Chapter 3

Random Vectors and Multivariate Normal Distributions

3.1 Random vectors

Definition 3.1.1. Random vector. Random vectors are vectors of random variables. For instance,

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_n \end{pmatrix},$$

where each element represent a random variable, is a random vector.

Definition 3.1.2. Mean and covariance matrix of a random vector. The mean (expectation) and covariance matrix of a random vector $\mathbf{X}$ is defined as follows:

$$E[\mathbf{X}] = \begin{pmatrix} E[\mathbf{X}_1] \\ E[\mathbf{X}_2] \\ \vdots \\ E[\mathbf{X}_n] \end{pmatrix},$$

and

$$\begin{aligned} cov(\mathbf{X}) &= E\left[\{\mathbf{X} - E(\mathbf{X})\}\{\mathbf{X} - E(\mathbf{X})\}^T\right] \\ &= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_n^2 \end{bmatrix}, \end{aligned} \tag{3.1.1}$$

where $\sigma_j^2 = var(\mathbf{X}_j)$ and $\sigma_{jk} = cov(\mathbf{X}_j, \mathbf{X}_k)$ for $j, k = 1, 2, \dots, n$.

Properties of Mean and Covariance.

1. If $\mathbf{X}$ and $\mathbf{Y}$ are random vectors and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are constant matrices, then

$$E\left[\mathbf{AXB} + \mathbf{CY} + \mathbf{D}\right] = \mathbf{A}E\left[\mathbf{X}\right]\mathbf{B} + \mathbf{C}E[\mathbf{Y}] + \mathbf{D}. \tag{3.1.2}$$

   Proof. Left as an exercise. □

2. For any random vector $\mathbf{X}$, the covariance matrix $cov(\mathbf{X})$ is symmetric.

   Proof. Left as an exercise. □

3. If $X_j, j = 1, 2, \ldots, n$ are independent random variables, then $cov(\mathbf{X}) = diag(\sigma_j^2, j = 1, 2, \ldots, n)$.

   Proof. Left as an exercise. □

4. $cov(\mathbf{X} + \mathbf{a}) = cov(\mathbf{X})$ for a constant vector $\mathbf{a}$.

   Proof. Left as an exercise. □

Properties of Mean and Covariance (cont.)

5. $cov(\mathbf{AX}) = \mathbf{A}cov(\mathbf{X})\mathbf{A}^T$ for a constant matrix $\mathbf{A}$.

   Proof. Left as an exercise. □

6. $cov(\mathbf{X})$ is positive semi-definite.

   Proof. Left as an exercise. □

7. $cov(\mathbf{X}) = E[\mathbf{X}\mathbf{X}^T] - E[\mathbf{X}]\left\{E[\mathbf{X}]\right\}^T$.

   Proof. Left as an exercise. □

Definition 3.1.3. Correlation Matrix.

A correlation matrix of a vector of random variable $\mathbf{X}$ is defined as the matrix of pairwise correlations between the elements of $\mathbf{X}$. Explicitly,

$$corr(\mathbf{X}) = \begin{bmatrix} 1 & \rho_{12} & \ldots & \rho_{1n} \\ \rho_{21} & 1 & \ldots & \rho_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{n1} & \rho_{n2} & \ldots & 1 \end{bmatrix}, \quad (3.1.3)$$

where $\rho_{jk} = corr(\mathbf{X}_j, \mathbf{X}_k) = \sigma_{jk}/(\sigma_j \sigma_k), j, k = 1, 2, \ldots, n.$

Example 3.1.1. If only successive random variables in the random vector $\mathbf{X}$ are correlated and have the same correlation ρ, then the correlation matrix $corr(\mathbf{X})$ is given by

$$corr(\mathbf{X}) = \begin{bmatrix} 1 & \rho & 0 & \ldots & 0 \\ \rho & 1 & \rho & \ldots & 0 \\ 0 & \rho & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \quad (3.1.4)$$

Example 3.1.2. If every pair of random variables in the random vector $\mathbf{X}$ have the same correlation ρ, then the correlation matrix $corr(\mathbf{X})$ is given by

$$corr(\mathbf{X}) = \begin{bmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \rho & \rho & 1 & \dots & \rho \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho & \rho & \rho & \dots & 1 \end{bmatrix}, \qquad (3.1.5)$$

and the random variables are said to be exchangeable.

