Introduction to spatial modeling

(a mostly geometrical presentation)

Alternatives

- $X = \Re^n$ e.g. $(x_1, x_2, \ldots, x_n) \in X$
- Alternatives are infinite set of “policies” in n-dimensional Euclidean space
- Each dimension is an issue or characteristic of policy:

Economic liberalism	Defense spending
Civil liberties	Welfare spending
Taxation	Trade protection
Redistribution	Immigration

Preferences

- Preferences are satiable
- Each agent has an ideal point
- Utility declines as a distance from ideal point increases

$$U(x) = -\sum_{j=1}^{k} \alpha_j \left| x_j - \theta_j \right|$$

Linear

$$U(x) = -\sum_{j=1}^{k} \alpha_j \left(x_j - \theta_j \right)^2$$

Quadratic

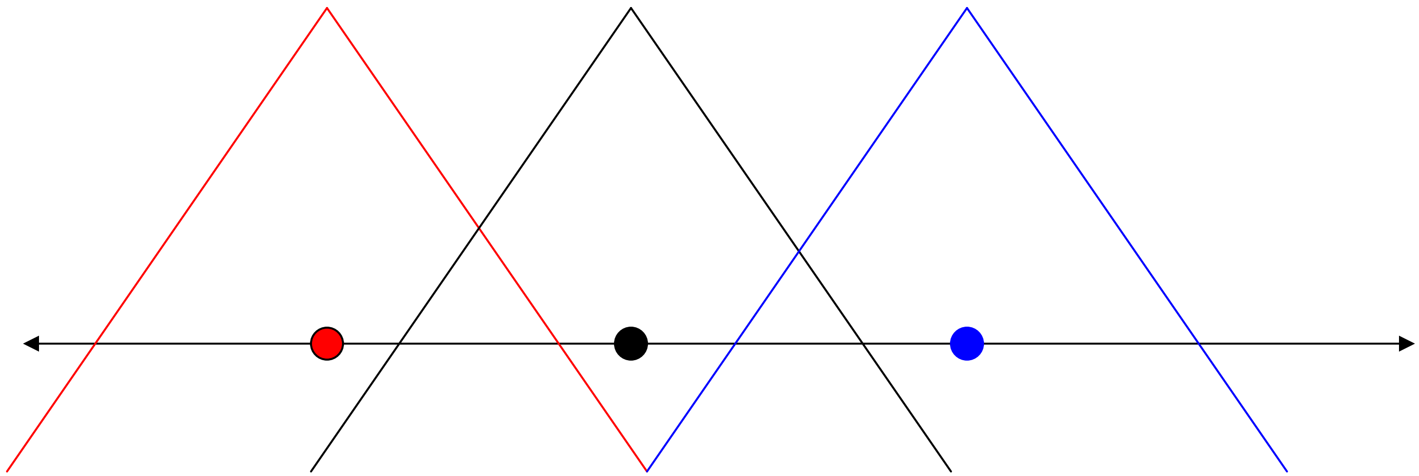
j indexes dimensions

α_j = weight on dimension j

θ_j = ideal policy on dimension j

One dimension

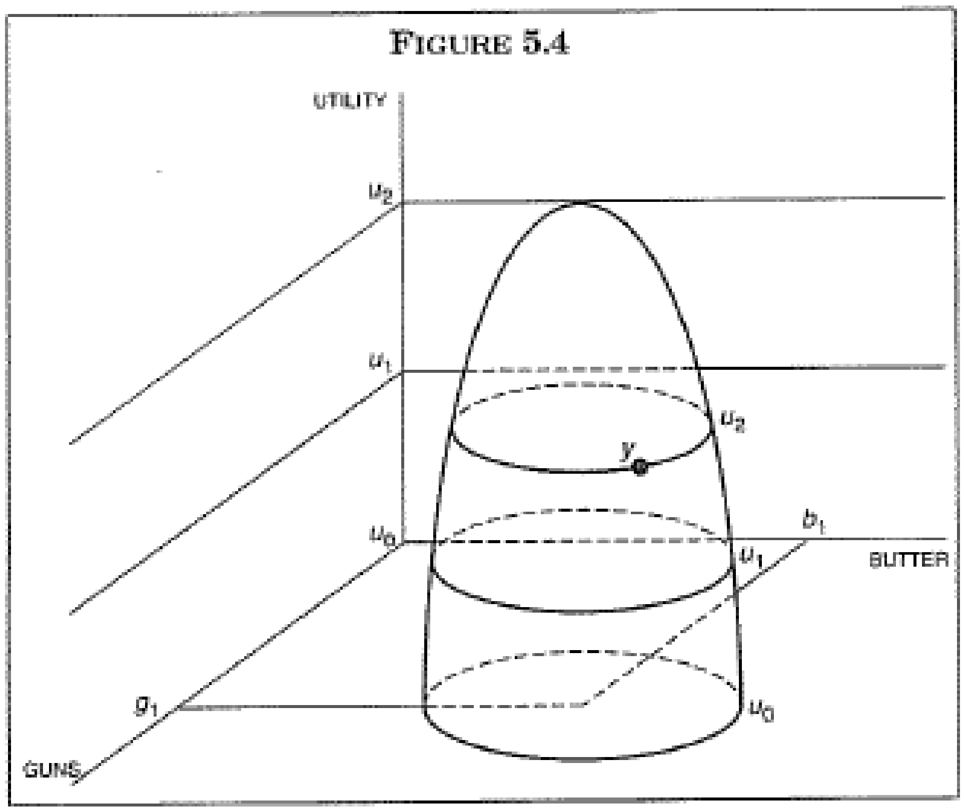
- Preferences satisfy single-peaked property
- Black's median voter theorem applies



