6.1 Numerical Methods. Euler's Method

Need to solve IVP

$$y' = f(t, y), \quad y(t_0) = y_0 \qquad (1)$$

We use the general solution (family of curves) and the direction field to find a numeric solution. In the direction field we use the tangent line passes through the point (t_0, y_0) as an approximate solution to the particular solution $y(t)$ of (1). The corresponding equation of the tangent line is

$$u(t) - y_0 = y'(t_0)(t - t_0)$$

where $y'(t_0)$ is the slope of the line.

Recall that $y'(t_0) = f(t_0, y_0)$ in (1). Then

$$u(t) = y_0 + f(t_0, y_0)(t - t_0) \qquad (2)$$

To find an approximate value of the solution to (1) at the point t_1 close to t_0 we use the tangent line (2) to get

$$u(t_1) = y_0 + f(t_0, y_0)(t_1 - t_0)$$

We denote this value by y_1, the value of numeric solution at t_1. Therefore, $y_1 = u(t_1)$ or

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0) \qquad (3)$$

Note that this formula lets us evaluate y_1 without solving the IVP (1).

To find a numeric value at the next point t_2 we continue to use a tangent line approach. For that we consider a new IVP

$$y' = f(t, y), \quad y(t_1) = y_1 \qquad (1')$$

and draw the tangent line $v(t)$ at the point (t_1, y_1).
Note that the actual solution to $(1')$ differs from the actual solution to (1) because of different initial conditions.

The equation of the tangent line to the solution curve of $(1')$ through the point (t_1, y_1) is

$$v(t) = y_1 + f(t_1, y_1)(t - t_1) \qquad (2')$$

At the point t_2 we have

$$v(t_2) = y_1 + f(t_1, y_1)(t_2 - t_1)$$

Denoting this result as before by y_2 we obtain

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1) \qquad (3')$$

We can continue this process and find numeric values y_3, y_4, and etc. Figure 1 shows the corresponding plots of actual solutions and tangent lines.

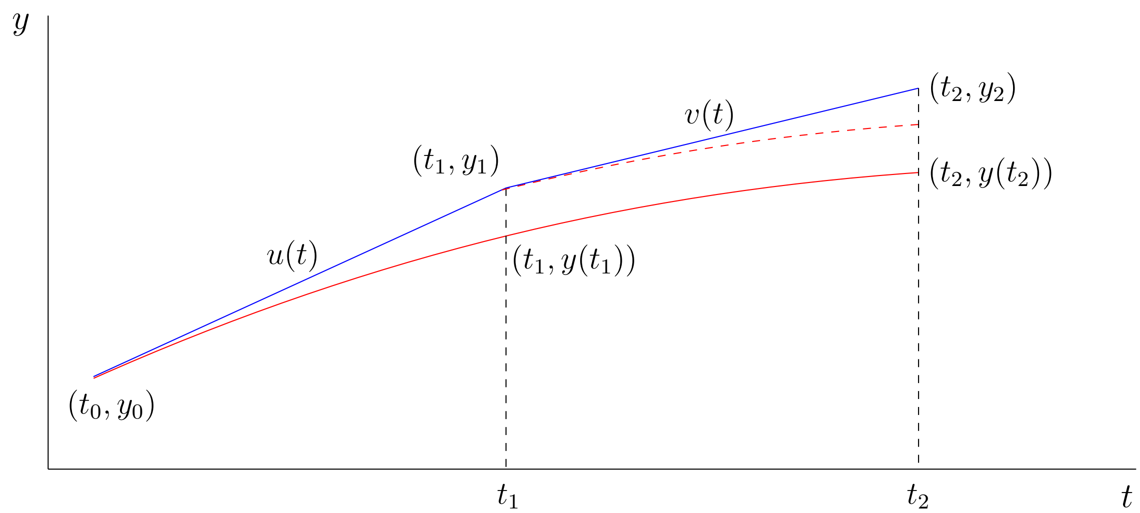

Figure 1: The bottom red line represents the actual solution to the IVP (1). The dashed red line that begins at the point (t_1, y_1) is the solution to the IVP (1′). The blue line $u(t)$ is the tangent line through the initial point (t_0, y_0) that gives the numeric solution y_1 at t_1. The blue line $v(t)$ is the tangent line through the point (t_1, y_1) that gives the numeric solution y_2 at t_2. The difference between the numeric value y_1 and the actual solution $y(t_1)$ gives the error of the method. As you can see the error at t_2 is a sum of two errors.

In general, let's find numeric solution for (1) on an interval $[a, b]$. We divide the interval into n small subintervals of equal length h that is called a step size and is found by the formula

$$h = \frac{b-a}{n}$$

and denote endpoints of the intervals by $t_0(= a)$, t_1, t_2, $, t_3, \dots, t_{n-1}$, $t_n(= b)$, where $t_i = a + i \cdot h$. Then we obtain intervals $[t_0, t_1]$, $[t_1, t_2]$, $[t_2, t_3]$, $\dots$, $[t_{n-1}, t_n]$.