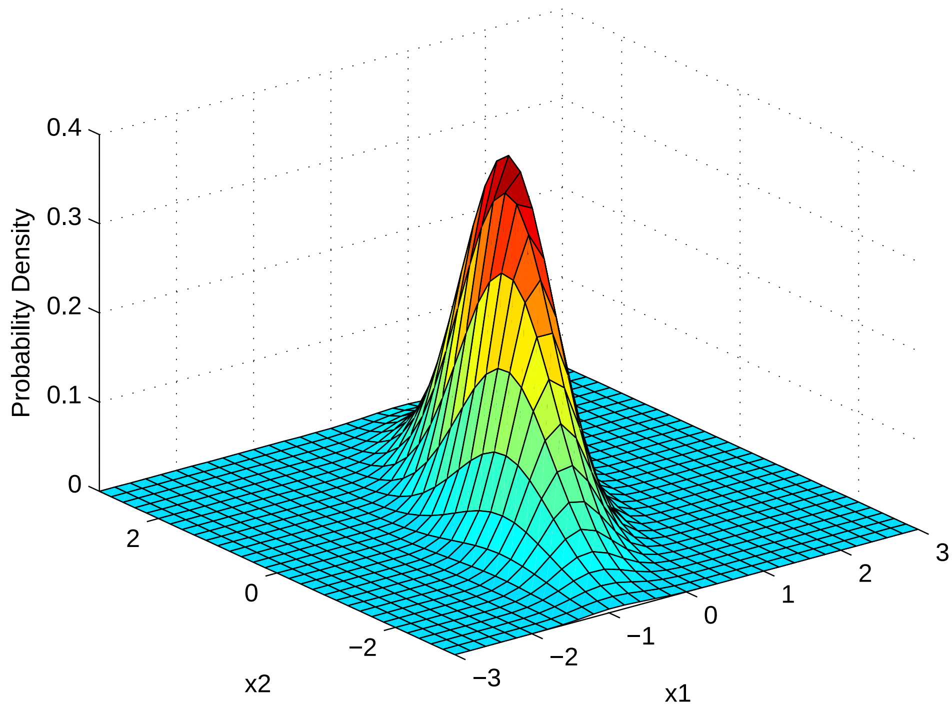
3.2 Multivariate Normal Distribution

Definition 3.2.1. Multivariate Normal Distribution. A random vector $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n)^T$ is said to follow a multivariate normal distribution with mean μ and covariance matrix $\mathbf{\Sigma}$ if $\mathbf{X}$ can be expressed as

$$\mathbf{X} = \mathbf{AZ} + \mu,$$

where $\mathbf{\Sigma} = \mathbf{AA}^T$ and $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n)$ with $\mathbf{Z}_i, i = 1, 2, \dots, n$ iid $N(0, 1)$ variables.

Bivariate normal distribution with mean $(0,0)^T$ and covariance matrix $\begin{bmatrix} 0.25 & 0.3 \\ 0.3 & 1.0 \end{bmatrix}$



Definition 3.2.2. Multivariate Normal Distribution. A random vector $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)^T$ is said to follow a multivariate normal distribution with mean μ and a positive definite covariance matrix $\mathbf{\Sigma}$ if $\mathbf{X}$ has the density

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\mathbf{\Sigma}|^{1/2}} exp\left[-\frac{1}{2}(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right] \qquad (3.2.1)$$

.

Properties

1. Moment generating function of a $N(\mu, \mathbf{\Sigma})$ random variable $\mathbf{X}$ is given by
$$M_{\mathbf{X}}(\mathbf{t}) = exp\left\{\mu^T\mathbf{t} + \frac{1}{2}\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}\right\}. \tag{3.2.2}$$

2. $E(\mathbf{X}) = \mu$ and $cov(\mathbf{X}) = \mathbf{\Sigma}$.

3. If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are i.i.d $N(0, 1)$ random variables, then their joint distribution can be characterized by $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)^T \sim N(0, \mathbf{I}_n)$.

4. $\mathbf{X} \sim N_n(\mu, \mathbf{\Sigma})$ if and only if all non-zero linear combinations of the components of $\mathbf{X}$ are normally distributed.

Linear transformation

5. If $\mathbf{X} \sim N_n(\mu, \mathbf{\Sigma})$ and $A_{m\times n}$ is a constant matrix of rank m, then $\mathbf{Y} = \mathbf{Ax} \sim N_p(\mathbf{A}\mu, \mathbf{A\Sigma A}^T)$.