Two dimensions

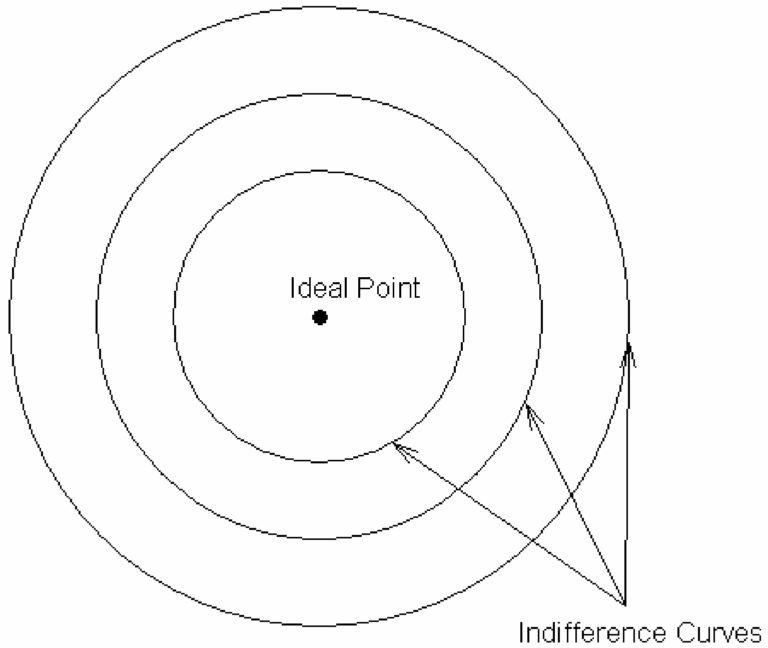
- Median voter theorem does not apply
- Can we guarantee transitivity of MR?
- Can it be generalized to 2 dimensions?

Utility function

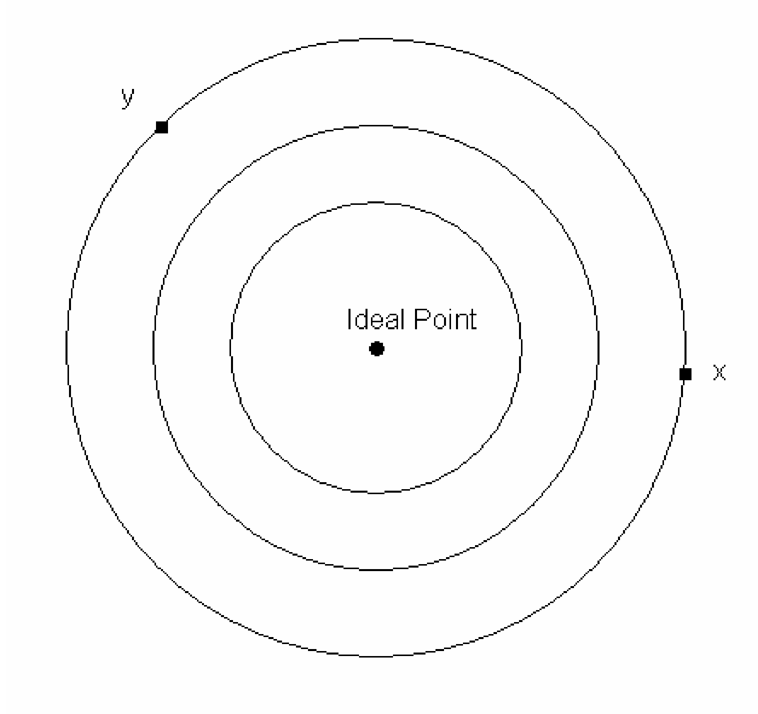


Projection onto policy space

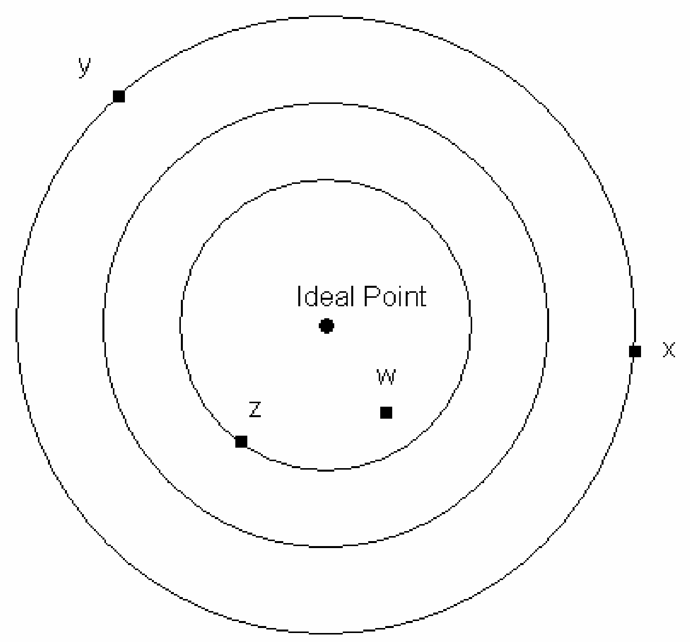
$$U(x,y) = -(x-\theta_1)^2 - (y-\theta_2)^2$$



Indifferent between x and y



Projection onto policy space



w P z P x I y

Effect of weights

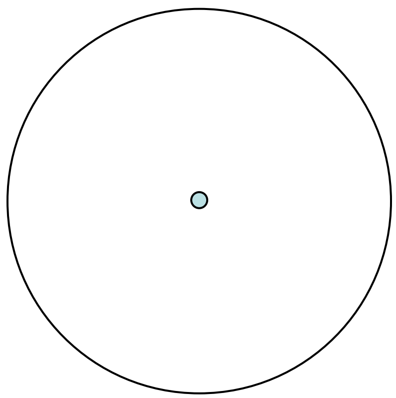
Equal weights:

Indifference circle

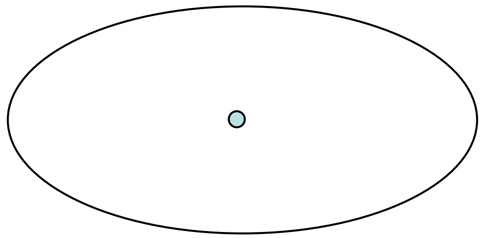
Different weights:

Indifference ellipse

$\alpha_1 = \alpha_2$



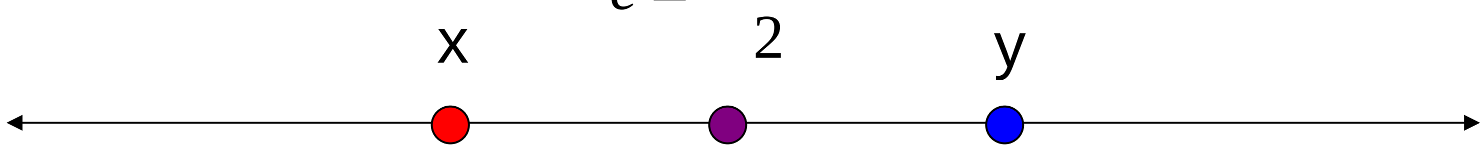
$\alpha_1 < \alpha_2$



Cut point

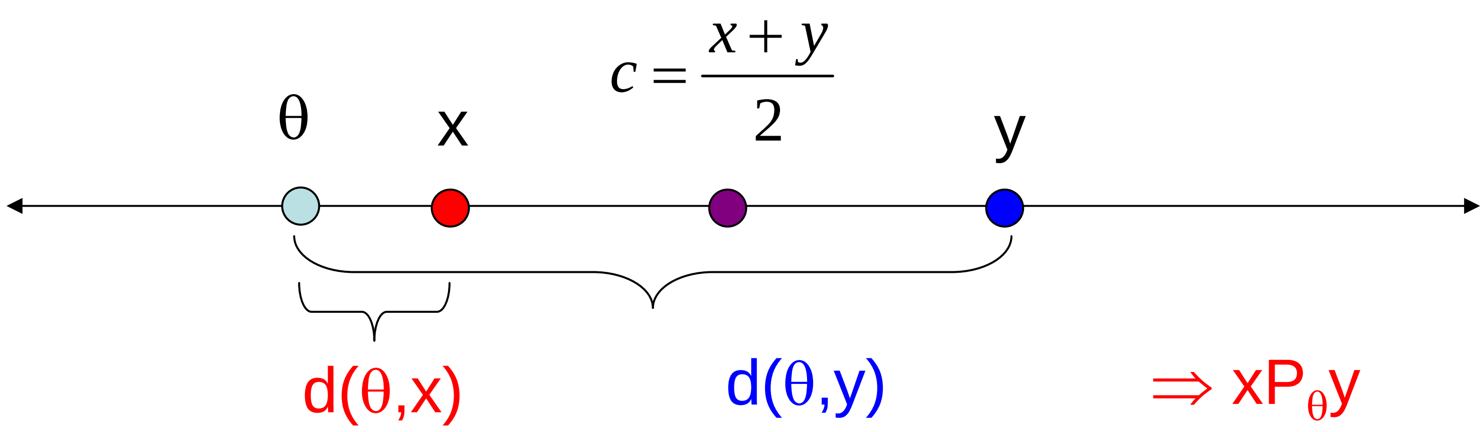
- Midpoint between two alternatives, divides ideal points

