Uniform inf-sup condition for the Brinkman problem in highly heterogeneous media

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Brinkman equation

$$
\begin{aligned}
-\tilde{\nu}\Delta \mathbf{u} + \nu K^{-1}\mathbf{u} + \nabla p &= \mathbf{f} \text{ in } \Omega, \\
\nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega, \\
\mathbf{u} &= 0 \text{ on } \partial\Omega,
\end{aligned}
$$

where

- $\mathbf{u}$ is the fluid velocity.
- p is the pressure.
- ν is the viscosity. We assume $\nu = 1$.
- $\tilde{\nu}$ is the effective viscosity. We assume $\tilde{\nu} = 1$.
- $\mathbf{f}$ is the external forcing term.
- $0 < K(x) < \infty$ is the permeability of the medium.

Brinkman equation

Brinkman can be used to model flows in:

- Pebble Bed Reactors,
- filtration,
- biological flows,
- oil/water reservoirs.

Flows in highly heterogeneous media

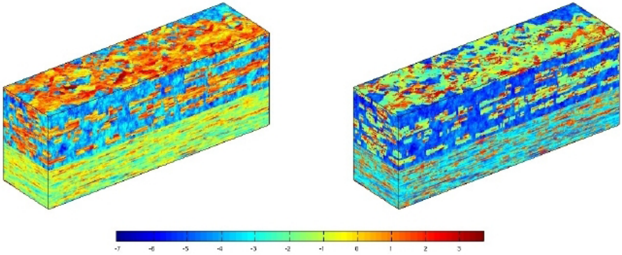

Figure: SPE10 3 dimensional permeability distributions, logscale.

Flows in highly heterogeneous media

- When permeability field $K(x)$ has large variations and jumps, the problem becomes more challenging.
- Contrast of the media: $\kappa_\Omega = \frac{\max\limits_{x \in \Omega} K(x)}{\min\limits_{x \in \Omega} K(x)}$.
- Exact solution has low regularity when $\kappa_\Omega \gg 1$.
- The known iterative methods, converge very slowly or practically do not converge when $\kappa_\Omega \gg 1$, due to the dependence of the condition number of the linear system on κ_Ω.

Preconditioners

- X - real, separable, Hilbert space,
- inner product on X is $(\cdot,\cdot)$, norm on X is $\|\cdot\|$,
- X^* be the dual of X, $\langle\cdot,\cdot\rangle$ be the duality pairing.

Given $\mathcal{A} \in \mathcal{L}(X, X^*)$, symmetric and $f \in X^*$, find $x \in X$ such that

$$\mathcal{A}x = f \Leftrightarrow$$

$$a(x,y) := \langle \mathcal{A}x, y\rangle = \langle f, y\rangle \;\; \forall y \in X.$$

Preconditioners cont-d

Definition

[3, Mardal & Winther (2011)] $\mathcal{B} \in \mathcal{L}(X^*, X)$ is a preconditioner for $\mathcal{A} \in \mathcal{L}(X, X^*)$ if $\mathcal{B}$ is symmetric and positive definite in the sense that

$$\langle \cdot, \mathcal{B} \cdot \rangle$$

is inner product on X^*.

$\mathcal{B}$ is a Riesz operator: Given $f \in X^*$

$$(\mathcal{B}f, y) = \langle f, y \rangle \quad \forall y \in X.$$

Preconditioned system

Usually

$$\mathcal{A}x = f$$

is preconditioned as

$$\mathcal{B}\mathcal{A}x = \mathcal{B}f.$$

Condition number of the system

$$cond\,(\mathcal{B}\mathcal{A}) := \|\mathcal{B}\mathcal{A}\|_{\mathcal{L}(X,X)} \|\,(\mathcal{B}\mathcal{A})^{-1}\,\|_{\mathcal{L}(X,X)}$$

Letting

$$\|a\| := \sup_{x,y \in X} \frac{a(x,y)}{\|x\|\|y\|}, \inf_{x \in X} \sup_{y \in Y} \frac{a(x,y)}{\|x\|\|y\|} \geq \gamma \Rightarrow$$

$$cond\,(\mathcal{B}\mathcal{A}) \leq \frac{\|a\|}{\gamma}.$$

Condition number of the Brinkman problem

Let

$$\mathcal{A} = \begin{pmatrix} -\Delta + \mathcal{I}K^{-1} & \nabla \\ -\nabla\cdot & 0 \end{pmatrix}.$$

Brinkman system:

$$\mathcal{A}\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}.$$

If $X = \left(H_0^1, \|\nabla\cdot\|\right), Q = \left(L_0^2, \|\cdot\|\right)$, then

$$\mathcal{B} = \begin{pmatrix} (-\Delta)^{-1} & 0 \\ 0 & \mathcal{I} \end{pmatrix}.$$

Condition number of the system

$$cond\,(\mathcal{B}\mathcal{A}) \leq \mathcal{O}\,(\kappa_\Omega)\,.$$

Question

Is it possible to establish well-posedness of the Brinkman problem, such that *cond* $(\mathcal{B}\mathcal{A})$ is independent of the media contrast κ_Ω?

