9.3 Lane Plane Portraits

$$y' = Ay \quad (1)$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $y = y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

$t = a + d$, $D = ad - bc$

Char polyn $p(\lambda) = \lambda^2 - T\lambda + D = 0$

Slu's (e-values):

$\lambda = \frac{1}{2}\left[T \pm \sqrt{T^2 - 4D}\right]$

(1) is an autonomous system $\Rightarrow$ we can draw solution curves in $y_1 y_2$ - plane (phase plane)

First, find equilibrium pts of (1) , i.e points $\bar{V} \epsilon R^2$ such that $A\bar{v} = 0$.

If A is nonsingular i.e. $\det A \neq 0$ then the origin $\bar{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only equilibrium point. We assume that A is non-singular and consider different types of its equilibrium pt $(0, 0)$:

Real e-values:

Saddle pt.

CHART

e-values are real and of different signs. $\lambda_1 < 0 < \lambda_1$

$T^2 - 4D > 0$

$\bar{V_1}, \bar{V_2}$ - e-vectors

$y(t) = C_1 e^{\lambda_1 t} \bar{v}_1 + C_2 e^{\lambda_2 t} \bar{v}_2$

No dal sink. E-values are real and both are negative $\lambda_1 < \lambda_2 < 0$

CHART

The general solution decays to the origin along a direction of the slower exponential solution $C_2 e^{\lambda_2 t} \bar{v}_2$

No dal source

CHART

E-values are real and positive $0 < \lambda_1 < \lambda_2$

<u>Complex e-values:</u>

$\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$

$y(t) = c_1 e^{\alpha t}(\cos\beta t \bar{v}_1 - \sin\beta t \bar{v}_2) + C_2 e^{\alpha t}(\sin\beta t \bar{v}_1 + \cos\beta t \bar{v}_2)$

<u>Center</u>

CHART

$\alpha = 0$ (pure imaginary e-values)

sin and cos are periodic functions

<u>Spiral sink</u>

$\lambda_{1,2} = \alpha \pm \beta i$ with $\alpha < 0$

CHART

<u>Spiral source</u>

$\lambda_{1,2} = \alpha \pm \beta i$, $\alpha > 0$

CHART

Direction of rotation for the case of complex e-values. Use the point $(1, 0)$ in $y_1 y_2$ plane. Then $A * \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

If $c > 0$ the vector $\begin{pmatrix} a \\ c \end{pmatrix}$ points into the upper half plane $\Rightarrow$ the rotation is counterclockwise.

If $c < 0$ this vector points into lower half -plane $\Rightarrow$ the rotation is clockwise.

Example:

$$\begin{aligned} y_1' &= 2y_1 \\ y_2' &= y_1 + y_2 \end{aligned} \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{aligned} T &= 3 \\ D &= 2 \end{aligned}$$

$T^2 - 4D = 1 \quad \lambda_{1,2} = \frac{1}{2}\left[T \pm \sqrt{1}\right] = \dfrac{3 \pm 1}{2} \quad \lambda_1 = 1, \lambda_2 = 2$

real, positive $\Rightarrow$ nodal source

$\bar{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \bar{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

CHART

Example:

CHART

$A = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$

$T = -2 \quad D = -3 + 8 = 5$

$T^2 - 4D = -16 < 0$ complex e-values

$\lambda = -1 \pm 2i$

$\alpha = -1 < 0 \quad \Rightarrow$

Spiral sink

$c = 4 > 0 \quad \nearrow$

counterclockwise rotation

- Exercise 20, page 402. Calculate the eigenvalues to determine whether the equilibrium point is a spiral sink or a source.
  Calculate and sketch the vector generated by the right-hand side of the system at the point $(1, 0)$.
  Use this to help sketch the solution trajectory for the system passing through the point $(1, 0)$.
  Draw arrows on the solution, indicating the direction of motion.
  Use your numerical solver to check your result.

$$\mathbf{y}' = \begin{pmatrix} -2 & 20 \\ -1 & 0 \end{pmatrix} \mathbf{y}$$

Solution:

$$0 = \lambda^2 - \lambda \mathrm{Tr}(A) + \det(A) = \lambda^2 - \lambda(-2) + (-2 \cdot 0 + 1 \cdot 2) = \lambda^2 + 2\lambda + 2,$$
$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i.$$

Since the real part of the eigenvalues is negative

$$Re(\lambda_{1,2}) = -1 < 0 \qquad \text{(spiral sink)}$$

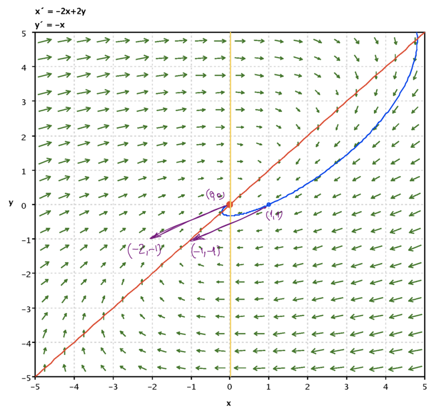

Figure 1: Exercise 20, x-nullclines, y-nullclines and equilibrium points.

the origin $(0, 0)$ is a spiral sink. To determine the sense of rotation (direction of motion) we evaluate the RHS at the point (1,0):

$$\begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

The vector starting at (0,0) pointing towards $(-2, -1)$ has to be translated to the vector starting from (1,0) to $(-1, 1)$. This vector gives the direction of motion/rotation at the point (1,0), i.e., clockwise motion.