$$c = \frac{x + y}{2}$$



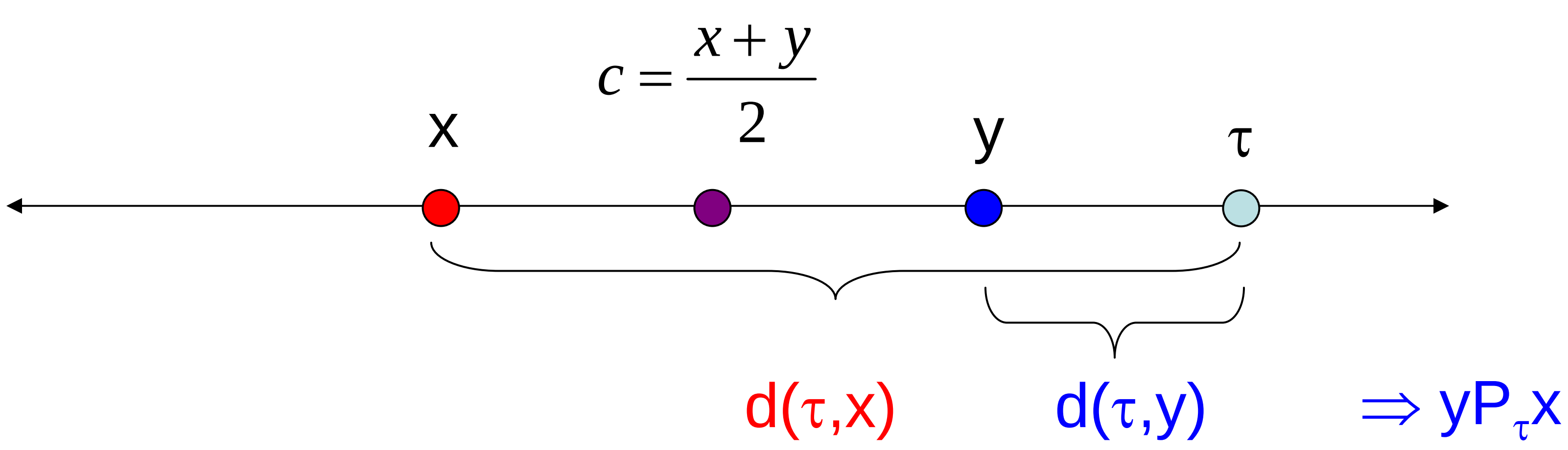
Cut point

- Midpoint between two alternatives, divides agents with opposing preferences



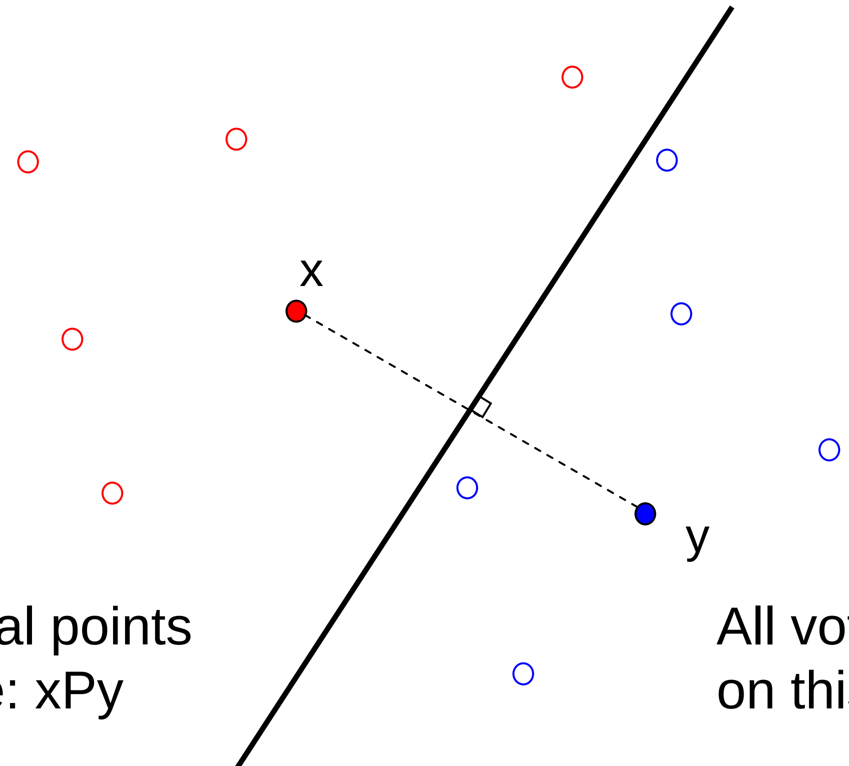
Cut point

- Midpoint between two alternatives, divides ideal points



Cutting lines

- Set of points equidistant between two alternatives
- Convenient way to determine preferences



All voters with ideal points on this side of line: xPy

All voters with ideal points on this side of line: yPx

Useful sets

$P_i(x)$ = i's **preferred-to set** of x

Set of policies an individual prefers to x

(Interior of indifference curve through x)

W(x) = **Majority rule winset** of x

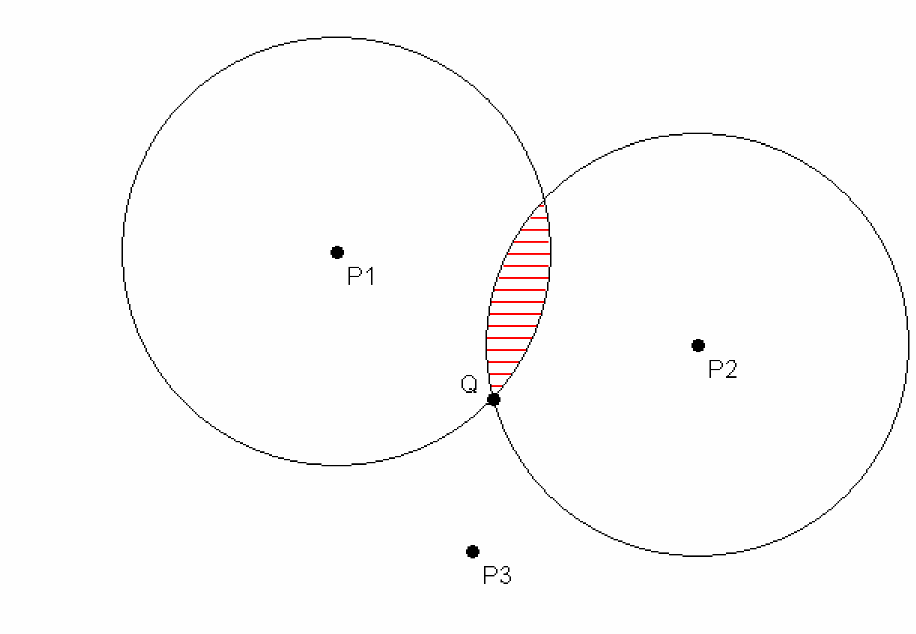
Set of all policies that some majority prefers to x

Finding winsets

Step 1. For each majority coalition, find intersection of preferred-to sets

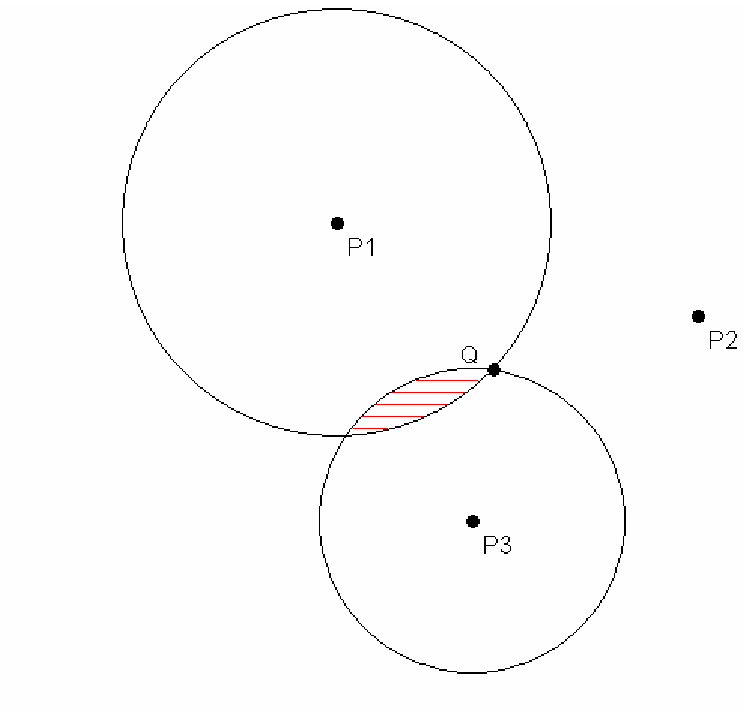
Step 2. Winset is **union** of sets in Step 1.

