Weierstrass Institute for
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Towards pressure-robust mixed methods for the incompressible Navier–Stokes equations

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Model

$$
\begin{aligned}
\mathbf{v}_t - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + 2\boldsymbol{\Omega} \times \mathbf{v} + \nabla p &= \mathbf{f} \\
-\nabla \cdot \mathbf{v} &= 0.
\end{aligned}
$$

Question

How many sources for velocity errors (space discretization)?

Discretizing the incompressible Navier–Stokes equations (II)

Model

$$\underline{\underline{\mathbf{v}_t}} - \nu \Delta \mathbf{v} + \underline{\underline{(\mathbf{v} \cdot \nabla)\mathbf{v}}} + \underline{\underline{2\boldsymbol{\Omega} \times \mathbf{v}}} + \nabla p = \underline{\mathbf{f}}$$

$$-\underline{\nabla \cdot \mathbf{v}} = 0.$$

Answer

- *Sources for velocity errors (space discretization) are difficult to count*
- *About 8 !*
- *Velocity errors = instabilities ???*

Model

$$\underline{\underline{\mathbf{v}_t}} - \nu\Delta\mathbf{v} + \underline{\underline{(\mathbf{v}\cdot\nabla)\mathbf{v}}} + \underline{\underline{2\boldsymbol{\Omega}\times\mathbf{v}}} + \nabla p = \underline{\mathbf{f}}$$
$$-\underline{\nabla\cdot\mathbf{v}} = 0.$$

Remark (Potential velocity errors)

- 3 *velocity errors known from scalar PDEs:*
  - *dominant advection*
  - *dominant 'reaction' in Coriolis term*
  - *'mass lumping' necessary in* $\mathbf{v}_t$
- $1+4=5$ *vectorial velocity errors:*
  - *discrete inf-sup stability*
  - L^2*-orthogonality of divergence-free vector fields and gradients fields*

Question (40 years ago)

Rather naive approach: quest for optimal (high order) methods!

- *Scalar-type sources for velocity errors were known*
- *But how to treat the vectorial character of the Navier–Stokes equations ?*

Answer (50 years ago)

Discrete inf-sup stability $\Rightarrow$ *convergence & optimal convergence order !*

Remark (Discrete divergence)

- *discrete divergence* $\operatorname{div}_h : X_h \to Q_h$
  - *conforming methods like Taylor–Hood:* $\operatorname{div}_h \mathbf{w}_h = \pi_{Q_h}(\nabla \cdot \mathbf{w}_h)$
  - *nonconforming methods like Crouzeix–Raviart, ...*
- *discrete inf-sup stability*

$$\forall g_h \in Q_h \, \exists_{\mathbf{w}_h \in X_h} : \qquad -\operatorname{div}_h \mathbf{w}_h = g_h \quad \wedge \quad \|\nabla \mathbf{w}_h\|_0 \leq \frac{1}{\beta_h} \|g_h\|_0$$

- $V_{0,h} = \{\mathbf{w}_h \in X_h : \operatorname{div}_h \mathbf{w}_h = 0\}$
- *discrete divergence replaces divergence !*
- *divergence no sense !*
- *problem:* $\phi \in H^1(D)$, $\mathbf{w}_h \in X_h$:
  - *conforming methods:* $\int_D \nabla \phi \cdot \mathbf{w}_h \, dx = -\int_D \phi \nabla \cdot \mathbf{w}_h \, dx \neq \int_D \phi \operatorname{div}_h \mathbf{w}_h \, dx$
  - *nonconforming methods:*

    $\int_D \nabla \phi \cdot \mathbf{w}_h \, dx = -\int_D \phi \operatorname{div}_h \mathbf{w}_h \, dx \quad + C h^{l+1} |\phi|_{l+1} \|\mathbf{w}_h\|_{1,h}$
- *vector L^2 scalar product can excite arbitrarily large velocity errors !*

A Stokes model

Model

$$
\begin{aligned}
-\nu \Delta \mathbf{v} + \nabla p &= \mathbf{f}, && \mathbf{x} \in \Omega \\
-\nabla \cdot \mathbf{v} &= g, && \mathbf{x} \in \Omega \\
\mathbf{v} &= \mathbf{v}_D, && \mathbf{x} \in \partial\Omega.
\end{aligned}
$$