   Proof. Use definition 3.2.1 or property 1 above. □

Orthogonal linear transformation

6. If $\mathbf{X} \sim N_n(\mu, \mathbf{I}_n)$ and $\mathbf{A}_{n\times n}$ is an orthogonal matrix and $\mathbf{\Sigma} = \mathbf{I}_n$, then $\mathbf{Y} = \mathbf{Ax} \sim N_n(\mathbf{A}\mu, \mathbf{I}_n)$.

Marginal and Conditional distributions

Suppose $\mathbf{X}$ is $N_n(\mu, \mathbf{\Sigma})$ and $\mathbf{X}$ is partitioned as follows,

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix},$$

where $\mathbf{X}_1$ is of dimension $p \times 1$ and $\mathbf{X}_2$ is of dimension $n-p \times 1$. Suppose the corresponding partitions for μ and $\mathbf{\Sigma}$ are given by

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \text{ and } \mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}$$

respectively. Then,

7. **Marginal distribution.** $\mathbf{X}_1$ is multivariate normal - $N_p(\mu_1, \mathbf{\Sigma}_{11})$.

   Proof. Use the result from property 5 above. □

8. **Conditional distribution.** The distribution of $\mathbf{X}_1|\mathbf{X}_2$ is p-variate normal - $N_p(\mu_{1|2}, \mathbf{\Sigma}_{1|2})$, where,

$$\mu_{1|2} = \mu_1 + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \mu_2),$$

   and

$$\mathbf{\Sigma}_{1|2} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21},$$

   provided $\mathbf{\Sigma}$ is positive definite.

   Proof. See Result 5.2.10, page 156 (Ravishanker and Dey). □

Uncorrelated implies independence for multivariate normal random variables

9. If $\mathbf{X}$, μ, and $\mathbf{\Sigma}$ are partitioned as above, then $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent if and only if $\mathbf{\Sigma}_{12} = 0 = \mathbf{\Sigma}_{21}^T$.

   Proof. We will use m.g.f to prove this result. Two random vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent iff

$$M_{(\mathbf{X}_1,\mathbf{X}_2)}(t_1, t_2) = M_{\mathbf{X}_1}(t_1)M_{\mathbf{X}_2}(t_2).$$

$\square$

3.3 Non-central distributions

We will start with the standard chi-square distribution.

Definition 3.3.1. Chi-square distribution. If $X_1, X_2, \ldots, X_n$ be n independent $N(0,1)$ variables, then the distribution of $\sum_{i=1}^n X_i^2$ is χ_n^2 (ch-square with degrees of freedom n).

χ_n^2-distribution is a special case of gamma distribution when the scale parameter is set to 1/2 and the shape parameter is set to be $n/2$. That is, the density of χ_n^2 is given by

$$f_{\chi_n^2}(x) = \frac{(1/2)^{n/2}}{\Gamma(n/2)} e^{-x/2} x^{n/2-1}, \;\; x \geq 0;\; n = 1, 2, \ldots, . \tag{3.3.1}$$

Example 3.3.1. The distribution of $(n-1)S^2/\sigma^2$, where $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$ is the sample variance of a random sample of size n from a normal distribution with mean μ and variance σ^2, follows a χ_{n-1}^2.

The moment generating function of a chi-square distribution with n d.f. is given by

$$M_{\chi_n^2}(t) = (1-2t)^{-n/2}, \; t < 1/2. \tag{3.3.2}$$

The m.g.f (3.3.2) shows that the sum of two independent ch-square random variables is also a ch-square. Therefore, differences of sequantial sums of squares of independent normal random variables will be distributed independently as chi-squares.

Theorem 3.3.2. *If* $\mathbf{X} \sim N_n(\mu, \mathbf{\Sigma})$ *and* $\mathbf{\Sigma}$ *is positive definite, then*

$$(\mathbf{X} - \mu)^T \mathbf{\Sigma}^{-1} (\mathbf{X} - \mu) \sim \chi^2_n. \tag{3.3.3}$$

Proof. Since $\mathbf{\Sigma}$ is positive definite, there exists a non-singular $\mathbf{A}_{n\times n}$ such that $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$ (Cholesky decomposition). Then, by definition of multivariate normal distribution,