Finding W(Q)



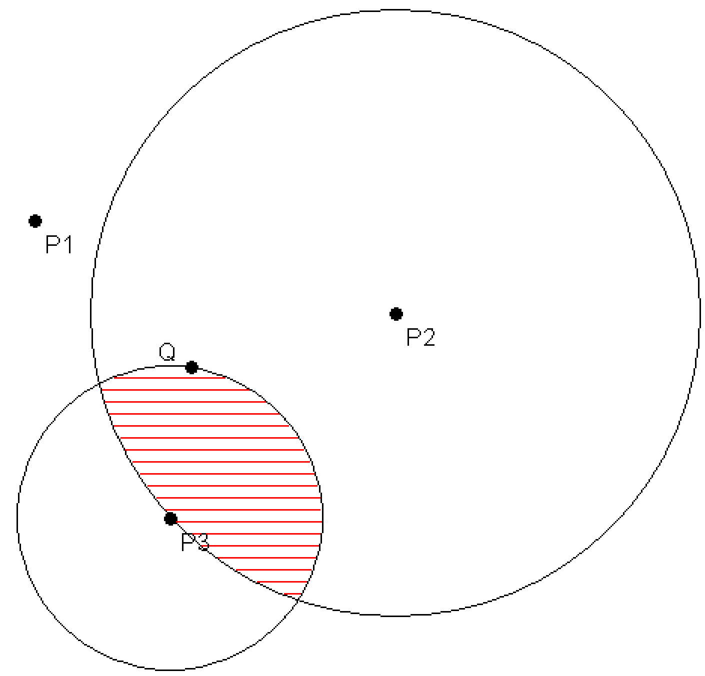
Set of policies coalition {1,2} prefers to Q

Finding W(Q)



Set of policies coalition {1,3} prefers to Q

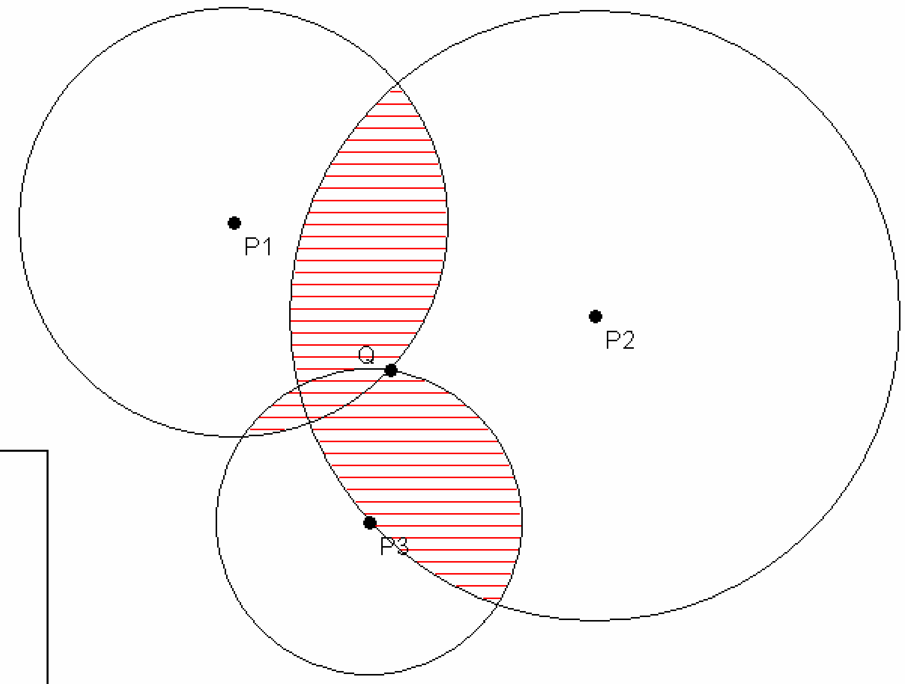
Finding W(Q)



NOTE: This figure is incorrect since P2's indifference curve should go through Q instead of P3.

Set of policies coalition {2,3} prefers to Q

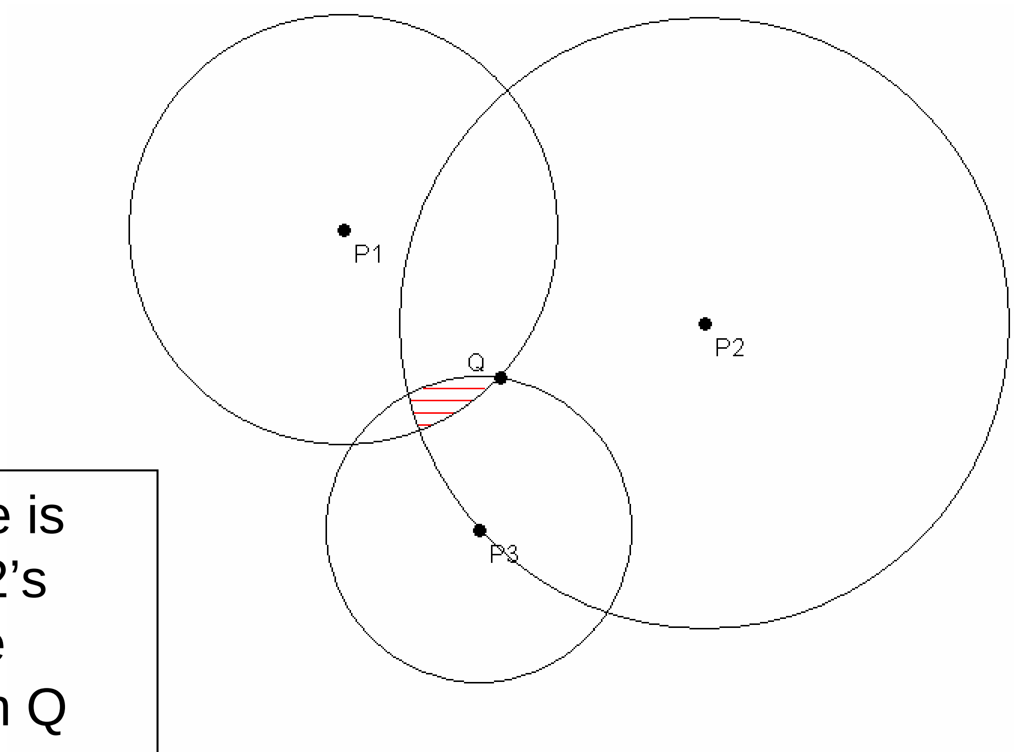
Finding W(Q)



