Chapter 8. An Introduction to Systems
8.2 Geometric interpretation of solutions

Phase Space

Let $\bar{\mathsf{x}}(t) = (x_1(t), x_2(t))$ be a solution of the planar system

$$\begin{aligned} x_1'(t) =& f_1(t, x_1(t), x_2(t)) \\ x_2'(t) =& f_2(t, x_1(t), x_2(t)) \end{aligned}$$

Let us look at a solution curve $t \to \bar{\mathsf{x}}(t)$ of the system in the x_1x_2-plane. The x_1x_2-plane in this case is called the phase plane and it does not contain t. This means that the solution curve is a parametric curve. It is called a phase plane plot.

For a general system $\bar{\mathsf{x}}'(t) = \bar{\mathsf{f}}(t, \bar{\mathsf{x}}(t))$ of dimension n the space consisting only of $\bar{\mathsf{x}}$ coordinates is called the phase space. A plot of a solution curve $t \to \bar{\mathsf{x}}(t)$ is called a phase space plot.

Autonomous Systems

An autonomous systems is a system which does not depend explicitly on the independent variable t:

$$\bar{\mathsf{x}}' = \bar{\mathsf{f}}(t, \bar{\mathsf{x}})$$

Direction Field

Consider the planar system

$$\begin{aligned} x' =& f(t, x, y) \\ y' =& g(t, x, y) \end{aligned} \tag{1}$$

Let (1) be defined in the 3-dim box $R = \{(t, x, y) \,|\, a_1 \le t \le b_1,\ a_2 \le x \le b_2,\ a_3 \le y \le b_3\}$.

Suppose $x(t)$ and $y(t)$ are solutions to (1) and consider the curve parametrized by $t \to (t, x(t), y(t))$ in the box R. At each point of the curve there is a tangent vector

$$(1, x', y') = (1, f(t, x, y), g(t, x, y)) \tag{2}$$

which can be computed without knowing the solution to the system (1). The set of vectors (2) form the direction field of the system (1) in the 3-dim space (t, x, y).