6.2 Runge-Kutta Methods

The step size $h = \dfrac{b-a}{N}$

$t_0 = a, \quad t_k = t_{k-1} + h = a + kh, \quad t_n = b$

IVP: $y' = f(t, y), \quad y(t_0) = y_0$

The second order RK Method (improved Euler's Method)

For $k = 1$ to N:

$s_1 = f(t_{k-1}, Y_{k-1})$

$s_2 = f(t_{k-1} + h, Y_{k-1} + hs,)$

$y_k = y_{k-1} + \dfrac{h}{2}(S_1 + S_2)$

$t_k = t_{k-1} + h$

Maximum error $\leq \dfrac{M}{L}(e^{L(b-a)} - 1)h^2$.

Where L is the same as for Euler's Method, M is not the same but depends only on $f(t, y)$.

Example: $y' = 3\sqrt{t}y$, $y(0) = 1$

Exact solution is $y(t) = e^{2t^{3/2}}$.

$t_0 = 0, \ \ y_0 = 1.$ Take $h = 0.2$. Then

Step 1: $(k = 1)$

$t_1 = 0.2$

$s_1 = 0$

$s_2 = 3\sqrt{0.2}(1 + 0) = 1.341641$

$y_1 = 1 + 0.1(s_1 + s_2) = 1.134164$

Step 2: $(k = 2)$

$t_2 = 0.4$

$s_1 = 3\sqrt{0.2}(1.34164) = 1.521641$

$s_2 = 3\sqrt{0.4}(y_1 + hs_1) = 2.729347$

$y_2 = 1.134164 + 0.1(s_1 + s_2) = 1.559263$

The fourth-order RK Method

For $k = 1$ to N

$s_1 = f(t_{k-1}, y_{k-1})$

$s_2 = f(t_{k-1} + \frac{h}{2}, y_{k-1} + \frac{h}{2}s_1)$

$s_3 = f(t_{k-1} + \frac{h}{2}, y_{k-1} + \frac{h}{2}s_2)$

$s_4 = f(t_{k-1} + h, y_{k-1} + hs_3)$

$y_k = y_{k-1} + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4)$

$t_k = t_{k-1} + h$

Exercise 5, page 259 Compute by hand the first five iterations using the second-order Runge-Kutta method with step-size $h = 0.1$. Arrange the results in a tabular form: $y' = t - 2y, y(0) = 1$. Solution: See the Table in the solution to **Exercise 23**.

Exercise 21, page 260 Compute by hand the first three iterations using the fourth-order Runge-Kutta method with step-size $h = 0.1$. Arrange the results in a tabular form: $y' = t + y, y(0) = 1$.
Solution: Use RK2and4Exercise21at6_2.m to obtain

time	Eulersolution	RK2solution	RK4solution	exact
0	1	1	1	1
0.1	1.1	1.1055	1.11034166666667	1.1103418361513
0.2	1.22	1.23165275	1.24280514170139	1.24280551632034
0.3	1.362	1.380533851375	1.39971699412508	1.39971761515201
0.4	1.5282	1.55442750343819	1.58364848016137	1.58364939528254
0.5	1.72102	1.75584746753373	1.79744127719368	1.79744254140026

Exercise 22, page 260 Compute by hand the first three iterations using the fourth-order Runge-Kutta method with step-size $h = 0.1$. Arrange the results in a tabular form: $z' = 5 - z, z(0) = 1$.

```
function Exercise_6_2_22()

tspan = [0.0,0.5];
y0 = 0;
ssize = 0.1;
f=@yprime;
tt=linspace(0,0.5,200);
[t,y] = rk4(f, tspan, y0, ssize);
format short
table (t,y)

figure(1)
hold all
plot(t,y,'r.-.')
plot(tt,5.*(1-exp(-tt)),'b-')
legend('rk4','exact')

end


function value = yprime(t,y) %the RHS of the ODE
value = 5 - y;
return
end
```

```
>> Exercise_6_2_22

ans =

  6×2 table

     t        y
    ---    -------

      0          0
    0.1    0.47581
    0.2    0.90635
    0.3     1.2959
    0.4     1.6484
    0.5     1.9673
```

Exercise 23, page 260 Compute by hand the first three iterations using the fourth-order Runge-Kutta method with step-size $h = 0.1$. Arrange the results in a tabular form: $y' = x - 2z, z(0) = 0$.

```
function Exercise_6_2_23()

tspan = [0.0,0.5];
y0 = 1;
ssize = 0.1;
f=@yprime;
tt=linspace(0,0.5,200);
[t,y] = rk4(f, tspan, y0, ssize);
format short
table (t,y)

figure(1)
hold all
plot(t,y,'ro-')
legend('rk4')
end

function value = yprime(t,y) %the RHS of the ODE
value = t- 2*y;
return
end
```

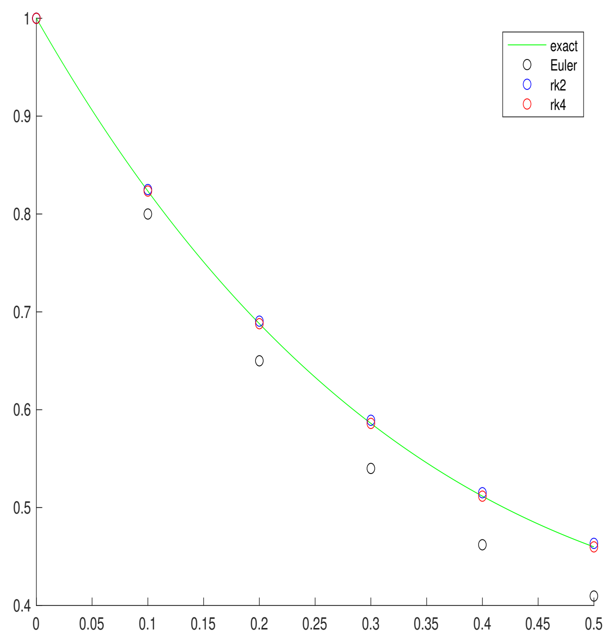
```
>>Exercise_6_2_5et23
```

time	Eulersolution	RK2solution	RK4solution	exact
----	-------------	------------	-----------------	----------------
0.0	1	1	1	1
0.1	0.8	0.825	0.823416666666667	1.1103418361513
0.2	0.65	0.6905	0.687905338888889	1.24280551632034
0.3	0.54	0.58921	0.586021031126296	1.39971761515201
0.4	0.462	0.51515	0.511668285550803	1.58364939528254
0.5	0.4096	0.46342	0.459856547656627	1.79744254140026

We note that the exact solution is

$$y(t) = \frac{5}{4}e^{-2t} + \frac{1}{4}(2t - 1).$$

Remark the plots of exact solution versus the Euler, RK2 and RK4 approximations.


exact
Euler
rk2
rk4
1
0.9
0.8
0.7
0.6
0.5
0.4
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5