9.2 Planar systems

Consider the equation

$$\bar{\mathsf{y}}' = A\bar{\mathsf{y}} \tag{1}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix and $\bar{\mathsf{y}}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$.

We find eigenvalues from the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc$$

Denote: $ad - bc = \det A = D$ is the determinant of A and $a + d = T$ is the trace of A, $\mathrm{tr}(A)$, the sum of diagonal elements.

Then

$$\det(A - \lambda I) = \lambda^2 - T\lambda + D = 0 \tag{2}$$

The roots of the equation are $\lambda = \frac{1}{2}(T \pm \sqrt{T^2 - 4D})$.

Cases:

Case 1. $T^2 - 4D > 0$.

There are two distinct real e-values λ_1 and λ_2.

Let $\bar{\mathsf{v}}_1$ and $\bar{\mathsf{v}}_2$ be two e-vectors associated with λ_1 and λ_2 correspondingly. Then the general solution to the equation (1) is

$$\bar{\mathsf{y}}(t) = C_1 e^{\lambda_1 t}\bar{\mathsf{v}}_1 = C_2 e^{\lambda_2 t}\bar{\mathsf{v}}_2$$

where C_1 and C_2 are arbitrary constants.

Case 2. $T^2 - 4D < 0$.

The equation (2) has two distinct complex e-values $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$,

where $\alpha = T/2 = \dfrac{a+d}{2}$ and $\beta = \frac{1}{2}\sqrt{4D - T^2} = \frac{1}{2}\sqrt{-4bc - (a-d)^2}$ are real constants.

Let $\bar{\mathsf{w}}$ be a complex-valued e-vector associated with λ, $\bar{\mathsf{w}} = \bar{\mathsf{v}}_1 + i\bar{\mathsf{v}}_2$. Then

$$\bar{\mathsf{w}} = C \begin{bmatrix} 1 \\ \dfrac{\lambda - a}{b} \end{bmatrix}$$

where C is any nonzero complex-valued constant.

One of possible solutions for $\bar{\mathsf{v}}_1$ and $\bar{\mathsf{v}}_2$ is

$$\bar{\mathsf{v}}_1 = c \begin{bmatrix} 1 \\ \dfrac{\alpha - a}{b} \end{bmatrix} \qquad \bar{\mathsf{v}}_2 = c \begin{bmatrix} 0 \\ \dfrac{\beta}{b} \end{bmatrix}$$

where c is any nonzero real-valued constant.

Then the general solution of the equation (1) can be written as

$$\bar{\mathsf{y}}(t) = C_1 e^{\alpha t}(\cos\beta t \cdot \bar{\mathsf{v}}_1 - \sin\beta t \cdot \bar{\mathsf{v}}_2) + C_2 e^{\alpha t}(\sin\beta t \cdot \bar{\mathsf{v}}_1 + \cos\beta t \cdot \bar{\mathsf{v}}_2)$$

where C_1, C_2 are arbitrary nonzero real-valued constants.

Case 3. $T^2 - 4D = 0$.

There is one real e-value of multiplicity two $\lambda = T/2$. It is also called repeated or double root.

Definition An eigenspace is a space that is formed by e-vectors.

Case 3(a). The complete case.

In this case $A = \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

The eigenspace has dimension 2 and every nonzero vector $\bar{\mathsf{v}}$ in $\mathbb{R}^2$ is an e-vector and the system (1) has the simple form

$$\bar{\mathsf{y}}' = \lambda \bar{\mathsf{y}}$$

For any vector $\bar{\mathsf{v}}$ the solution of the equation with the initial value $\bar{\mathsf{Y}}(0) = \bar{\mathsf{v}}$ is

$$\bar{\mathsf{Y}}(t) = e^{\lambda t} \cdot \bar{\mathsf{v}}$$

Case 3(b). The defective case.

In this case $A \neq \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

The dimension of the eigenspace is 1 and the equation

$$(A - \lambda I)\bar{\mathsf{v}} = \bar{0}$$

has only one nonzero solution $\bar{\mathsf{v}}_1$.

In this case we have to find $\bar{\mathsf{v}}_2$ s.t. $(A - \lambda I)\bar{\mathsf{v}}_2 = \bar{\mathsf{v}}_1$.