Theorem (Stokes error estimate)

$$
\|\nabla \mathbf{v} - \nabla \mathbf{v}_h\|_{L^2} \leq 2\,(1 + C_F) \inf_{\mathbf{w}_h \in \mathbf{X}_h} \|\nabla \mathbf{v} - \nabla \mathbf{w}_h\|_{L^2} + \frac{1}{\nu} \inf_{q_h \in Q_h} \|p - q_h\|_{L^2}
$$

Remark

- *Locking for $\nu \ll 1$!!!*
- *Velocity error small only, when $\mathbf{v}$ and $\frac{1}{\nu} p$ well-resolved simultaneously !*

Model

Steady incompressible Stokes equations:

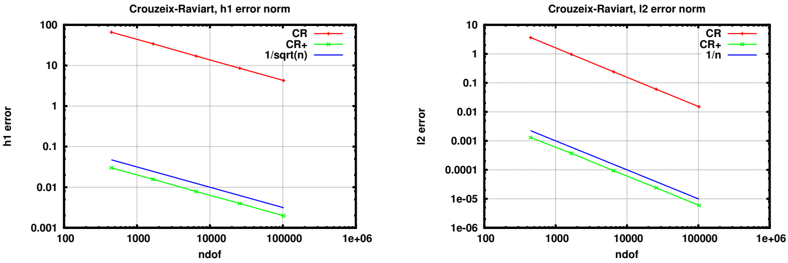
$$\begin{aligned} -\nu\Delta\mathbf{v}+\nabla p &= \underline{\mathbf{f}}, & \mathbf{x}\in\Omega \\ \nabla\cdot\mathbf{v} &= 0, & \mathbf{x}\in\Omega \\ \mathbf{v} &= 0 & \mathbf{x}\in\partial\Omega. \end{aligned}$$

- $\nu = 10^{-3}$
-

$$\begin{aligned} \xi &= x^2(1-x)^2y^2(1-y)^2, \\ \mathbf{v} &= \nabla\times\xi, \\ p &= x^3+y^3-\frac{1}{2}. \end{aligned}$$

- $\mathbf{f} := -\nu\Delta\mathbf{v}+\nabla p$ $\quad$ ν *small* $\Rightarrow$ $\mathbf{f}$ *nearly a gradient !*

Two Crouzeix–Raviart mixed FEMs (I)

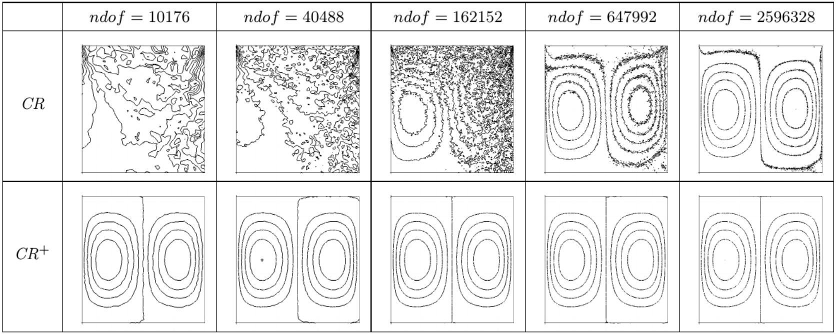


Remark

- *classical Crouzeix–Raviart element:*
  - *first-order convergent*
  - *right hand side:* $\int \mathbf{f} \cdot \mathbf{w}_h \, dx$
- *(modified) pressure-robust Crouzeix–Raviart element (A. L.: CMAME 2014):*
  - *first-order convergent*
  - *right hand side:* $\int \mathbf{f} \cdot \Pi_F(\mathbf{w}_h) \, dx$

Two Crouzeix–Raviart mixed FEMs (II)

Isolines of the vertical velocity component:



Remark

Example: Pressure-robust mixed method is 10 refinement levels more accurate.
2D: reduction of numerical effort: $4^{10} \approx 10^6$ *!*

Classical & pressure-robust mixed methods (I)

Model

$$
\begin{aligned}
-\nu \Delta \mathbf{v} + \nabla p &= \mathbf{f}, \qquad \mathbf{x} \in \Omega \\
-\nabla \cdot \mathbf{v} &= g, \qquad \mathbf{x} \in \Omega \\
\mathbf{v} &= \mathbf{v}_D, \qquad \mathbf{x} \in \partial\Omega.
\end{aligned}
$$