NOTE: This figure is incorrect since P2's indifference curve should go through Q instead of P3.

Majority rule winset of Q

Finding W(Q)



NOTE: This figure is incorrect since P2's indifference curve should go through Q instead of P3.

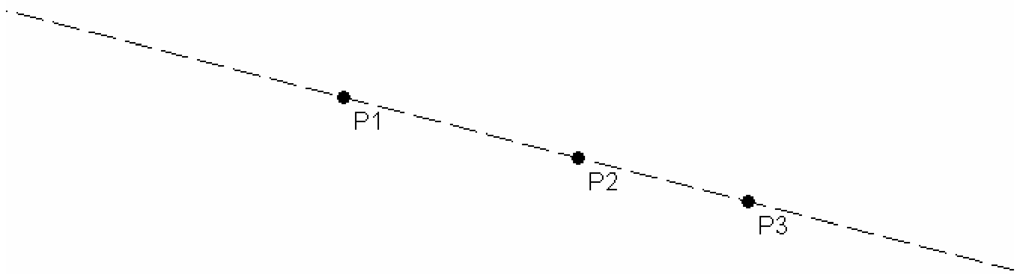
Unanimity rule winset of Q

Plott conditions

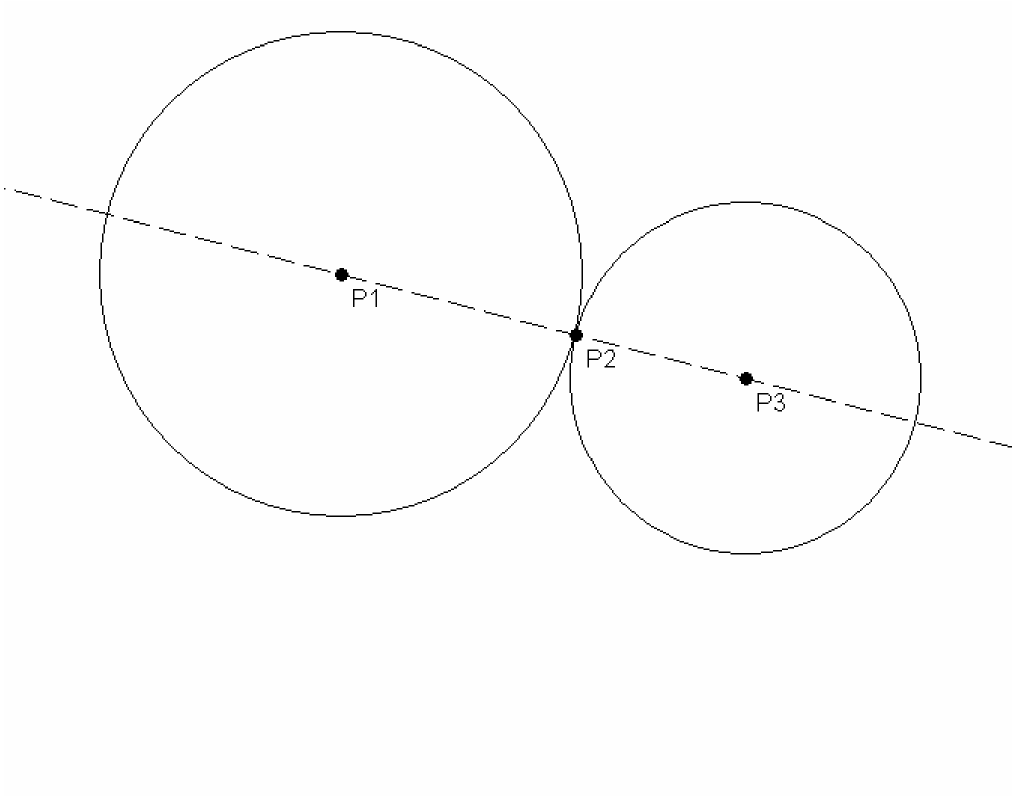
- The core is non-empty if and only if ideal points are distributed in a “radially symmetric” fashion around a policy x* and x* is a voter’s ideal point

- Radial symmetry means that pairs of ideal points form lines that intersect x* with x* between the pair of ideal points