On each interval $[t_i, t_{i+1}]$ using formula similar to (3) we can find an approximate (numeric) solution at the point t_{i+1} to the IVP

$$y' = f(t, y), \quad y(t_i) = y_i$$

That gives $y_{i+1} = y(t_{i+1}) = y_i + f(t_i, y_i)(t_{i+1} - t_i)$ or

$$\boxed{y_{i+1} = y_i + h \cdot f(t_i, y_i) \qquad (4)}$$

Formula (4) defines the **Euler's Method** (EM).

Example: $y' = 2t, \quad y(0) = 0$

Euler's Method: $y_{i+1} = y_i + h \cdot 2t_i$

Need an interval.

Consider $[0, 1]$ with $N = 10 \;\Rightarrow\;$ step size $h = \frac{1}{10} = 0.1$

Hence EM: $\quad y_{i+1} = y_i + 0.2t_i$

$t_0 = 0, \;\; t_1 = 0.1, \;\; t_2 = 0.2, \; \ldots \;, \; t_{10} = 1.0$ (11 points, 10 subintervals)

$y_1 = y_0 + 0.2t_0 = 0 + 0 = 0$

$y_2 = y_1 + 0.2t_1 = 0 + 0.2 \cdot 0.1 = 0.02$

$y_3 = y_2 + 0.2t_2 = 0.02 + 0.2 \cdot 0.2 = 0.06$

$y_4 = y_3 + 0.2t_3 = 0.06 + 0.2 \cdot 0.3 = 0.12$

$y_5 = 0.20, \;\; y_6 = 0.30, \;\; y_7 = 0.42, \;\; y_8 = 0.56, \;\; y_9 = 0.72, \;\; y_{10} = 0.9.$

The actual (exact) solution is $y = t^2$.

i	t_i	Numeric Solution	Actual Solution	Error
0	0	0	0	0
1	0.1	0	0.01	0.01
2	0.2	0.02	0.04	0.02
3	0.3	0.06	0.09	0.03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
8	0.8	0.56	0.64	0.08
9	0.9	0.72	0.81	0.09
10	1.0	0.9	1.0	0.1

The error in calculating the numeric solution for this example is

$$err = h\, t_i$$

(It can be shown analytically).