Singularly perturbed Stokes system

In [2, Mardal & Winther (2004)], authors consider

$$\begin{aligned} -\varepsilon\Delta\mathbf{u} + \mathbf{u} + \nabla p &= \mathbf{f} \text{ in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \text{ in } \Omega, \\ \mathbf{u} &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Authors establish well-posedness in ε-dependent norms, both in continuous and discrete cases.

Intersection and sum of Hilbert spaces

If X and Y are Hilbert spaces, then $X+Y$ and $X\cap Y$ are also Hilbert spaces [1, Bergh, Löfström], with the following norms:

$$\|z\|_{X\cap Y} = \sqrt{\|z\|_X^2 + \|z\|_Y^2} \sim \max\left(\|z\|_X, \|z\|_Y\right),$$

$$\|z\|_{X+Y} = \inf_{\substack{z=x+y \\ x\in X, y\in Y}} \sqrt{\|x\|_X^2 + \|y\|_Y^2}.$$

If $X\cap Y$ is dense in both X and Y, then

$$(X\cap Y)^* = X^* + Y^*, \text{ and } (X+Y)^* = X^*\cap Y^*.$$

The continuous, weak formulation

$$a(\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) = (\mathbf{f},\mathbf{v})$$
$$b(q,\mathbf{u}) = 0,$$

$$a(\mathbf{u},\mathbf{v}) := (\nabla\mathbf{u},\nabla\mathbf{v}) + (\mathbf{u},\mathbf{v})_\alpha\,,$$

$$b(p,\mathbf{v}) := (p,\nabla\cdot\mathbf{v})\,,$$

where $\alpha = K^{-1}$.
The natural space for the velocity field is

$$X := \left(H_0^1 \cap L_\alpha^2\right)^d, \|\mathbf{u}\|_X := \sqrt{\|\nabla\mathbf{u}\|^2 + \|\mathbf{u}\|_\alpha^2}.$$

The pressure norm

$\|\cdot\|_Q$ should be chosen, so that the Brezzi theory for well posedness holds:

- $a(\mathbf{u},\mathbf{v}) \le \|a\|\|\mathbf{u}\|_X\|\mathbf{v}\|_X$,
- $a(\mathbf{u},\mathbf{u}) \ge a_0\|\mathbf{u}\|_X^2$,
- $b(p,\mathbf{v}) \le \|b\|\|p\|_Q\|\mathbf{v}\|_X$,
- $\inf\limits_{p\in Q}\sup\limits_{v\in X} \frac{b(p,\mathbf{v})}{\|p\|_Q\|\mathbf{v}\|_X} \ge \beta$.

For our applications, we need to ensure that *cond* $(\mathcal{B}\mathcal{A})$ is independent of κ_Ω.

The pressure norm

Let

$$Q := L_0^2 + H_K^1 \cap L_0^2 = \left\{ q \in L_0^2 : q = q_1 + q_2,\ q_1 \in L_0^2,\ q_2 \in H_K^1 \cap L_0^2 \right\},$$

with the associated norm

$$\|q\|_Q = \inf_{\substack{q = q_1 + q_2 \\ q_1 \in L_0^2, q_2 \in H_K^1 \cap L_0^2}} \sqrt{\|q_1\|^2 + \|\nabla q_2\|_K^2}.$$

The pressure norm

Lemma

Given $q \in L^2_0$, let $q_2 \in H^1_K \cap L^2_0$ be the solution of the following elliptic problem:

$$\begin{cases} -\nabla \cdot (K \nabla q_2) + q_2 = q \text{ in } \Omega, \\ K \nabla q_2 \cdot \mathbf{n} = 0 \text{ on } \partial\Omega. \end{cases}$$

Then

$$\begin{aligned} \|q\|_{L^2_0 + H^1_K \cap L^2_0} &= \sqrt{\|q_1\|^2 + \|\nabla q_2\|^2_K} \\ &= \sqrt{\|q - q_2\|^2 + \|\nabla q_2\|^2_K} \\ &= \sqrt{\|q\|^2 - \|q_2\|^2_{H^1_K \cap L^2_0}}. \end{aligned}$$

Well-posedness: Continuity of $b(\cdot,\cdot)$

Lemma

The bilinear form $b(\cdot,\cdot) : Q \times X \to R$ is continuous.