Theorem (Stokes velocity error estimate)

Classical: $\|\nabla \mathbf{v} - \nabla \mathbf{v}_h\|_{L^2} \leq 2\,(1 + C_F) \inf_{\mathbf{w}_h \in \mathbf{X}_h} \|\nabla \mathbf{v} - \nabla \mathbf{w}_h\|_{L^2} + \frac{1}{\nu} \inf_{q_h \in Q_h} \|p - q_h\|_{L^2}$

Pressure-robust: $\|\nabla \mathbf{v} - \nabla \mathbf{v}_h\|_{L^2} \leq 2\,(1 + C_F) \inf_{\mathbf{w}_h \in \mathbf{X}_h} \|\nabla \mathbf{v} - \nabla \mathbf{w}_h\|_{L^2} + C_\# h^k |\mathbf{v}|_{k+1}$

Remark

- *Pressure-robust mixed method* $\approx$ *velocity error is pressure-independent!*
- *velocity error small, when only* $\mathbf{v}$ *is well-resolved !*

Remark

- *'poor mass conservation': grad-div stabilization !*
- *incompressible Navier–Stokes equations: two momentum balances for divergence-free & irrotational forces*
- *orthogonal in L^2 scalar product (up to gradients of harmonic potentials)*
- *'poor mass conservation': $V_{0,h} \not\perp_{L^2} \{\nabla\phi\}$!*
- *pressure-robust methods: $V_{0,h} \perp_{L^2} \{\nabla\phi\}$!*
- $$\int_D \mathbf{f} \cdot \mathbf{w}_h \, dx = \int_D \underbrace{(\nabla\phi + \nabla \times \boldsymbol{\xi})}_{\text{Helmholtz decomposition}} \cdot \mathbf{w}_h \, dx$$
  $$= -\int_D \phi \underbrace{\nabla \cdot \mathbf{w}_h}_{\neq \mathrm{div}_h \mathbf{w}_h} dx + \int_D (\nabla \times \boldsymbol{\xi}) \cdot \mathbf{w}_h \, dx$$
- *message: 'poor mass conservation' can be repaired without stabilization !*

Remark

- *construction of pressure-robust mixed methods rather easy*
- *since 70ies: it was thought: 'pressure-robust' = divergence-free*
- *first divergence-free / pressure-robust 3D Stokes discretization in 2005 (!) by S. Zhang: Math. Comp., 2005.*
- *three classes of inf-sup stable, pressure-robust methods:*
  - *'divergence-free' H^1-conforming*, $\mathrm{div}_h \mathbf{w}_h = \nabla \cdot \mathbf{w}_h, \quad \nabla \cdot X_h = Q_h$:
    S. Zhang, M. Neilan, J. Guzman, R. Falk, T. Hughes, J. Evans, A. Buffa, ...
  - *'divergence-free' $H(\mathrm{div})$-conforming*, $\mathrm{div}_h \mathbf{w}_h = \nabla \cdot \mathbf{w}_h, \quad \nabla \cdot X_h = Q_h$:
    G. Kanschat, B. Cockburn, D. Schötzau, J. Wang, J. Schöberl, C. Lehrenfeld, classical MAC scheme (V. Girault)
  - *inf-sup stable pressure-robust sibling methods: use $H(\mathrm{div})$-conforming velocity reconstructions Π_F with $\mathrm{div}_h \mathbf{w}_h = \nabla \cdot (\Pi_F \mathbf{w}_h)$:*
    FVM: A. L.
    FEM: C. Merdon, J. Schöberl, G. Matthies, L. Tobiska, W. Wollner, A. L.
    HDG: A. Ern, D. di Pietro, F. Schieweck, A. L.