Then FSS is $\bar{\mathsf{Y}}_1(t) = e^{\lambda t}\bar{\mathsf{v}}_1, \quad \bar{\mathsf{Y}}_2(t) = e^{\lambda t}(\bar{\mathsf{v}}_2 + t\bar{\mathsf{v}}_1)$

and the general solution is

$$\bar{\mathsf{y}}(t) = C_1\bar{\mathsf{Y}}_1(t) + C_2\bar{\mathsf{Y}}_2(t) = e^{\lambda t}\left[(C_1 + C_2 t)\bar{\mathsf{v}}_1 + C_2\bar{\mathsf{v}}_2\right]$$

Example 1 *Solve the system*

$$\begin{cases} y_1' = y_2 \\ y_2' = y_1 \end{cases}$$

Solution: Here $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $T = 0$, $D = -1$, $\lambda = \frac{1}{2}(0 \pm \sqrt{0 - 4(-1)}) = \frac{1}{2}(\pm 2)$

$\lambda_1 = -1$, $\lambda_2 = 1$.

For $\lambda_1 = -1$: $(A - \lambda_1 I)\bar{\mathsf{v}}_1 = \bar{0}$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $v_1 + v_2 = 0$, $v_2 = -v_1$.

Take $v_1 = 1$ to get $\bar{\mathsf{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $\lambda_2 = 1$: $(A - \lambda_1 I)\bar{\mathsf{v}}_1 = \bar{0}$, $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $v_1 - v_2 = 0$, $v_2 = v_1$.

Take $v_1 = 1$ to get $\bar{\mathsf{v}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then FSS: $\bar{\mathsf{Y}}_1(t) = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\bar{\mathsf{Y}}_2(t) = e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The general solution: $\bar{\mathsf{y}}(t) = C_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 e^{-t} + C_2 e^{t} \\ -C_1 e^{-t} + C_2 e^{t} \end{bmatrix}$

$y_1(t) = C_1 e^{-t} + C_2 e^{t}$, $y_2(t) = -C_1 e^{-t} + C_2 e^{t}$.

Example 2 *Solve the system*

$$\begin{cases} y_1' = -2y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$$

Solution: $A = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$, $\quad T = 2,\ D = 2, \quad \lambda = 1 \pm \sqrt{1-2} = 1 \pm i$

Therefore, $\lambda = 1 + i,\ \bar{\lambda} = 1 - i.$

$(A - \lambda I)\bar{\mathsf{w}} = \bar{0}, \quad \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad w_1 + (1-i)w_2 = 0, \quad w_1 = (-1+i)w_2.$

Take $w_1 = -1 + i,\ w_2 = 1$ to get $\bar{\mathsf{w}} = \begin{bmatrix} -1+i \\ 1 \end{bmatrix} = \begin{bmatrix} -1+i \\ 1+0i \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Therefore $\bar{\mathsf{v}}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \bar{\mathsf{v}}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Then FSS: $\bar{\mathsf{Y}}_1(t) = e^{-t}\left(\cos t \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \quad \bar{\mathsf{Y}}_2(t) = e^{-t}\left(\sin t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

The general solution is

$$\bar{\mathsf{y}}(t) = C_1 e^{-t}\left(\cos t \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + C_2 e^{-t}\left(\sin t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\bar{\mathsf{y}}(t) = \begin{bmatrix} C_1 e^{-t}(-\cos t - \sin t) + C_2 e^{-t}(-\sin t + \cos t) \\ C_1 e^{-t}\cos t + C_2 e^{-t}\sin t \end{bmatrix}$$

$y_1(t) = C_1 e^{-t}(-\cos t - \sin t) + C_2 e^{-t}(-\sin t + \cos t),$

$y_2(t) = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t.$

Example 3 *Solve the system*

$$\begin{cases} y_1' = -4y_1 - 17y_2 \\ y_2' = 2y_1 + 2y_2 \end{cases}$$

Solution: $A = \begin{bmatrix} -4 & -17 \\ 2 & 2 \end{bmatrix}$, $\quad T = -2,\ D = 26, \quad \lambda = -1 \pm \sqrt{1-26} = -1 \pm 5i$

Therefore, $\lambda = -1 + 5i,\ \bar{\lambda} = -1 - 5i.$

$(A - \lambda I)\bar{\mathsf{w}} = \bar{0}, \quad \begin{bmatrix} -3-5i & -17 \\ 2 & 3-5i \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 2w_1 + (3-5i)w_2 = 0, \quad 2w_1 = (-3+5i)w_2.$