Proof.

$\forall q \in Q$, let $q = q_1 + q_2$ with $q_1 \in L_0^2$ and $q_2 \in H_K^1 \cap L_0^2$. Then

$$\begin{aligned} b(q, \mathbf{v}) &= b(q_1, \mathbf{v}) + b(q_2, \mathbf{v}) \\ &= -(q_1, \nabla \cdot \mathbf{v}) + (\nabla q_2, \mathbf{v}) \\ &= -(q_1, \nabla \cdot \mathbf{v}) + \left(K^{\frac{1}{2}} \nabla q_2, \alpha^{\frac{1}{2}} \mathbf{v}\right) \\ &\le \sqrt{d}\|q_1\|\|\nabla \mathbf{v}\| + \|\nabla q_2\|_K \|\mathbf{v}\|_\alpha \\ &\le \sqrt{d}\sqrt{\|q_1\|^2 + \|\nabla q_2\|_K^2}\|\mathbf{v}\|_X. \end{aligned}$$

Taking infimum over all q_1, q_2 gives $b(q, \mathbf{v}) \le \sqrt{d}\|q\|_Q \|\mathbf{v}\|_X$.

Well-posedness: Coercivity of $b(\cdot,\cdot)$

Lemma

There exists a constant $\beta > 0$, *independent of* $0 < K(x) < \infty$, *such that*

$$\inf_{q \in Q} \sup_{\mathbf{v} \in X} \frac{b(q, \mathbf{v})}{\|q\|_Q \|\mathbf{v}\|_X} \geq \beta.$$

Proof of the coercivity of $b(\cdot,\cdot)$

Proof.

The result of Nečas: $\forall q \in L_0^2(\Omega) : \|\nabla q\|_{H^{-1}(\Omega)} \sim \|q\|$.
$\forall q \in Q$ with $q = q_1 + q_2$ and $\int\limits_\Omega q_i = 0$. Assuming that the duality pairing $\langle\cdot,\cdot\rangle_{X^*\times X}$ is an extension of the L^2 inner product, one obtains that:

$$
\begin{aligned}
\sup_{\mathbf{v}\in X} \frac{b(q,\mathbf{v})}{\|\mathbf{v}\|_X} = \|\nabla q\|_{X^*} &= \|\nabla q\|_{H^{-1}+L^2_K} \\
&= \inf_{\substack{q=q_1+q_2 \\ \nabla q_1\in H^{-1},\nabla q_2\in L^2_K}} \sqrt{\|\nabla q_1\|^2_{H^{-1}} + \|\nabla q_2\|^2_K} \\
&\geq C \inf_{\substack{q=q_1+q_2 \\ q_1\in L^2_0, q_2\in H^1_K\cap L^2_0}} \sqrt{\|q_1\|^2 + \|\nabla q_2\|^2_K} \\
&= C\|q\|_Q.
\end{aligned}
$$

Assumption

Assumption. The mesh has resolved the heterogeneity of the medium so that elementwise contrast

$$\kappa_E = \frac{\max_{x \in E} K(x)}{\min_{x \in E} K(x)}$$

is a moderate constant. We will also set

$$\kappa_{\mathcal{T}_h} = \max_{E \in \mathcal{T}_h} \kappa_E .$$

Inf-sup with conforming subspaces

Fortin's Lemma for Mini-element:

- $\Pi_h = \Pi_h^b(I - C_h) + C_h$.
- Π_h^b satisfies $\left(\nabla \cdot \Pi_h^b \mathbf{v}, q_h\right) = (\nabla \cdot \mathbf{v}, q_h)$.
- C_h is Clement or Scott-Zhang (quasilocal) interpolant.
- In particular need, $\|C_h \mathbf{v}\|_\alpha \leq c\|\mathbf{v}\|_\alpha$, with c independent of κ_Ω.

For any $E \in \mathcal{T}_h$:

$$\|C_h \mathbf{v}\|^2_{\alpha,E} \leq \max_{x\in E} \alpha(x) \|C_h \mathbf{v}\|^2_E \leq C \max_{x\in E} \alpha(x) \|\mathbf{v}\|^2_{\Omega_E}$$
$$\leq C \frac{\max_{x\in E} \alpha(x)}{\min_{x\in \Omega_E} \alpha(x)} \|\mathbf{v}\|^2_{\alpha,\Omega_E}.$$