Pressure-robust sibling methods (I)

Promise

- *you give me an inf-sup stable Stokes discretization*
- *I give you back a pressure-robust Stokes discretization !*
- *does not add any artificial viscosity !*

Remark

- *works for conforming/nonconforming FEM discretizations with discontinuous pressures on simplices (Crouzeix–Raviart, Bernardi–Raugel, P2b-P_1^{disc}, ...)*
- *works for conforming/nonconforming FEM discretizations with discontinuous pressures on bricks (Q_{k+1}-P_k^{disc}, ...)*
- *(under construction for mapped elements on quadrilaterals/hexahedra)*
- *seems to work for conforming/nonconforming FEM discretizations with continuous pressures on simplices (Mini, Taylor–Hood, ...)*
- *probably useful in practice, at least good for comparison with well-established methods, ...*

References

- *A. L.: On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime. CMAME, 2014.*
- *A. L., G. Matthies, L. Tobiska: Robust arbitrary order mixed finite element methods for the incompressible Stokes equations. M2AN, 2016.*

Idea

- *Crouzeix–Raviart, Bernardi–Raugel, P2b-P_{-1}, all simplicial H^1-conforming, inf-sup stable FEMs with discontinuous pressures, ...*
- *major difficulty in Navier–Stokes equations: double orthogonality !*
- *H^1-orthogonality: appropriate by definition*
- *L^2-orthogonality: $\mathbf{f}$, $(\mathbf{v}_h \cdot \nabla)\mathbf{v}_h, 2\Omega \times \mathbf{v}_h, \ldots$ must be improved !*
- *mixed methods &relaxation of divergence condition:*
  - *good idea in trial functions*
  - *bad idea in some test functions !*

Scheme (Pressure-robust sibling method)

For all $\mathbf{w}_h \in \mathbf{X}_h$ *holds*

$$\nu \int \nabla \mathbf{v}_h : \nabla \mathbf{w}_h \, dx - \int p_h \mathrm{div}_h \mathbf{w}_h dx = \int \mathbf{f} \cdot \Pi_F(\mathbf{w}_h) \, dx, \qquad \mathrm{div}_h \mathbf{v}_h = 0.$$

Pressure-robust sibling methods (III)

Scheme (Pressure-robust sibling method)

For all $\mathbf{w}_h \in \mathbf{X}_h$ *holds*

$$\nu \int \nabla \mathbf{v}_h : \nabla \mathbf{w}_h \, dx - \int p_h \mathrm{div}_h \mathbf{w}_h dx = \int \mathbf{f} \cdot \Pi_F(\mathbf{w}_h) \, dx, \qquad \mathrm{div}_h \mathbf{v}_h = 0.$$

Properties of Π_F *:*

- $\Pi_F : \mathbf{X}_h \to H(\mathrm{div}, D)$
- $\forall q_h \in Q_h \forall \mathbf{w}_h \in \mathbf{X}_h : \quad (\nabla \cdot \mathbf{w}_h, q_h) = (\nabla \cdot \Pi_F \mathbf{w}_h, q_h)$
- $|(\Delta \mathbf{v}, \Pi_F \mathbf{w}_h) + (\nabla \mathbf{v}, \nabla \mathbf{w}_h)| \leq C h^k |\mathbf{v}|_{k+1} \cdot |\mathbf{w}_h|_1$
- *new pressure-robust sibling methods deliver in the case* $\mathbf{f} = \mathbf{0}$ *exactly the same discrete solutions like classical inf-sup stable mixed methods, since* $\Pi_F \mathbf{0} = \mathbf{0}$ *!*

Remark

$$\|\nabla \mathbf{v} - \nabla \mathbf{v}_h\|_{L^2} \leq 2\,(1 + C_F) \inf_{\mathbf{w}_h \in \mathbf{X}_h} \|\nabla \mathbf{v} - \nabla \mathbf{w}_h\|_{L^2} + C_\# \, h^k |\mathbf{v}|_{k+1}$$

Remark

- *pressure-robust convection form (not skew-symmetric)*

$$\int_D (\mathbf{v}_h \cdot \nabla)\mathbf{v}_h \cdot \Pi_F \mathbf{w}_h \, dx$$

- *pressure-robust rotational form (skew-symmetric)*

$$\int_D ((\nabla \times \mathbf{v}_h) \times \Pi_F \mathbf{v}_h) \cdot \Pi_F \mathbf{w}_h \, dx - \frac{1}{2} \int_D \mathbf{v}_h^2 \mathrm{div}_h \mathbf{w}_h \, dx$$