$$\mathbf{X} = \mathbf{A}\mathbf{Z} + \mu,$$

where $\mathbf{Z}$ is a random sample from a $N(0, 1)$ distribution. Now, □

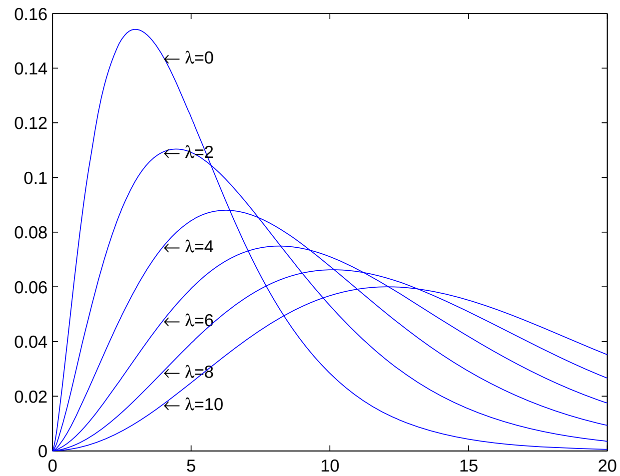


Figure 3.1: Non-central chi-square densities with df 5 and non-centrality parameter λ.

Definition 3.3.2. Non-central chi-square distribution. Suppose X's are as in Definition (3.3.1) except that each X_i has mean μ_i, $i = 1, 2, \ldots, n$. Equivalently, suppose, $\mathbf{X} = (X_1, \ldots, X_n)^T$ be a random vector distributed as $N_n(\mu, \mathbf{I}_n)$, where $\mu = (\mu_1, \ldots, \mu_n)^T$. Then the distribution of $\sum_{i=1}^n X_i^2 = \mathbf{X}^T\mathbf{X}$ is referred to as non-central chi-square with d.f. n and non-centrality parameter $\lambda = \sum_{i=1}^n \mu_i^2/2 = \frac{1}{2}\mu^T\mu$. The density of such a non-central chi-square variable $\chi_n^2(\lambda)$ can be written as a infinite poisson mixture of central chi-square densities as follows:

$$f_{\chi_n^2(\lambda)}(x) = \sum_{j=1}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \frac{(1/2)^{(n+2j)/2}}{\Gamma((n+2j)/2)} e^{-x/2} x^{(n+2j)/2-1}. \tag{3.3.4}$$

Properties

1. The moment generating function of a non-central chi-square variable $\chi^2_n(\lambda)$ is given by
$$M_{\chi^2_n(n,\lambda)}(t) = (1-2t)^{-n/2} exp\left\{\frac{2\lambda t}{1-2t}\right\},\ t < 1/2. \tag{3.3.5}$$
2. $E\left[\chi^2_n(\lambda)\right] = n + 2\lambda$.
3. $Var\left[\chi^2_n(\lambda)\right] = 2(n + 4\lambda)$.
4. $\chi^2_n(0) \equiv \chi^2_n$.
5. For a given constant c,
   (a) $P(\chi^2_n(\lambda) > c)$ is an increasing function of λ.
   (b) $P(\chi^2_n(\lambda) > c) \geq P(\chi^2_n > c)$.

Theorem 3.3.3. *If* $\mathbf{X} \sim N_n(\mu, \mathbf{\Sigma})$ *and* $\mathbf{\Sigma}$ *is positive definite, then*

$$\mathbf{X}^T\mathbf{\Sigma}^{-1}\mathbf{X} \sim \chi^2_n(\lambda = \mu^T\mathbf{\Sigma}^{-1}\mu/2). \tag{3.3.6}$$

Proof. Since $\mathbf{\Sigma}$ is positive definite, there exists a non-singular matrix $\mathbf{A}_{n\times n}$ such that $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$ (Cholesky decomposition). Define,

$$\mathbf{Y} = \{\mathbf{A}^T\}^{-1}\mathbf{X}.$$

Then, □

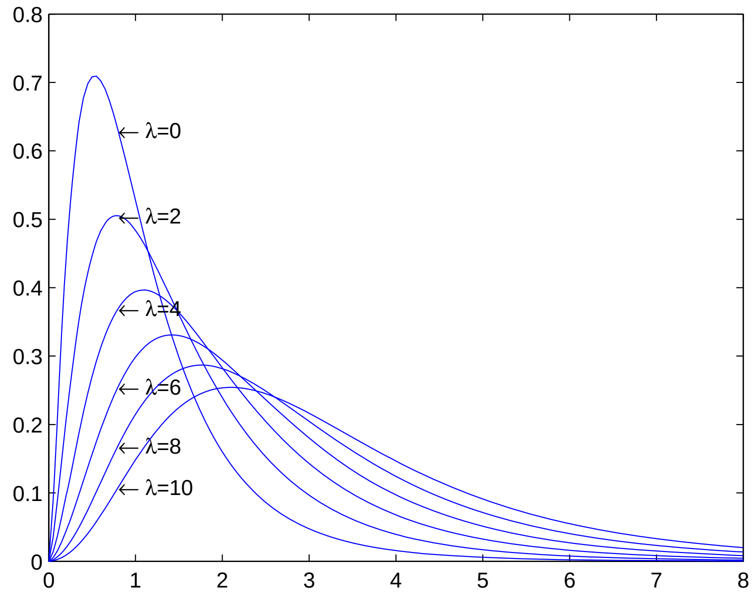


Figure 3.2: Non-central F-densities with df 5 and 15 and non-centrality parameter λ.