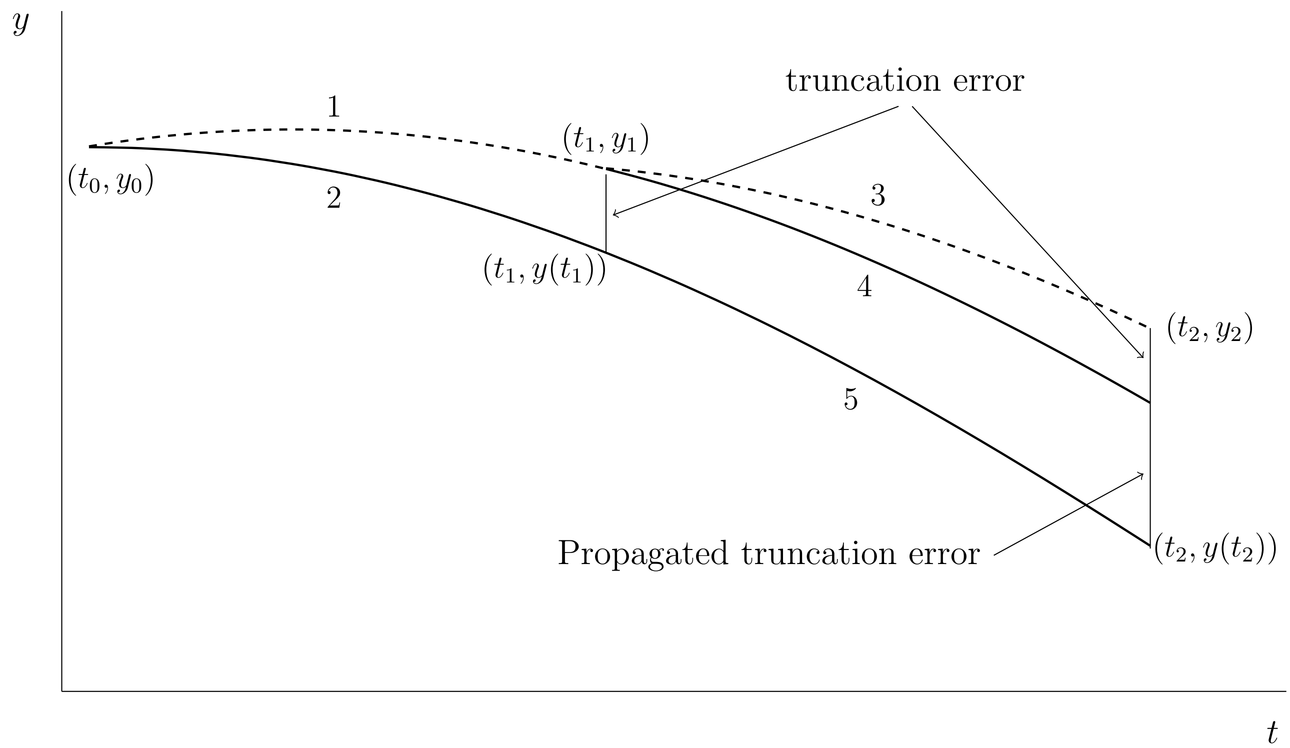

Figure 2: Dashed curves 1 and 3 represent numeric solutions found by a numeric method. For the Euler's method these curves are straight lines. Solid curves 2 and 5 represent the exact solution of the equation $y' = f(t, y)$ with the IC $y(t_0) = y_0$. Curve 4 is an exact solution of the equation with the IC $y(t_1) = y_1$. Truncation error is the difference between an exact solution and the corresponding numeric solution. Propagated truncation error is the difference between two exact solutions at t_2 obtained from two different initial conditions at t_1 (the difference between curves 4 and 5).

Error analysis

$$\text{Maximum error} \;\leq\; \frac{M}{L}\left(e^{L(b-a)} - 1\right) h \qquad (5)$$

where $L = \max\limits_{(t,y)\in R} \left| \dfrac{\partial f}{\partial y}(t, y) \right| \qquad M = \frac{1}{2} \max\limits_{(t,y)\in R} \left| \dfrac{\partial f}{\partial t} + f \dfrac{\partial f}{\partial y} \right|$

R is the rectangle which contains the solution curve. For different types of errors see figure 2.

Example: $y' = t\,y^2, \quad y(0) = 2, \quad h = 0.2.$

EM: $y_{i+1} = y_i + 0.2 \cdot t_i \cdot y_i^2$

$y_i = 2 + 0 = 2$

$y_2 = 2 + 0.2 \cdot 0.2 \cdot 2^2 = 2.16$

$y_3 = 2.5332, \quad y_4 = 3.3033, \quad y_5 = 5.0492, \quad y_6 = 10.1482$ and etc.

How good or trustful are these results?

The **actual Solution** is $y(t) = \dfrac{2}{1-t^2}$. Obviously it blows up at $t = 1$.

The interval of existence is $(-1, 1)$ and for the problem we have to assume $0 \le t < 1$, since the initial value for t is 0.

Consider the maximum possible interval $[0, 0.8]$ for t taking into account that the step size is 0.2. The maximum value of y is attained at the point 0.8 and equals $\frac{50}{9}$.

Let us find the maximum error using formula (4)

$R = \{(t, y) \mid 0 \le t \le 0.8,\ 2 \le y \le \frac{50}{9}\}$

Then $L = \max\limits_{(t,y)\in R} |2ty| = 2 \cdot 0.8 \cdot \frac{50}{9} = \frac{80}{9} \approx 8.8889, \quad M = \frac{1}{2} \max\limits_{(t,y)\in R} |y^2 + 2t^2y^3| \approx 152.6063$

Maximum error $\le \dfrac{152.6063}{8.8889}\,(e^{8.8889 \cdot 0.8} - 1) \cdot 0.2 = 4204.5337 \approx 4200$

The maximum error is too big because the solution blows up at $t = 1$ and $t = 0.8$ is close to 1.

It means that if we did not know the actual solution then using the numeric value y_4 we would say that the actual value of the solution at $t = 0.8$ has to be in the interval

$$[3.3033 - 4204.5337,\ 3.3033 + 4204.5337] = [-4201.23,\ 4207.837]$$

Obviously that does not make sense and the numeric result y_4 is not trustful.

But what if we did not know the actual solution $y(t) = \dfrac{2}{1-t^2}$? Could we still find out the error and show that the results are not trustful?

Yes, we could. First, the equation $y' = t\,y^2$ tells us that y is an increasing function because $t \ge 0$, $y^2 \ge 0$ and therefore $y' \ge 0$. Then $y \ge 2$, its initial value.

So, we can replace the upper bound $\frac{50}{9}$ for y by 2 in all previous formulas and obtain

Then $L = \max_{(t,y)\in R} |2ty| = 2 \cdot 0.8 \cdot 2 = 3.2, \quad M = \frac{1}{2} \max_{(t,y)\in R} |y^2 + 2t^2y^3| = 7.12$

Maximum error $\leq \dfrac{7.12}{3.2}\,(e^{3.2\cdot 0.8} - 1)\cdot 0.2 \approx 5.3$ which is still a large value for neat calculations.