- *pressure-robust methods: rotational form can be safely used !*
- *note for conforming inf-sup stable methods we have*

$$\int_D (\mathbf{v}_h \cdot \nabla)\mathbf{v}_h \cdot \Pi_F \mathbf{w}_h \, dx = \int_D ((\nabla \times \mathbf{v}_h) \times \mathbf{v}_h) \cdot \Pi_F \mathbf{w}_h \, dx - \frac{1}{2} \int_D \mathbf{v}_h^2 \mathrm{div}_h \mathbf{w}_h \, dx$$

- *pressure-robust, skew-symmetric Coriolis term*

$$2 \int_D (\mathbf{\Omega} \times \Pi_F \mathbf{v}_h) \cdot \Pi_F \mathbf{w}_h \, dx.$$

Idea (Benchmark construction)

- *construct physically stable (!) benchmarks, where pressure-robust discretizations are more efficient than classical discretizations !*
- *better pressure approximation, smaller 'poor mass conservation'*
- *first order method and $\mathbf{v} \in H^2$, $p \in H^1$: maximize*

$$\mathscr{F}_1 := \frac{1}{\nu} \frac{\|p - \pi_h^{L^2} p\|_{L^2}}{\|\nabla \mathbf{v} - \nabla \pi_h^{H^1} \mathbf{v}\|} \approx \frac{1}{\nu} \frac{|p|_1}{|\mathbf{v}|_2}.$$

- *second order method and $\mathbf{v} \in H^3$, $p \in H^2$: maximize*

$$\mathscr{F}_2 := \frac{1}{\nu} \frac{\|p - \pi_h^{L^2} p\|_{L^2}}{\|\nabla \mathbf{v} - \nabla \pi_h^{H^1} \mathbf{v}\|} \approx \frac{1}{\nu} \frac{|p|_2}{|\mathbf{v}|_3}.$$

- *third order method and $\mathbf{v} \in H^4$, $p \in H^3$: maximize*

$$\mathscr{F}_3 := \frac{1}{\nu} \frac{\|p - \pi_h^{L^2} p\|_{L^2}}{\|\nabla \mathbf{v} - \nabla \pi_h^{H^1} \mathbf{v}\|} \approx \frac{1}{\nu} \frac{|p|_3}{|\mathbf{v}|_4}.$$

Conjecture

Classical inf-sup stable mixed methods perform well for momentum balance:

- $$-\nu\Delta\mathbf{v} + \underbrace{(\mathbf{v}\cdot\nabla)\mathbf{v}}_{\approx\mathbf{0}} + \nabla p = \mathbf{0}.$$
- $$\mathscr{F}_1 = \frac{1}{\nu}\frac{|p|_1}{|\mathbf{v}|_2} \approx \frac{1}{\nu}\frac{|\nu\mathbf{v}|_2}{|\mathbf{v}|_2} = 1!$$
- $\Rightarrow \quad p = \mathscr{O}(\nu)$ *and easier than* $\mathbf{v}$
- $\Rightarrow$ *error term*
  $$\frac{C_2}{\nu}\inf_{q_h\in Q_h}\|p-q_h\|_{L^2}$$
  is relatively small !
- *new pressure-robust sibling methods deliver in this case exactly the same discrete solutions like classical mixed methods, since* $\Pi_F\mathbf{0} = \mathbf{0}$ *!*
- *Stokes flow & low Reynolds number flows*
- *not to speak about: dominant advection, nonlinear convection, turbulence, ...*

Conjectures

Classical mixed methods perform poorly, when pressure large & complicated:

- *(quasi-)hydrostatics with complicated* $\mathbf{f}$*:*

$$\underbrace{-\nu\Delta\mathbf{v}}_{\approx\mathbf{0}}+\underbrace{(\mathbf{v}\cdot\nabla)\mathbf{v}}_{\approx\mathbf{0}}+\nabla p=\mathbf{f}.$$

  Reference: K. Galvin, A. L., L. Rebholz, N. Wilson: CMAME, 2012.