Definition 3.3.3. Non-central F-distribution. If $U_1 \sim \chi^2_{n_1}(\lambda)$ and $U_2 \sim \chi^2_{n_2}$ and U_1 and U_2 are independent, then, the distribution of

$$F = \frac{U_1/n_1}{U_2/n_2} \tag{3.3.7}$$

is referred to as non-central F-distribution with df n_1 and n_2, and non-centrality parameter λ.

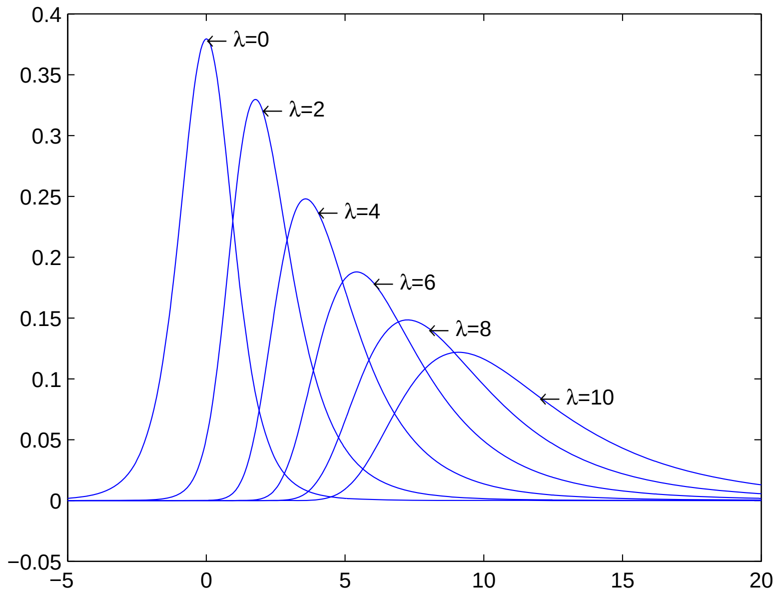


Figure 3.3: Non-central t-densities with df 5 and non-centrality parameter λ.

Definition 3.3.4. Non-central t-distribution. If $U_1 \sim N(\lambda, 1)$ and $U_2 \sim \chi^2_n$ and U_1 and U_2 are independent, then, the distribution of

$$T = \frac{U_1}{\sqrt{U_2/n}} \tag{3.3.8}$$

is referred to as non-central t-distribution with df n and non-centrality parameter λ.

3.4 Distribution of quadratic forms

Caution: We assume that our matrix of quadratic form is symmetric.

Lemma 3.4.1. *If $\mathbf{A}_{n\times n}$ is symmetric and idempotent with rank r, then r of its eigenvalues are exactly equal to 1 and $n-r$ are equal to zero.*

Proof. Use spectral decomposition theorem. (See Result 2.3.10 on page 51 of Ravishanker and Dey). □

Theorem 3.4.2. *Let $\mathbf{X} \sim N_n(0, \mathbf{I}_n)$. The quadratic form $\mathbf{X}^T\mathbf{A}\mathbf{X} \sim \chi^2_r$ iff $\mathbf{A}$ is idempotent with rank$(\mathbf{A}) = r$.*

Proof. Let $\mathbf{A}$ be (symmetric) idempotent matrix of rank r. Then, by spectral decomposition theorem, there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{\Lambda} = \begin{bmatrix} \mathbf{I}_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.4.1}$$

Define $\mathbf{Y} = \mathbf{P}^T\mathbf{X} = \begin{bmatrix} \mathbf{P}_1^T\mathbf{X} \\ \mathbf{P}_2^T\mathbf{X} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix}$, so that $\mathbf{P}_1^T\mathbf{P}_1 = \mathbf{I}_r$. Thus, $\mathbf{X} =$

