$

  (ii) Use any techniques, other than $\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s)$, to find the Laplace transform of the convolution found in part (i).

  (iii) Compute $F(s) = \mathcal{L}\{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$, and the product $F(s)G(s)$. Compare this result with that found in part (ii).

  <u>Solution</u>:

  (i) Using (convolution) we have

  $$\begin{aligned}(f * g)(t) &= \int_0^t f(u)g(t-u)\,du = \int_0^t \underbrace{e^{3u}}_{=(\frac{e^{3u}}{3})'} (t-u)^2\,du \\ &= \frac{1}{3}e^{3u}(t-u)^2\Big|_{u=0}^{u=t} + \frac{2}{3}\int_0^t \underbrace{e^{3u}}_{=(\frac{e^{3u}}{3})'} (t-u)\,du \\ &= -\frac{1}{3}t^2 + \frac{2}{9}\left(e^{3u}(t-u)\Big|_{u=0}^{u=t} + \int_0^t \underbrace{e^{3u}}_{=(\frac{e^{3u}}{3})'} du\right) \\ &= -\frac{1}{3}t^2 + \frac{2}{9}\left(-t + \frac{1}{3}(e^{3t}-1)\right) \\ &= \frac{2}{27}e^{3t} - \frac{1}{3}t^2 - \frac{2}{9}t - \frac{2}{27}.\end{aligned}$$

  (ii)

  $$\mathcal{L}\left\{\frac{2}{27}e^{3t} - \frac{1}{3}t^2 - \frac{2}{9}t - \frac{2}{27}\right\}(s) = \frac{2}{27}\frac{1}{s-3} - \frac{1}{3}\frac{2}{s^3} - \frac{2}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s} = \frac{2}{s^3(s-3)}.$$

  (iii)

  $$\begin{aligned}F(s) &= \frac{1}{s-3}, \\ G(s) &= \frac{2}{s^3}, \\ F(s)G(s) &= \frac{2}{s^3(s-3)}.\end{aligned}$$

- Exercise 17, page 241. Use $\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s)$ (DO NOT USE PARTIAL FRACTIONS) to find the Laplace transform of the $s-$domain function

  $$\frac{1}{s(s+1)}$$

Solution:

$$\frac{1}{s(s+1)} = \underbrace{\frac{1}{s}}_{=F(s)} \cdot \underbrace{\frac{1}{s+1}}_{=G(s)}, \qquad (f(t)=1, g(t)=e^{-t})$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = 1 * e^{-t} = \int_0^t 1e^{-(t-u)}\,du = e^{-t}\int_0^t e^u\,du = e^{-t}(e^t-1) = 1-e^{-t}.$$

- Exercise 18, page 241. Use $\mathcal{L}\{f*g\}(s) = \mathcal{L}\{f\}(s)\cdot\mathcal{L}\{g\}(s)$ (DO NOT USE PARTIAL FRACTIONS) to find the Laplace transform of the $s-$domain function

$$\frac{1}{s^2-3s}$$

Solution:

$$\frac{1}{s^2-3s} = \frac{1}{s(s-3)} = \underbrace{\frac{1}{s}}_{=F(s)} \cdot \underbrace{\frac{1}{s-3}}_{=G(s)}, \qquad (f(t)=1, g(t)=e^{3t})$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = 1 * e^{3t} = \int_0^t 1e^{3(t-u)}\,du = e^{3t}\int_0^t e^{-3u}\,du = -e^{3t}\frac{1}{3}(e^{-3t}-1) = \frac{1}{3}(e^{3t}-1).$$

- Exercise 19, page 241. Use $\mathcal{L}\{f*g\}(s) = \mathcal{L}\{f\}(s)\cdot\mathcal{L}\{g\}(s)$ (DO NOT USE PARTIAL FRACTIONS) to find the Laplace transform of the $s-$domain function

$$\frac{1}{(s+1)(s-2)}$$

Solution:

$$\frac{1}{(s+1)(s-2)} = \underbrace{\frac{1}{s+1}}_{=F(s)} \cdot \underbrace{\frac{1}{s-2}}_{=G(s)}, \qquad (f(t)=e^{-t}, g(t)=e^{2t})$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = e^{-t} * e^{2t} = \int_0^t e^{-u}e^{2(t-u)}\,du = e^{2t}\int_0^t e^{-3u}\,du = -e^{2t}\frac{1}{3}(e^{-3t}-1)$$
$$= \frac{1}{3}(e^{2t}-e^{-t}).$$

- Exercise 20, page 241. Use $\mathcal{L}\{f*g\}(s) = \mathcal{L}\{f\}(s)\cdot\mathcal{L}\{g\}(s)$ (DO NOT USE PARTIAL FRACTIONS) to find the Laplace transform of the $s-$domain function

$$\frac{1}{s^2-2s-3}$$

Solution:

$$\frac{1}{s^2-2s-3} = \frac{1}{(s+1)(s-3)} = \underbrace{\frac{1}{s+1}}_{=F(s)} \cdot \underbrace{\frac{1}{s-3}}_{=G(s)}, \qquad (f(t)=e^{-t}, g(t)=e^{3t})$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-3)}\right\} = e^{-t} * e^{3t} = \int_0^t e^{-u}e^{3(t-u)}\,du = e^{3t}\int_0^t e^{-4u}\,du = -e^{3t}\frac{1}{5}(e^{-4t}-1)$$
$$= \frac{1}{4}(e^{3t}-e^{-t}).$$