- *(quasi-)geostrophic flows (meteorology):*

$$\underbrace{-\nu\Delta\mathbf{v}}_{\approx\mathbf{0}}+\underbrace{(\mathbf{v}\cdot\nabla)\mathbf{v}}_{\approx\mathbf{0}}+\underbrace{fy\mathbf{v}^{\perp}}_{\text{latitude-dependent Coriolis force}}+\nabla p=\mathbf{0}.$$

- *high Reynolds number flows with momentum balance:*

$$\underbrace{-\nu\Delta\mathbf{v}}_{\approx\mathbf{0}}+(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla p=\mathbf{0}.$$

Example

(Quasi-)hydrostatics for: $-\nu\Delta\mathbf{v}+\nabla p=\mathbf{f}$:

- $\mathbf{v}=(0,0)^T, \quad \mathbf{f}=(3x^2,0)^T$
- $\mathbf{f}\in(P_k)^d \quad \Rightarrow \quad p\in P_{k+1}$.
- *Taylor–Hood element:*

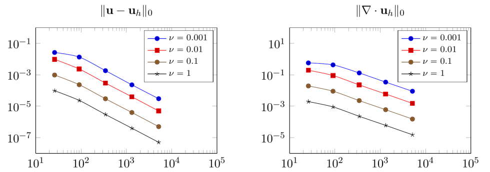


FIGURE 1. Convergence history of the L^2 velocity error (left) and the divergence error (right) of the Taylor-Hood finite element method versus the number of degrees of freedom in Section 3.1 for different values of ν.

Reference: A. L., C. Merdon: JCP, 2016.

- $\mathscr{F}_1=\mathscr{F}_2=\mathscr{F}_3=\infty$, $\mathscr{F}_4=0$

Some simple benchmarks (II)

Example

West wind for: $-\nu\Delta\mathbf{v} + y\mathbf{v}^{\perp} + \nabla p = \mathbf{0}$:

- $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad p = y^2$
- $\mathbf{v} \in (P_k)^d \quad \Rightarrow \quad p \in P_{k+2}$
- *Taylor–Hood element:*

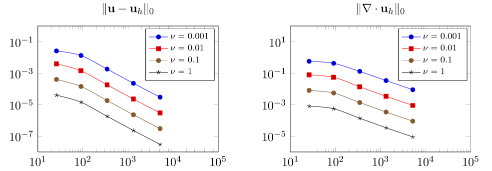


FIGURE 2. Convergence history of the L^2 velocity error (left) and the divergence error (right) of the Taylor-Hood finite element method versus the number of degrees of freedom in Section 3.2 for different values of ν.

Reference: A. L., C. Merdon: JCP, 2016.

- $\mathscr{F}_1 = \mathscr{F}_2 = \infty$, $\mathscr{F}_3 = \mathscr{F}_4 = 0$

Example

Rigid body rotation for $-\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{0}$:

- $$\mathbf{v} = \begin{pmatrix} -y \\ x \end{pmatrix}, \quad p = \frac{1}{2}(x^2 + y^2)$$
- $\mathbf{v} \in (P_k)^d \quad \Rightarrow \quad p \in P_{2k}$
- *Taylor–Hood element (*p *more complicated than* $\mathbf{v}$*):*

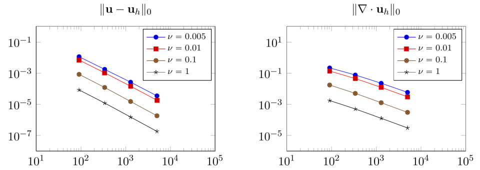


FIGURE 3. Convergence history of the L^2 velocity error (left) and the divergence error (right) of the Taylor-Hood finite element method versus the number of degrees of freedom in Section 3.3 for different values of ν.

Reference: *A. L., C. Merdon: JCP, 2016.*

- $\mathscr{F}_1 = \mathscr{F}_2 = \infty$, $\mathscr{F}_3 = \mathscr{F}_4 = 0$

Remark

- *many exact Navier–Stokes solutions fulfill* $(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\phi$*,*
- *potential flows:* $(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v}^2)$
- *generic case:* $\mathbf{v} \in (P_k)^d \quad \Rightarrow \quad p \in P_{2k}$
- *a classical (huge) class of flow problems: potential flows*
- h*: harmonic function with* $\Delta h = 0$
- $\mathbf{v} := \nabla h \qquad \Rightarrow \qquad \nabla\cdot\mathbf{v} = \Delta h = 0$
- $-\nu\Delta\mathbf{v} = \mathbf{0}$
- $$(\mathbf{v}\cdot\nabla)\mathbf{v} = (\nabla\times\mathbf{v})\times\mathbf{v} + \frac{1}{2}\nabla(\mathbf{v}^2) = \frac{1}{2}\nabla(\mathbf{v}^2)$$
- $\mathbf{v}_t = \nabla(h_t) \qquad \Rightarrow \qquad p := -h_t - \frac{1}{2}\mathbf{v}^2$
- $$\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{0}.$$