$\mathbf{PY}$ and $\mathbf{Y}_1 \sim N_r(0, \mathbf{I}_r)$. Now,

$$\begin{aligned}
\mathbf{X}^T\mathbf{A}\mathbf{x} &= (\mathbf{PY})^T\mathbf{APY} \\
&= \mathbf{Y}^T \begin{bmatrix} \mathbf{I}_r & 0 \\ 0 & 0 \end{bmatrix} \mathbf{Y} \\
&= \mathbf{Y}_1^T\mathbf{Y}_1 \sim \chi_r^2. \qquad (3.4.2)
\end{aligned}$$

Now suppose $\mathbf{X}^T\mathbf{AX} \sim \chi_r^2$. This means that the moment generating function of $\mathbf{X}^T\mathbf{AX}$ is given by

$$M_{\mathbf{X}^T\mathbf{AX}}(t) = (1-2t)^{-r/2}. \qquad (3.4.3)$$

But, one can calculate the m.g.f. of $\mathbf{X}^T\mathbf{AX}$ directly using the multivariate normal density as

$$\begin{aligned}
M_{\mathbf{X}^T\mathbf{AX}}(t) &= E\left[exp\left\{(\mathbf{X}^T\mathbf{AX})t\right\}\right] \\
&= \int exp\left\{(\mathbf{X}^T\mathbf{AX})t\right\} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \\
&= \int exp\left\{(\mathbf{X}^T\mathbf{AX})t\right\} \frac{1}{(2\pi)^{n/2}} exp\left[-\frac{1}{2}\mathbf{x}^T\mathbf{x}\right] d\mathbf{x} \\
&= \int \frac{1}{(2\pi)^{n/2}} exp\left[-\frac{1}{2}\mathbf{x}^T(\mathbf{I}_n - 2t\mathbf{A})\mathbf{x}\right] d\mathbf{x} \\
&= |\mathbf{I}_n - 2t\mathbf{A}|^{-1/2} \\
&= \prod_{i=1}^{n}(1-2t\lambda_i)^{-1/2}. \qquad (3.4.4)
\end{aligned}$$

□

Equate (3.4.3) and (3.4.4) to obtain the desired result.

Theorem 3.4.3. *Let* $\mathbf{X} \sim N_n(\mu, \mathbf{\Sigma})$ *where* $\mathbf{\Sigma}$ *is positive definite. The quadratic form* $\mathbf{X}^T\mathbf{A}\mathbf{X} \sim \chi^2_r(\lambda)$ *where* $\lambda = \mu^T\mathbf{A}\mu/2$, *iff* $\mathbf{A}\mathbf{\Sigma}$ *is idempotent with* $rank(\mathbf{A}\mathbf{\Sigma}) = r$.

Proof. Omitted. □

Theorem 3.4.4. ***Independence of two quadratic forms.*** *Let* $\mathbf{X} \sim N_n(\mu, \mathbf{\Sigma})$ *where* $\mathbf{\Sigma}$ *is positive definite. The two quadratic forms* $\mathbf{X}^T\mathbf{A}\mathbf{X}$ *and* $\mathbf{X}^T\mathbf{B}\mathbf{X}$ *are independent if and only if*

$$\mathbf{A}\mathbf{\Sigma}\mathbf{B} = 0 = \mathbf{B}\mathbf{\Sigma}\mathbf{A}. \tag{3.4.5}$$

Proof. Omitted. □

Remark 3.4.1. Note that in the above theorem, the two quadratic forms need not have a chi-square distribution. When they are, the theorem is referred to as **Craig's theorem.**

Theorem 3.4.5. ***Independence of linear and quadratic forms.*** *Let* $\mathbf{X} \sim N_n(\mu, \mathbf{\Sigma})$ *where* $\mathbf{\Sigma}$ *is positive definite. The quadratic form* $\mathbf{X}^T\mathbf{A}\mathbf{X}$ *and the linear form* $\mathbf{B}\mathbf{X}$ *are independently distributed if and only if*

$$\mathbf{B}\mathbf{\Sigma}\mathbf{A} = 0. \tag{3.4.6}$$

Proof. Omitted. □

Remark 3.4.2. Note that in the above theorem, the quadratic form need not have a chi-square distribution.

Example 3.4.6. Independence of sample mean and sample variance. Suppose $\mathbf{X} \sim N_n(0, \mathbf{I}_n)$. Then $\bar{X} = \sum_{i=1}^{n} X_i/n = 1^T\mathbf{X}/n$ and $S_X^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$ are independently distributed.