Examples

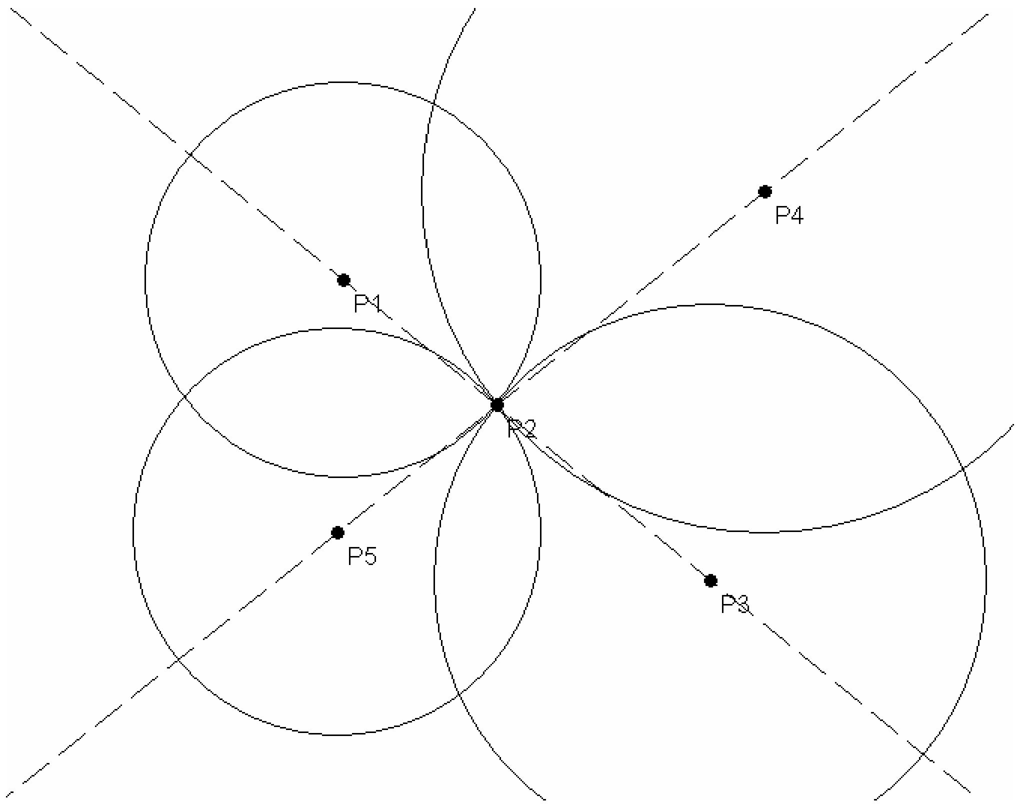


Examples: Plott conditions hold



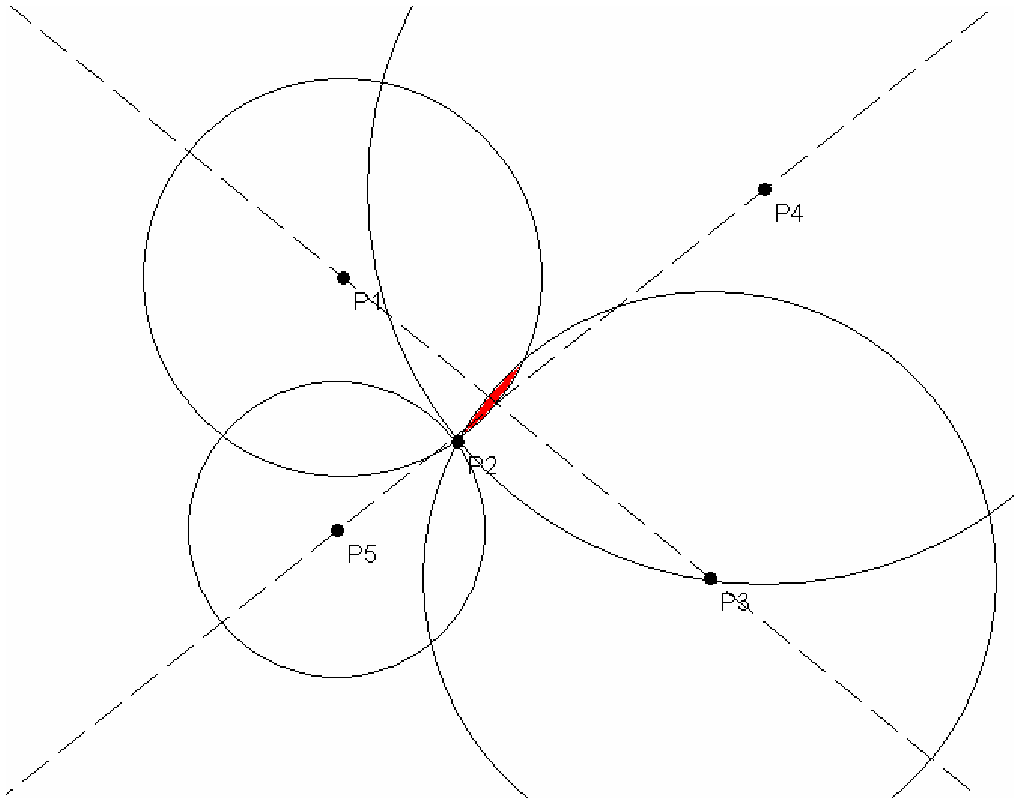
P2 has an empty winset $\Rightarrow$ Condorcet Winner

Examples: Plott conditions hold



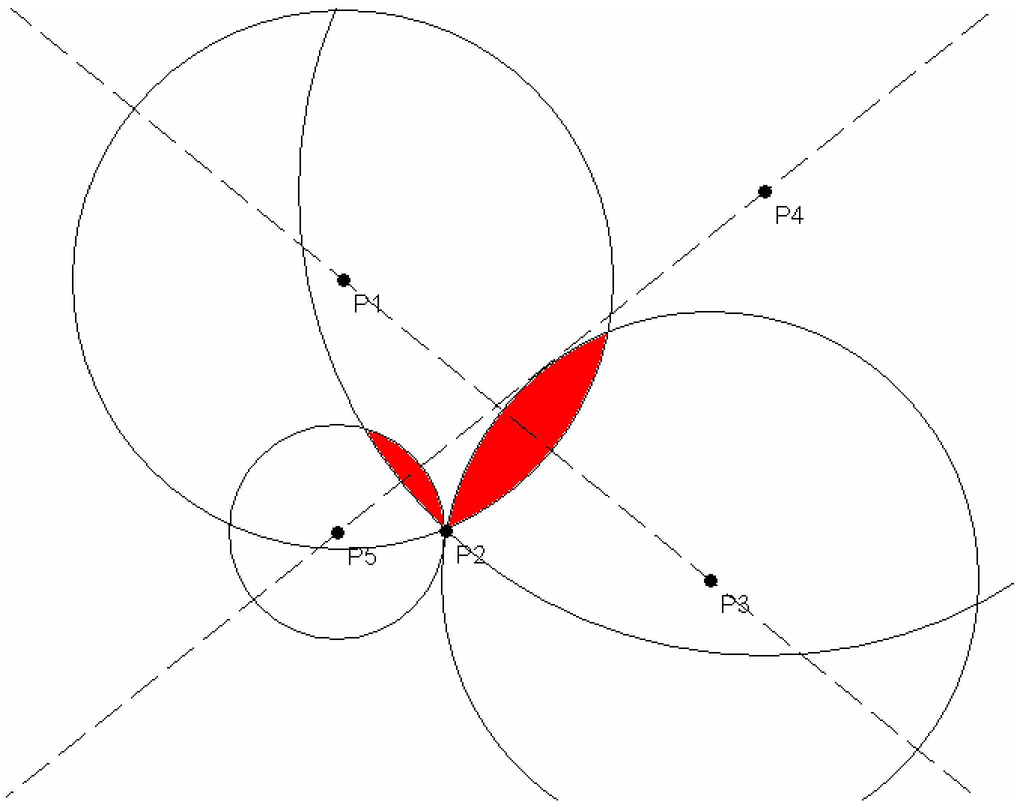
P2 has an empty winset $\Rightarrow$ Condorcet Winner