Remark

- *Compare Bernardi–Raugel element with pressure-robust sibling (with RT_0)*
- $$\int_D (\mathbf{v}_h \cdot \nabla)\mathbf{v}_h \cdot \mathbf{w}_h \, dx \qquad vs. \qquad \int_D (\mathbf{v}_h \cdot \nabla)\mathbf{v}_h \cdot \Pi_F(\mathbf{w}_h) \, dx$$
- *2D:* $h = y^5 - 5x^4y - 10x^2y^3$
-

ν	ndof					
	491	1808	7161	28123	112212	446447
$1e+01$	1.00	1.00	1.00	1.00	1.00	1.00
$1e+00$	1.00	1.00	1.00	1.00	1.00	1.00
$1e-01$	1.01	1.00	1.01	1.01	1.01	1.01
$1e-02$	1.30	1.34	1.72	1.70	1.79	1.82
$2e-03$	-	-	7.08	6.61	7.26	7.59
$1e-03$	-	-	-	12.74	13.96	14.88
$5e-04$	-	-	-	-	27.37	28.73
$2e-04$	-	-	-	-	-	73.21

Table: Reduction of L^2 gradient error

- *Speedup: 6 refinement levels* $\approx$ *factor* $4^6 = 4096$ *!*

Remark

- *Compare P2b-P_1^{disc} with pressure-robust sibling (with RT_1)*
- $$\int_D (\mathbf{v}_h \cdot \nabla)\mathbf{v}_h \cdot \mathbf{w}_h \, dx \qquad vs. \qquad \int_D (\mathbf{v}_h \cdot \nabla)\mathbf{v}_h \cdot \Pi_F(\mathbf{w}_h) \, dx$$
- *2D:* $h = y^5 - 5x^4y - 10x^2y^3$
ν	ndof						
	304	1200	4529	18175	71847	287593	1146124
$1e+00$	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$1e-01$	1.01	1.00	1.00	1.00	1.01	1.01	1.01
$1e-02$	1.31	1.34	1.21	1.41	1.44	1.45	1.47
$2e-03$	-	3.67	3.28	4.72	5.03	5.31	5.48
$1e-03$	-	-	5.12	8.48	9.39	10.20	10.72
$2e-04$	-	-	-	22.46	33.79	41.26	47.57
$1e-04$	-	-	-	-	49.01	68.30	84.17
$5e-05$	-	-	-	-	-	-	139.19

  Table: Reduction of L^2 gradient error
- *Speedup: 3 refinement levels $\approx$ factor $4^3 = 64$!*

Remark

- *Compare Bernardi–Raugel element with pressure-robust sibling (with RT_0)*
- $$\int_D (\mathbf{v}_h \cdot \nabla)\mathbf{v}_h \cdot \mathbf{w}_h \, dx \qquad vs. \qquad \int_D (\mathbf{v}_h \cdot \nabla)\mathbf{v}_h \cdot \Pi_F(\mathbf{w}_h) \, dx$$
- *3D:* $h = xyz$
ν	*ndof*			
	884	*5124*	*36555*	*277056*
$1e+00$	1.01	1.01	1.02	1.03
$1e-01$	1.63	2.19	2.42	2.58
$1e-02$	12.31	19.46	21.95	23.84
$2e-03$	35.69	71.82	97.94	114.61
$1e-03$	40.19	93.73	156.28	208.62
$5e-04$	38.57	102.71	203.99	328.12
$2e-04$	-	-	133.33	441.19

  Table: Reduction of L^2 gradient error
- *Speedup: 8 refinement levels* $\approx$ *factor* $8^8 \approx 16$ *millions !*

Remark

- *classical idea against 'poor mass conservation': grad-div stabilization,* $\gamma = \mathcal{O}(1)$*:*

$$-\nu\Delta\mathbf{v} - \gamma\nabla\nabla\cdot\mathbf{v} + \nabla p = \mathbf{f}$$