Proof. □

Theorem 3.4.7. *Let* $\mathbf{X} \sim N_n(\mu, \mathbf{\Sigma})$. *Then*

$$E\left[\mathbf{X}^T\mathbf{A}\mathbf{X}\right] = \mu^T\mathbf{A}\mu + trace(\mathbf{A}\mathbf{\Sigma}). \tag{3.4.7}$$

Remark 3.4.3. Note that in the above theorem, the quadratic form need not have a chi-square distribution.

Proof. □

Theorem 3.4.8. ***Fisher-Cochran theorem.*** *Suppose* $\mathbf{X} \sim N_n(\mu, \mathbf{I}_n)$. *Let* $\mathbf{Q}_j = \mathbf{X}^T\mathbf{A}_j\mathbf{X}, j = 1, 2, \ldots, k$ *be* k *quadratic forms with* $rank(\mathbf{A}_j) = r_j$ *such that* $\mathbf{X}^T\mathbf{X} = \sum_{j=1}^k \mathbf{Q}_j$. *Then,* Q_j*'s are independently distributed as* $\chi^2_{r_j}(\lambda_j)$ *where* $\lambda_j = \mu^T\mathbf{A}_j\mu/2$ *if and only if* $\sum_{j=1}^k r_j = n$.

Proof. Omitted. □

Theorem 3.4.9. ***Generalization of Fisher-Cochran theorem.*** *Suppose* $\mathbf{X} \sim N_n(\mu, \mathbf{I}_n)$. *Let* $\mathbf{A}_j, j = 1, 2, \ldots, k$ *be* k $n \times n$ *symmetric matrices with* $rank(\mathbf{A}_j) = r_j$ *such that* $\mathbf{A} = \sum_{j=1}^k \mathbf{A}_j$ *with* $rank(\mathbf{A}) = r$. *Then,*

1. $\mathbf{X}^T\mathbf{A}_j\mathbf{X}$*'s are independently distributed as* $\chi^2_{r_j}(\lambda_j)$ *where* $\lambda_j = \mu^T\mathbf{A}_j\mu/2$, *and*

2. $\mathbf{X}^T\mathbf{A}\mathbf{X} \sim \chi^2_r(\lambda)$ *where* $\lambda = \sum_{j=1}^k \lambda_j$

if and only if ANY ONE *of the following conditions is satisfied.*

C1. $\mathbf{A}_j\mathbf{\Sigma}$ *is idempotent for all* j *and* $\mathbf{A}_j\mathbf{\Sigma}\mathbf{A}_k = 0$ *for all* $j < k$.

C2. $\mathbf{A}_j\mathbf{\Sigma}$ *is idempotent for all* j *and* $\mathbf{A}\mathbf{\Sigma}$ *is idempotent.*

C3. $\mathbf{A}_j\mathbf{\Sigma}\mathbf{A}_k = 0$ *for all* $j < k$ *and* $\mathbf{A}\mathbf{\Sigma}$ *is idempotent.*

C4. $r = \sum_{j=1}^k r_j$ *and* $\mathbf{A}\mathbf{\Sigma}$ *is idempotent.*

C5. the matrices $\mathbf{A}\mathbf{\Sigma}$, $\mathbf{A}_j\mathbf{\Sigma}, j = 1, 2, \ldots, k-1$ *are idempotent and* $\mathbf{A}_k\mathbf{\Sigma}$ *is non-negative definite.*

3.5 Problems

1. Consider the matrix

$$A = \begin{pmatrix} 8 & 4 & 4 & 2 & 2 & 2 & 2 \\ 4 & 4 & 0 & 2 & 2 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

   (a) Find the rank of this matrix.

   (b) Find a basis for the null space of A.

   (c) Find a basis for the column space of A.

2. Let $X_i, i = 1, 2, 3$ are independent standard normal random variables. Show that the variance-covariance matrix of the 3-dimensional vector $\mathbf{Y}$, defined as

$$\mathbf{Y} = \begin{pmatrix} 5X_1 \\ 1.6X_1 - 1.2X_2 \\ 2X_1 - X_2 \end{pmatrix},$$

   is not positive definite.

3. Let

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3 \left[\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix} \right].$$

   (a) Find the marginal distribution of X_2.

   (b) What is the conditional distribution of X_2 given $X_1 = x_1$ and $X_3 = x_3$? Under what condition does this distribution coincide with the marginal distribution of X_2?

4. If $\mathbf{X} \sim N_n(\mu, \mathbf{\Sigma})$, then show that $(\mathbf{X} - \mu)^T \mathbf{\Sigma}^{-1} (\mathbf{X} - \mu) \sim \chi^2_n$.