Take $w_1 = -3+5i$, $w_2 = 2$ to get $\bar{\mathsf{w}} = \begin{bmatrix} -3+5i \\ 2 \end{bmatrix} = \begin{bmatrix} -3+5i \\ 2+0i \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} + i \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Therefore $\bar{\mathsf{v}}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $\bar{\mathsf{v}}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Then FSS: $\bar{\mathsf{Y}}_1(t) = e^{-t}\left(\cos 5t \begin{bmatrix} -3 \\ 2 \end{bmatrix} - \sin 5t \begin{bmatrix} 5 \\ 0 \end{bmatrix}\right)$, $\bar{\mathsf{Y}}_2(t) = e^{-t}\left(\sin 5t \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \cos 5t \begin{bmatrix} 5 \\ 0 \end{bmatrix}\right)$

The general solution is

$$\bar{\mathsf{y}}(t) = C_1 e^{-t}\left(\cos 5t \begin{bmatrix} -3 \\ 2 \end{bmatrix} - \sin 5t \begin{bmatrix} 5 \\ 0 \end{bmatrix}\right) + C_2 e^{-t}\left(\sin 5t \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \cos 5t \begin{bmatrix} 5 \\ 0 \end{bmatrix}\right)$$

$$\bar{\mathsf{y}}(t) = \begin{bmatrix} C_1 e^{-t}(-3\cos 5t - 5\sin 5t) + C_2 e^{-t}(-3\sin 5t + 5\cos 5t) \\ 2C_1 e^{-t}\cos 5t + 2C_2 e^{-t}\sin 5t \end{bmatrix}$$

$y_1(t) = C_1 e^{-t}(-3\cos 5t - 5\sin 5t) + C_2 e^{-t}(-3\sin 5t + 5\cos 5t)$,

$y_2(t) = 2C_1 e^{-t}\cos 5t + 2C_2 e^{-t}\sin 5t$.

Example 4 *Solve the system*

$$\begin{cases} y_1' = 2y_1 \\ y_2' = 2y_2 \end{cases}$$

Solution: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $T = 4$, $D = 4$, $\lambda = 2 \pm \sqrt{4-4} = 2$.

$\bar{\mathsf{v}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\bar{\mathsf{v}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Then FSS: $\bar{\mathsf{Y}}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\bar{\mathsf{Y}}_2(t) = e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The general solution: $\bar{\mathsf{y}}(t) = C_1 e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 e^{2t} \\ C_2 e^{2t} \end{bmatrix}$

$y_1(t) = C_1 e^{2t}$, $y_2(t) = C_2 e^{2t}$.

Example 5 *Solve the system*

$$\begin{cases} y_1' = -2y_1 + y_2 \\ y_2' = -y_1 \end{cases}$$

Solution: $A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$, $T = -2$, $D = 1$, $\lambda = -1 \pm \sqrt{1-1} = -1$.

$(A - \lambda I)\bar{\mathsf{v}}_1 = \bar{0}$, $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $-v_1 + v_2 = 0$, $v_2 = v_1$.

Take $v_1 = 1$ to get $\bar{\mathsf{v}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We find $\bar{\mathsf{v}}_2$ from the equation $(A - \lambda I)\bar{\mathsf{v}}_2 = \bar{\mathsf{v}}_1$, $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$-v_1 + v_2 = 1$, $v_2 = v_1 + 1$.

Take $v_1 = 0$ to get $\bar{\mathsf{v}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Then FSS: $\bar{\mathsf{Y}}_1(t) = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\bar{\mathsf{Y}}_2(t) = e^{-t} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

The general solution: $\bar{\mathsf{y}}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} (C_1 + C_2 t)\, e^{-t} \\ (C_1 + C_2 + C_2 t)\, e^{-t} \end{bmatrix}$

$y_1(t) = (C_1 + C_2 t)\, e^{-t}$, $y_2(t) = (C_1 + C_2 + C_2 t)\, e^{-t}$.

Example 6 *Solve the system*

$$\begin{cases} y_1' = 2y_1 + y_2 \\ y_2' = -y_1 + 4y_2 \end{cases}$$

Solution: $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$, $T = 6$, $D = 9$, $\lambda = 3 \pm \sqrt{9-9} = 3$.

$(A - \lambda I)\bar{\mathsf{v}}_1 = \bar{0}$, $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $-v_1 + v_2 = 0$, $v_2 = v_1$.