- Exercise 21, page 241. Use $\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s)$ (DO NOT USE PARTIAL FRACTIONS) to find the Laplace transform of the $s-$domain function

$$\frac{s}{(s-1)(s^2+1)}$$

<u>Solution</u>:

$$\frac{s}{(s-1)(s^2+1)} == \underbrace{\frac{1}{s-1}}_{=F(s)} \cdot \underbrace{\frac{s}{s^2+1}}_{=G(s)}, \qquad (f(t) = e^t, g(t) = \cos t)$$

$$I := \mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s^2+1)}\right\} = e^{-t} * \cos t = \int_0^t e^u \cos(t-u)\, du$$

$$= e^u \cos(t-u)\Big|_{u=0}^{u=t} - \int_0^t e^u \sin(t-u)\, du$$

$$= e^t - \cos t - e^u \sin(t-u)\Big|_{u=0}^{u=t} - I = e^t - \cos t + \sin t - I,$$

$$I = \frac{1}{2}\big(e^t - \cos t + \sin t\big).$$

- Exercise 22, page 241. Use $\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s)$ (DO NOT USE PARTIAL FRACTIONS) to find the Laplace transform of the $s-$domain function

$$\frac{1}{(s+1)(s^2+4)}$$

<u>Solution</u>:

$$\frac{1}{(s+1)(s^2+4)} = \underbrace{\frac{1}{s+1}}_{=F(s)} \cdot \underbrace{\frac{1}{s^2+4}}_{=G(s)}, \qquad (f(t) = e^{-t}, g(t) = \tfrac{1}{2}\sin 2t)$$

$$I := \mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\} = e^{-t} * \frac{1}{2}\sin 2t = \frac{1}{2}\int_0^t \underbrace{e^{-u}}_{=-(e^{-u})'} \sin 2(t-u)\, du$$

$$= -\frac{1}{2}e^{-u}\sin 2(t-u)\Big|_{u=0}^{u=t} - \int_0^t \underbrace{e^{-u}}_{=-(e^{-u})'} \cos 2(t-u)\, du$$

$$= \frac{1}{2}\sin 2t + e^{-u}\cos 2(t-u)\Big|_{u=0}^{u=t} - 2\int_0^t \underbrace{e^{-u}}_{=-(e^{-u})'} \sin 2(t-u)\, du$$

$$= \frac{1}{2}\sin 2t + e^{-t} - \cos 2t - 4I,$$

$$I = \frac{1}{5}\Big(\frac{1}{2}\sin 2t + e^{-t} - \cos 2t\Big).$$

- Exercise 23, page 241. Use $\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s)$ (DO NOT USE PARTIAL FRACTIONS) to find the Laplace transform of the $s-$domain function

$$\frac{1}{(s^2+1)^2}$$

Solution:

$$\frac{1}{(s^2+1)^2} = \underbrace{\frac{1}{s^2+1}}_{=F(s)} \cdot \underbrace{\frac{1}{s^2+1}}_{=G(s)}, \qquad (f(t) = g(t) = \sin t)$$

$$I := \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \sin t * \sin t = \int_0^t \sin u \sin(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \big(\cos(u-t+u) - \cos(u+t-u)\big)\, du$$

$$\left(\text{Use: } \sin a \sin b = \tfrac{1}{2}\big(\cos(a-b) - \cos(a+b)\big)\right)$$

$$= \frac{1}{2}\int_0^t \big(\underbrace{\cos(2u-t)}_{\frac{1}{2}\left(\sin(2u-t)\right)'} - \cos t\big)\, du = \frac{1}{4}\sin(2u-t)\Big|_{u=0}^{u=t} - \frac{t}{2}\cos t$$

$$= \frac{1}{4}2\sin t - \frac{t}{2}\cos t = \frac{1}{2}\sin t - \frac{t}{2}\cos t.$$

- Exercise 24, page 241. Use $\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s)$ (DO NOT USE PARTIAL FRACTIONS) to find the Laplace transform of the $s-$domain function

$$\frac{s^2}{(s^2+9)^2}$$

Solution:

$$\frac{s^2}{(s^2+9)^2} = \underbrace{\frac{s}{s^2+9}}_{=F(s)} \cdot \underbrace{\frac{s}{s^2+9}}_{=G(s)}, \qquad (f(t) = g(t) = \cos 3t)$$

$$I := \mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+9)^2}\right\} = \cos 3t * \cos 3t = \int_0^t \cos 3u \cos 3(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \big(\cos 3(u+t-u) + \cos 3(u-t+u)\big)\, du$$

$$\left(\text{Use: } \cos a \cos b = \tfrac{1}{2}\big(\cos(a+b) + \cos(a-b)\big)\right)$$

$$= \frac{1}{2}\int_0^t \big(\cos 3t + \underbrace{\cos 3(2u-t)}_{\frac{1}{6}\left(\sin 3(2u-t)\right)'}\big)\, du = \frac{t}{2}\cos 3t + \frac{1}{12}\sin 3(2u-t)\Big|_{u=0}^{u=t}$$

$$= \frac{t}{2}\cos 3t + \frac{1}{12}2\sin 6t = \frac{t}{2}\cos 3t + \frac{1}{6}\sin 6t.$$