- Exercise 22, page 402. Calculate the eigenvalues to determine whether the equilibrium point is a spiral sink or a source.
  Calculate and sketch the vector generated by the right-hand side of the system at the point $(1, 0)$.
  Use this to help sketch the solution trajectory for the system passing through the point $(1, 0)$.
  Draw arrows on the solution, indicating the direction of motion.
  Use your numerical solver to check your result.

$$\mathbf{y}' = \begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix} \mathbf{y}$$

<u>Solution</u>:

$$0 = \lambda^2 - \lambda \text{Tr}(A) + \det(A) = \lambda^2 - \lambda(2) + (-5 \cdot 7 - 4 \cdot (-10)) = \lambda^2 - 2\lambda + 5,$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i.$$

Since the real part of the eigenvalues is positive

$$Re(\lambda_{1,2}) = 1 > 0 \qquad \text{(spiral sink)}$$

the origin $(0, 0)$ is a spiral source. To determine the sense of rotation (direction of motion) we evaluate the RHS at the point (1,0):

$$\begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

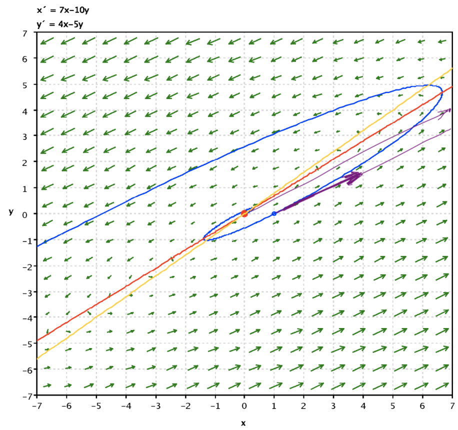

Figure 2: Exercise 22, x-nullclines, y-nullclines and equilibrium points.

The vector starting at (0,0) pointing towards $(7,4)$ has to be translated to the vector starting from (1,0) to $(8,4)$. This vector gives the direction of motion/rotation at the point (1,0), i.e., counter-clockwise motion.

- Exercise 23, page 402. Calculate the eigenvalues to determine whether the equilibrium point is a spiral sink or a source.
  Calculate and sketch the vector generated by the right-hand side of the system at the point $(1,0)$.
  Use this to help sketch the solution trajectory for the system passing through the point $(1,0)$.
  Draw arrows on the solution, indicating the direction of motion.
  Use your numerical solver to check your result.

$$\mathbf{y}' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \mathbf{y}$$

<u>Solution</u>:

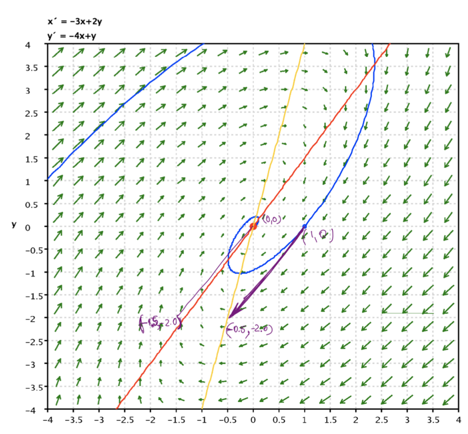

Figure 3: Exercise 23, x-nullclines, y-nullclines and equilibrium points.

$$0 = \lambda^2 - \lambda \mathrm{Tr}(A) + \det(A) = \lambda^2 - \lambda(-2) + (-3 \cdot 1 - (-4) \cdot 2) = \lambda^2 + 2\lambda + 5,$$
$$\lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = -1 \pm 2i.$$

Since the real part of the eigenvalues is negative

$$Re(\lambda_{1,2}) = -1 < 0 \qquad \text{(spiral sink)}$$

the origin $(0,0)$ is a spiral sink. To determine the sense of rotation (direction of motion) we evaluate the RHS at the point (1,0):

$$\begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

The vector starting at (0,0) pointing towards $(-3,-4)$ has to be translated to the vector starting from (1,0) to $(-2,-4)$. This vector gives the direction of motion/rotation at the point (1,0), i.e., clockwise motion.