Take $v_1 = 1$ to get $\bar{\mathsf{v}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We find $\bar{\mathsf{v}}_2$ from the equation $(A - \lambda I)\bar{\mathsf{v}}_2 = \bar{\mathsf{v}}_1$, $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$-v_1 + v_2 = 1$, $v_2 = v_1 + 1$.

Take $v_1 = 0$ to get $\bar{\mathsf{v}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Then FSS: $\bar{\mathsf{Y}}_1(t) = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\bar{\mathsf{Y}}_2(t) = e^{3t} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

The general solution: $\bar{y}(t) = C_1 e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + C_2 e^{3t}\left(\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}(C_1 + C_2 t)\, e^{3t}\\(C_1 + C_2 + C_2 t)\, e^{3t}\end{bmatrix}$

$y_1(t) = (C_1 + C_2 t)\, e^{3t}, \quad y_2(t) = (C_1 + C_2 + C_2 t)\, e^{3t}.$

- Exercise 1, page 389. The matrix A has real eigenvalues. Find the general solution to the system $\mathbf{y}' = A\mathbf{y}$.

$$A = \begin{pmatrix} 2 & -6 \\ 0 & -1 \end{pmatrix}$$

Solution:

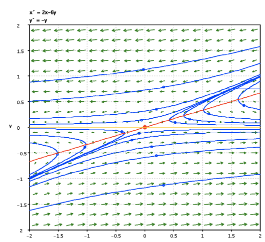


Figure 1: Exercise 1, x-nullclines, y-nullclines and equilibrium points.

$\lambda_1 = 2, \lambda_2 = -1$. ($-1 < 0 < 2 \Longrightarrow$ *The origin* $(0,0)$ *is a saddle point.*) $\lambda_1 = 2$:

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix} 2-2 & -6 \\ 0 & -1-2 \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 0 & -3 \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix}1\\0\end{pmatrix}$$

$\lambda_2 = -1$:

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix} 2+1 & -6 \\ 0 & -1+1 \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 0 & 0 \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix}2\\1\end{pmatrix}$$

The solution is

$$\mathbf{y}(t) = c_1 e^{2t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}2\\1\end{pmatrix}.$$

- Exercise 6, page 389. The matrix A has real eigenvalues. Find the general solution to the system $\mathbf{y}' = A\mathbf{y}$.

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Solution:

$$0 = \lambda^2 - \lambda \mathrm{Tr} + \det(A) = \lambda^2 - \lambda(-2) + (1-1) = \lambda(\lambda+2), \qquad \lambda_{1,2} = 0, -2.$$

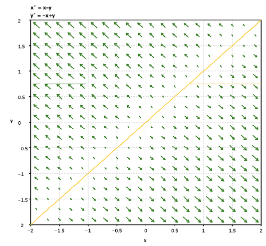


Figure 2: Exercise 6, x-nullclines, y-nullclines and equilibrium points.

(*There are no equilibrium points: matrix A is singular,* $\det(A) = 0$*!*)
$\lambda_1 = 0$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1-0 & 1 \\ 0 & -1-0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+y \\ x-y \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = -2$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1+2 & 1 \\ 1 & -1+2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The solution is

$$\mathbf{y}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- Exercise 12, page 390. Find the general solution of the initial value problem for the system $\mathbf{y}' = A\mathbf{y}$ with matrix

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the initial value $\mathbf{y}(0) = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

<u>`Solution`</u>:

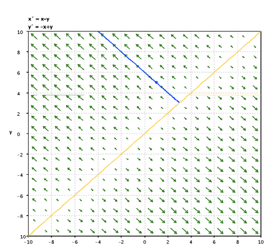


Figure 3: Exercise 12, x-nullclines, y-nullclines and equilibrium points.

The general solution is

$$\mathbf{y}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and substituting the initial condition we get

$$\begin{pmatrix}1\\5\end{pmatrix} = \begin{pmatrix}c_1+c_2\\c_1-c_2\end{pmatrix}, \qquad \begin{cases} c_1+c_2=1\\ c_1-c_2=5\end{cases}, \qquad c_1=3, c_2=-2,$$

$$\mathbf{y}(t) = \begin{pmatrix}3\\3\end{pmatrix} - 2e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}3-2e^{-2t}\\3+2e^{-2t}\end{pmatrix}.$$

- Exercise 14, page 390. A complex vector valued function $\mathbf{z}(t)$ is given. Find the real and the imaginary part of $\mathbf{z}(t)$.