- *pressure-robust sibling methods show: artificial diffusion not necessary !*
- *trade-off, conceptually strange idea:*
  - *divergence has no sense in inf-sup stable methods*
  - *discrete divergence is zero (stable!)*
- *non-locking-free error estimate:*

$$\|\nabla\mathbf{v} - \nabla\mathbf{v}_h\|_{L^2} \le \frac{C_1}{\sqrt{\nu}} \inf_{\mathbf{w}_h\in\mathbf{X}_h} \|\nabla\mathbf{v} - \nabla\mathbf{w}_h\|_{L^2} + \frac{C_2}{\sqrt{\nu}} \inf_{q_h\in Q_h} \|p - q_h\|_{L^2}$$

- *can safely work, when additional inf-sup condition holds!*
  M. Case, V. Ervin, A. L., L. Rebholz : SINUM 2011,
  E. Jenkins, V. John, A. L., L. Rebholz: Adv. Comput. Math., 2014.
- *however: '*div_h*-*div_h*-term' can be good for linear solvers (does not change the discrete velocity solution)!*

Conjecture

- *mixed methods for Navier–Stokes: lot of confusion concerning divergence !!!*
- *discrete inf-sup stability: part of the solution*
- *Navier–Stokes: two momentum balances, no 'poor mass conservation'*
- *relaxing divergence constraint: very dangerous, but only in test functions !*
- L^2*-orthogonality of divergence-free and gradients fields important, can be repaired via changing some test functions !*
- *no artificial viscosity necessary !*
- *dozens/hundreds of new pressure-robust sibling methods !*
- *pressure-robust sibling mixed methods: universal tool (FEM, FVM, DG, ...)*
- *pressure-robust Bernardi–Raugel (3D) !*
- *classification of flows according to momentum balance* $\Rightarrow$ *better benchmarks !*
- *large to arbitrary speedups in some benchmarks in reach*
- *high Reynolds numbers: nonlinear convection excites two different oscillations !*
- *pressure-robust mixed methods: new research field in CFD !*

Open questions

Questions

- *pressure-robust siblings on general quadrilaterals & hexahedra (work in progress: G. Matthies, L. Tobiska, N. Ahmed, A. L.)*
- *pressure-robust siblings for methods with continuous pressures like Taylor–Hood (work in progress: J. Schöberl, P. Lederer, C. Merdon, A. L.)*
- *numerical analysis for time-dependent Navier–Stokes (work in progress: G. Matthies, C. Merdon, N. Ahmed, A. L.)*
- *more and better benchmarks in 2D and 3D*
- *quadratures & coupled problems*
- *compressible Euler/Navier–Stokes: low Mach numbers, well-balanced schemes, ...*

References (Pressure-robust mixed siblings)

- *A. L.: On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime. CMAME, 2014.*
- *A. L., G. Matthies, L. Tobiska: Robust arbitrary order mixed finite element methods for the incompressible Stokes equations. M2AN, 2016.*
- *A. L.: A divergence-free velocity reconstruction for incompressible flows. CRAS, 2012.*
- *C. Brennecke, A. L., C. Merdon, J. Schöberl: Optimal and pressure-independent L2 velocity error estimates for a modified Crouzeix-Raviart Stokes element with BDM reconstructions. JCM, 2015.*
- *A. L., C. Merdon: Guaranteed energy error estimators for a modified robust Crouzeix-Raviart Stokes element. J. Sci. Comp., 2015.*
- *D. di Pietro, A. Ern, A. L., F. Schieweck: A discontinuous skeletal method for the viscosity-dependent Stokes problem. CMAME, 2016.*

References (Poor mass conservation & benchmarks)

- *V. John, A. L., C. Merdon, M. Neilan, L. Rebholz: On the divergence constraint in mixed finite element methods for incompressible flows. WIAS Preprint 2177, 2015.*
- *A. L., C. Merdon: On velocity errors due to irrotational forces in the Navier-Stokes momentum balance. JCP, 2016.*
- *A. L., C. Merdon: On pressure robustness and discrete Helmholtz projectors in mixed methods for the incompressible Navier–Stokes equations. WIAS Preprint 2250, 2016.*
- *K. Galvin, A. L., L. Rebholz, N. Wilson: Stabilizing poor mass conservation in incompressible flow problems with large irrotational forcing and application to thermal convection. CMAME, 2012.*
- *A. L.: Collision in a cross-shaped domain - A steady 2d Navier-Stokes example demonstrating the importance of mass conservation in CFD. CMAME, 2009.*