5. Suppose $\mathbf{Y} = (Y_1, Y_2, Y_3)^T$ be distributed as $N_3(0, \sigma^2 I_3)$.

   (a) Consider the quadratic form:

$$Q = \frac{(Y_1 - Y_2)^2 + (Y_2 - Y_3)^2 + (Y_3 - Y_1)^2}{3}. \tag{3.5.1}$$

   Write Q as $\mathbf{Y}^T \mathbf{A} \mathbf{Y}$ where $\mathbf{A}$ is symmetric. Is $\mathbf{A}$ idempotent? What is the distribution of Q/σ^2? Find $E(Q)$.

   (b) What is the distribution of $L = Y_1 + Y_2 + Y_3$? Find $E(L)$ and $Var(L)$.

   (c) Are Q and L independent? Find $E(Q/L^2)$

6. Write each of the following quadratic forms in $\mathbf{X}^T \mathbf{A} \mathbf{X}$ form:

   (a) $\frac{1}{6}X_1^2 + \frac{2}{3}X_2^2 + \frac{1}{6}X_3^2 - \frac{2}{3}X_1X_2 + \frac{1}{3}X_1X_3 - \frac{2}{3}X_2X_3$

(b) $\sum_{i=1}^{n} X_i^2$

(c) $\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2$

(d) $\sum_{i=1}^{2}\sum_{j=1}^{2} \left(X_{ij} - \bar{X}_{i.}\right)^2$, where $\bar{X}_{i.} = \frac{X_{i1}+X_{i2}}{2}$

(e) $2\sum_{i=1}^{2} \left(\bar{X}_{i.} - \bar{X}_{..}\right)^2$, where $\bar{X}_{..} = \frac{X_{11}+X_{12}+X_{21}+X_{22}}{4}$.

In each case, determine if A is idempotent. If A is idempotent, find $rank(A)$.

7. Let $X \sim N_2(\mu, \Sigma)$, where $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, and $\Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$. Show that $Q_1 = (X_1 - X_2)^2$ and $Q_2 = (X_1 + X_2)^2$ are independently distributed. Find the distribution of Q_1, Q_2, and $\frac{Q_2}{3Q_1}$.

8. Assume that $Y \sim N_3(0, I_3)$. Define $Q_1 = Y^T AY$ and $Q_2 = Y^T BY$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{and, } B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.5.2}$$

Are Q_1 and Q_2 independent? Do Q_1 and Q_2 follow χ^2 distribution?

9. Let $Y \sim N_3(0, I_3)$. Let $U_1 = Y^T A_1 Y$, $U_2 = Y^T A_2 Y$, and $V = BY$

where

$$A_1 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{and}, B = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}.$$

(a) Are U_1 and U_2 independent?

(b) Are U_1 and V independent?

(c) Are U_2 and V independent?

(d) Find the distribution of V.

(e) Find the distribution of $\frac{U_2}{U_1}$. (Include specific values for any parameters of the distribution.)

10. Suppose $X \sim N_3(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \text{and } \Sigma = \begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix}.$$

Find the distribution of $Q = \sum_{i=1}^{3} \frac{X_i^2}{\sigma_i^2}$. Express the parameters of its distribution in terms of μ_i and $\sigma_i^2, i = 1, 2, 3$. What is the variance of Q?

11. Suppose $X \sim N(0, 1)$ and $Y = UX$, where U follows a uniform distribution on the discrete space $\{-1, 1\}$ independently of X.

(a) Find $E(Y)$ and $cov(X,Y)$.

(b) Show that Y and X are not independent.

12. Suppose $X \sim N_4(\mu, I_4)$, where

$$X = \begin{pmatrix} X_{11} \\ X_{12} \\ X_{21} \\ X_{22} \end{pmatrix} \mu = \begin{pmatrix} \alpha + a_1 \\ \alpha + a_1 \\ \alpha + a_2 \\ \alpha + a_2 \end{pmatrix}.$$

(a) Find the distribution of $E = \sum_{i=1}^{2} \sum_{j=1}^{2} \left(X_{ij} - \bar{X}_{i.}\right)^2$, where $\bar{X}_{i.} = \frac{X_{i1}+X_{i2}}{2}$.

(b) Find the distribution of $Q = 2\sum_{i=1}^{2} \left(\bar{X}_{i.} - \bar{X}_{..}\right)^2$, where $\bar{X}_{..} = \frac{X_{11}+X_{12}+X_{21}+X_{22}}{4}$.

(c) Use Fisher-Cochran theorem to prove that E and Q are independently distributed.

(d) What is the distribution of Q/E?