$$\mathbf{z}(t) = e^{(1+i)t}\begin{pmatrix}-1+i\\2\end{pmatrix}.$$

<u>Solution</u>:

$$\begin{aligned}\mathbf{z}(t) &= e^{(1+i)t}\begin{pmatrix}-1+i\\2\end{pmatrix} = e^t(\cos t + i\sin t)\begin{pmatrix}-1+i\\2\end{pmatrix}\\ &= e^t\begin{pmatrix}-\cos t + i\cos t - i\sin t - \sin t\\ 2\cos t + 2i\sin t\end{pmatrix}\\ &= e^t\left\{\begin{pmatrix}-\cos t - \sin t\\ 2\cos t\end{pmatrix} + i\begin{pmatrix}\cos t - \sin t\\ 2\sin t\end{pmatrix}\right\}\\ &= e^t\begin{pmatrix}-\cos t - \sin t\\ 2\cos t\end{pmatrix} + ie^t\begin{pmatrix}\cos t - \sin t\\ 2\sin t\end{pmatrix}.\end{aligned}$$

- Exercise 20, page 390. The matrix A has complex eigenvalues. Find a fundamental set of **real** solutions to the system $\mathbf{y}' = A\mathbf{y}$.

$$A = \begin{pmatrix}-1 & 3\\ -3 & -1\end{pmatrix}$$

<u>Solution</u>:

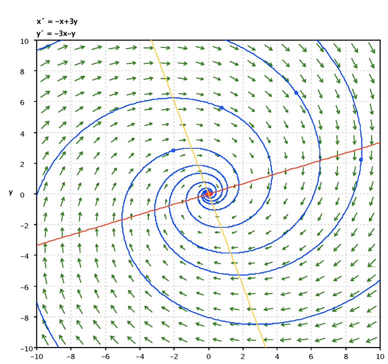


Figure 4: Exercise 20, x-nullclines, y-nullclines and equilibrium points.

$$0 = \lambda^2 - \lambda\mathrm{Tr}(A) + \det(A) = \lambda^2 - \lambda(-2) + (1+9) = \lambda^2 + 2\lambda + 10,$$

$$\lambda_{1,2} = \frac{-2\pm\sqrt{4-40}}{2} = \frac{-2\pm 6i}{2} = -1\pm 3i.$$

($\mathrm{Re}(\lambda_{1,2}) = -1 < 0$($\Longrightarrow$ *The origin* $(0,0)$ *is a spiral sink.*)
$\lambda_1 = -1 - 3i$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1+1+3i & 3 \\ -3 & -1+1+3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3i & 3 \\ -3 & 3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3ix + 3y \\ -3x + 3iy \end{pmatrix},$$

$$\mathbf{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Then solution is

$$\mathbf{y}(t) = e^{(-1-3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^{-t}\big(\cos 3t - i \sin 3t\big) \begin{pmatrix} i \\ 1 \end{pmatrix} = e^{-t} \begin{pmatrix} i\cos 3t + \sin 3t \\ \cos 3t - i \sin 3t \end{pmatrix}$$
$$= e^{-t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} + ie^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}$$

and therefore the real solution writes

$$\mathbf{y}(t) = c_1 e^{-t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix},$$

with the fundamental set of real solutions

$$e^{-t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}, \qquad e^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}.$$

- Exercise 26, page 390. Find the solution of the initial value problem for system $\mathbf{y}' = A\mathbf{y}$ with the matrix

$$A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$$

and the initial value $\mathbf{y}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

<u>Solution</u>: Since real solution is

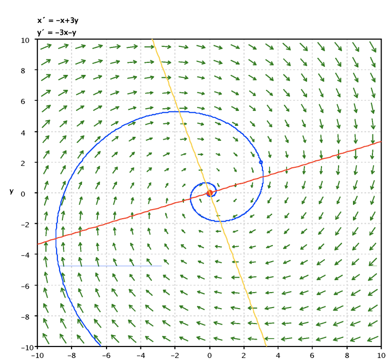


Figure 5: Exercise 26, x-nullclines, y-nullclines and equilibrium points.

$$\mathbf{y}(t) = c_1 e^{-t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix},$$

using the initial condition we obtain

$$\mathbf{y}(0) = \begin{pmatrix}3\\2\end{pmatrix} = c_1 1 \cdot \begin{pmatrix}0\\1\end{pmatrix} + c_2 1 \cdot \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}c_2\\c_1\end{pmatrix} \Longrightarrow c_1 = 2, c_2 = 3.$$