Non-conforming velocity space

$$X_h := \left(P_k^d \oplus \mathrm{x} P_k\right) \cap H_0\left(div, \Omega\right), Q_h := P_{k-1} \subset H^1\left(\Omega\right).$$

$$J\left(\mathbf{u}_h, \mathbf{v}_h\right) := \sum_{e \in \Gamma_h \cup \Gamma} \frac{\sigma_e}{|e|} \int_e [\mathbf{u}_h][\mathbf{v}_h],$$

$$a_h\left(\mathbf{u}_h, \mathbf{v}_h\right) := \sum_{E \in \mathcal{T}_h} \left(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h\right)_E + \left(\alpha \mathbf{u}_h, \mathbf{v}_h\right) + J\left(\mathbf{u}_h, \mathbf{v}_h\right)$$

$$- \sum_{e \in \Gamma_h \cup \Gamma} \int_e \left\{\nabla \mathbf{u}_h \cdot \mathbf{n}\right\} \left[\mathbf{v}_h\right] - \sum_{e \in \Gamma_h \cup \Gamma} \int_e \left\{\nabla \mathbf{v}_h \cdot \mathbf{n}\right\} [\mathbf{u}_h],$$

$$b_h\left(p_h, \mathbf{v}_h\right) := - \sum_{E \in \mathcal{T}_h} \left(p_h, \nabla \cdot \mathbf{v}_h\right)_E$$

Discrete weak formulation

$$
\begin{aligned}
a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(p_h, \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) \\
b_h(q_h, \mathbf{u}_h) &= 0.
\end{aligned}
$$

$$
\|\mathbf{u}_h\|_{X_h} := \sqrt{\sum_{E \in \mathcal{T}_h} \|\nabla \mathbf{u}_h\|_E^2 + \|\mathbf{u}_h\|_\alpha^2 + J(\mathbf{u}_h, \mathbf{u}_h)},
$$

$$
\|q_h\|_{\color{red}{Q}} := \inf_{q_h = q_1 + q_2} \sqrt{\|q_1\|^2 + \|\nabla q_2\|_K^2} = \sqrt{\|q_h - q_2\|^2 + \|\nabla q_2\|_K^2}.
$$

Discrete inf-sup

Lemma

The following inf-sup condition holds: There exists a constant $\beta_h > 0$*, independent of* κ_Ω *and* h*, such that*

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in X_h} \frac{b_h(q_h, \mathbf{v}_h)}{\|q_h\|_Q \|\mathbf{v}_h\|_{X_h}} \geq \beta_h.$$

Proof of discrete inf-sup

Proof.

Let $\forall q_h \in Q_h$. By continuous inf-sup condition, there exists $\mathbf{v} \in X$ such that

$$b\left(q_h, \mathbf{v}\right) \geq \beta \|q_h\|_Q \|\mathbf{v}\|_X .$$

Let $\mathbf{v}_h = \pi_h \mathbf{v} \in X_h$ be the Raviart-Thomas interpolant of $\mathbf{v}$. By definition of the Raviart-Thomas interpolant:

$$\begin{aligned} b_h\left(q_h, \mathbf{v}_h\right) &= -\sum_{E \in \mathcal{T}_h} \left(\nabla \cdot \mathbf{v}_h, q_h\right)_E = -\sum_{E \in \mathcal{T}_h} \left(\nabla \cdot \mathbf{v}, q_h\right)_E \\ &= b\left(q_h, \mathbf{v}\right) \geq \beta \|q_h\|_Q \|\mathbf{v}\|_X . \end{aligned}$$

One can show that

$$\|\mathbf{v}_h\|_\alpha \leq C \kappa_{\mathcal{T}_h} \|\mathbf{v}\|_\alpha \Rightarrow \|\mathbf{v}_h\|_X \leq C\left(\kappa_{\mathcal{T}_h}\right) \|\mathbf{v}\|_X .$$

□

Continuous vs. discrete pressure norms

Recall that for $p_h \in L_0^2$,

$$\|p_h\|_Q = \sqrt{\|p_h\|^2 - \|p_2\|^2_{H_K^1 \cap L_0^2}},$$

where

$$(K\nabla p_2, \nabla q) + (p_2, q) = (p_h, q) \ \ \forall q \in H_K^1 \cap L_0^2.$$

So in general, $p_2 \notin Q_h$, and therefore $\|p_h\|_Q$ is not computable.

Continuous vs. discrete pressure norms

Let

$$\|p_h\|_{Q_h} = \sqrt{\|p_h\|^2 - \|p_{2,h}\|^2_{H^1_K \cap L^2_0}},$$

where

$$(K\nabla p_{2,h}, \nabla q_h) + (p_{2,h}, q_h) = (p_h, q_h) \;\; \forall q_h \in Q_h,$$

Then

$$\|p_{2,h}\|_{H^1_K \cap L^2_0} \leq \|p_2\|_{H^1_K \cap L^2_0} \Rightarrow$$
$$\|p_h\|_{Q_h} \geq \|p_h\|_{Q}.$$

The end

THANK YOU!

References

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