Hence

$$\begin{aligned}\mathbf{y}(t) &= 2e^{-t}\begin{pmatrix}\sin 3t\\ \cos 3t\end{pmatrix} + 3e^{-t}\begin{pmatrix}\cos 3t\\ -\sin 3t\end{pmatrix}\\ &= e^{-t}\begin{pmatrix}3\cos 3t + 2\sin 3t\\ 2\cos 3t - 3\sin 3t\end{pmatrix} \xrightarrow{t\nearrow\infty} \begin{pmatrix}0\\0\end{pmatrix} \qquad \text{the origin of the plane!!!}\end{aligned}$$

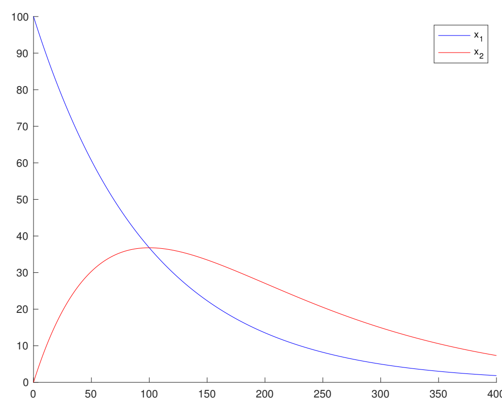
- Exercise 58, page 391. Two tanks - each holds 500 gallons o a salt solution. Pure water pours into the top tank at a rate of 5gal/s. Salt solution pours out of the bottom of the tank and into the tank below at a rate of 5 gal/s. There is a drain at the bottom of the second tank, out of which salt solution flows at a rate of 5gal/s. As a result, the amount of solution in each tank remains constant at 500 gallons. Initially (time $t = 0$) there is 100 pounds of salt present in the first tank, and zero pounds of salt present in the tank immediately below.

  (a) Set up, in matrix-vector form, an initial value problem that models the salt content in each tank.

  (b) Find the eigenvalues and eigenvectors of the coefficient matrix in part (a), then find the general solution in vector form. Find the solution that satisfies the initial condition posed in part (a).

  (c) Plot each component of your solution in part (b) $[0, 4T_c], (T_c = \frac{1}{c}$ if $x(t) = e^{-ct})$. What is the eventual salt content in each tank? Why? Give both a physical and mathematical reason for your answer.

<u>`Solution`</u>:



(a)

$$\begin{aligned}\frac{dx}{dt} &= \text{rate salt in } - \text{rate salt out } = 0 - \frac{x}{500}\cdot 5 = -\frac{x}{100},\\ \frac{dy}{dt} &= \text{rate salt in } - \text{rate salt out } = \frac{x}{100} - \frac{y}{500}\cdot 5 = \frac{x}{100} - \frac{y}{100},\end{aligned}$$

hence the system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -\frac{1}{100} & 0 \\ \frac{1}{100} & -\frac{1}{100} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

with the initial condition

$$\begin{pmatrix} x \\ y \end{pmatrix}(0) = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

(b)

$$0 = \lambda^2 - \lambda \mathrm{Tr}(A) + \det(A) = \lambda^2 - \lambda \cdot \frac{-2}{100} + \frac{1}{10^4} = \left(\lambda + \frac{1}{10^2}\right)^2, \qquad \lambda_{1,2} = -\frac{1}{100}.$$

For the eigenvectors

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{100} + \frac{1}{100} & 0 \\ \frac{1}{100} & -\frac{1}{100} + \frac{1}{100} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1}{100} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{u}{100} \end{pmatrix},$$

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1}{100} & \end{pmatrix} \begin{pmatrix} \mathbf{v}_2^1 \\ \mathbf{v}_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\mathbf{v}_2^1}{100} \end{pmatrix},$$

$$\mathbf{v}_2 = \begin{pmatrix} 100 \\ 0 \end{pmatrix}.$$

(c)

$$\mathbf{x}(t) = e^{-\frac{1}{100}t} \begin{pmatrix} 100 \\ 0 \end{pmatrix} + te^{-\frac{1}{100}t}(A - \lambda I)\mathbf{x}(0) = e^{-\frac{1}{100}t}\mathbf{x}(0) + te^{-\frac{1}{100}t} \begin{pmatrix} 0 & 0 \\ \frac{1}{100} & 0 \end{pmatrix} \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 100e^{-\frac{1}{100}t} \\ te^{-\frac{1}{100}t} \end{pmatrix} \xrightarrow{t \nearrow \infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$