Examples: Plott conditions violated



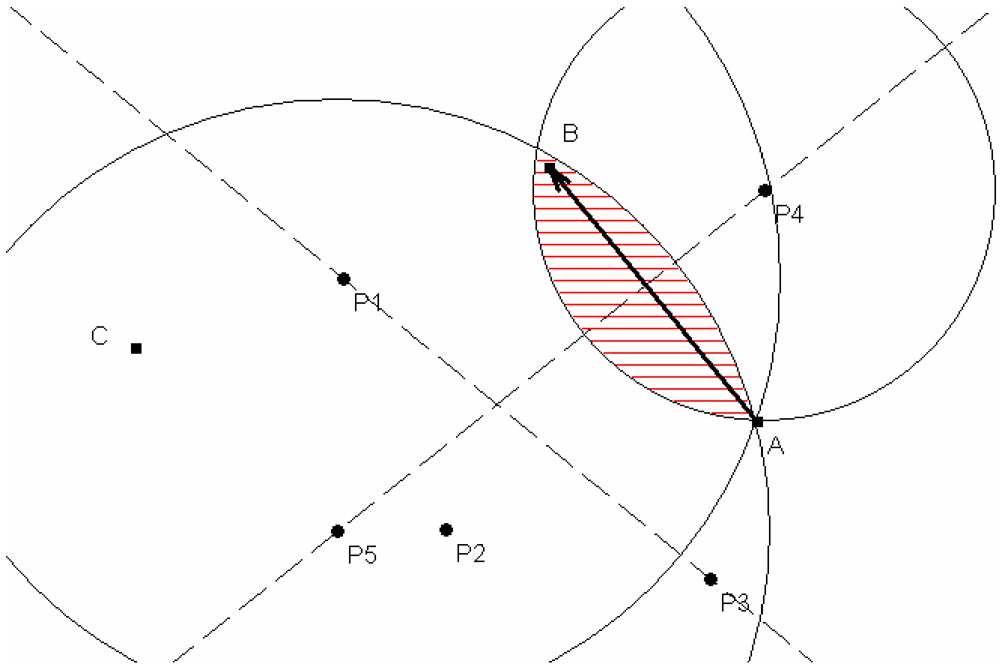
Plott conditions are violated $\Rightarrow$ W(P2) nonempty

Example: Plott conditions violated



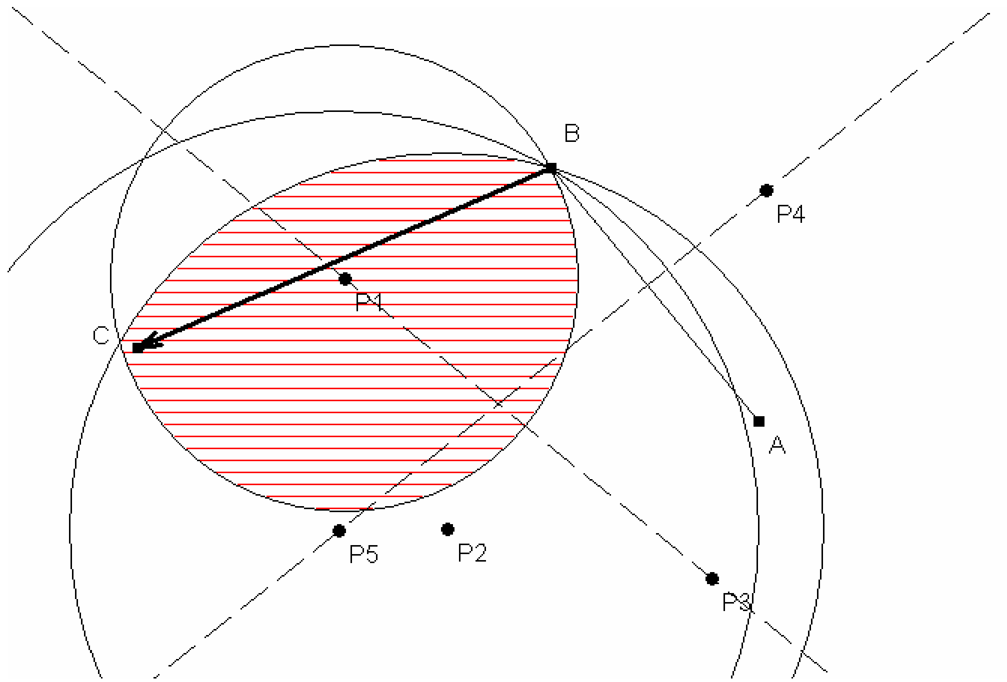
Plott conditions are violated $\Rightarrow$ W(P2) nonempty

Constructing a preference cycle



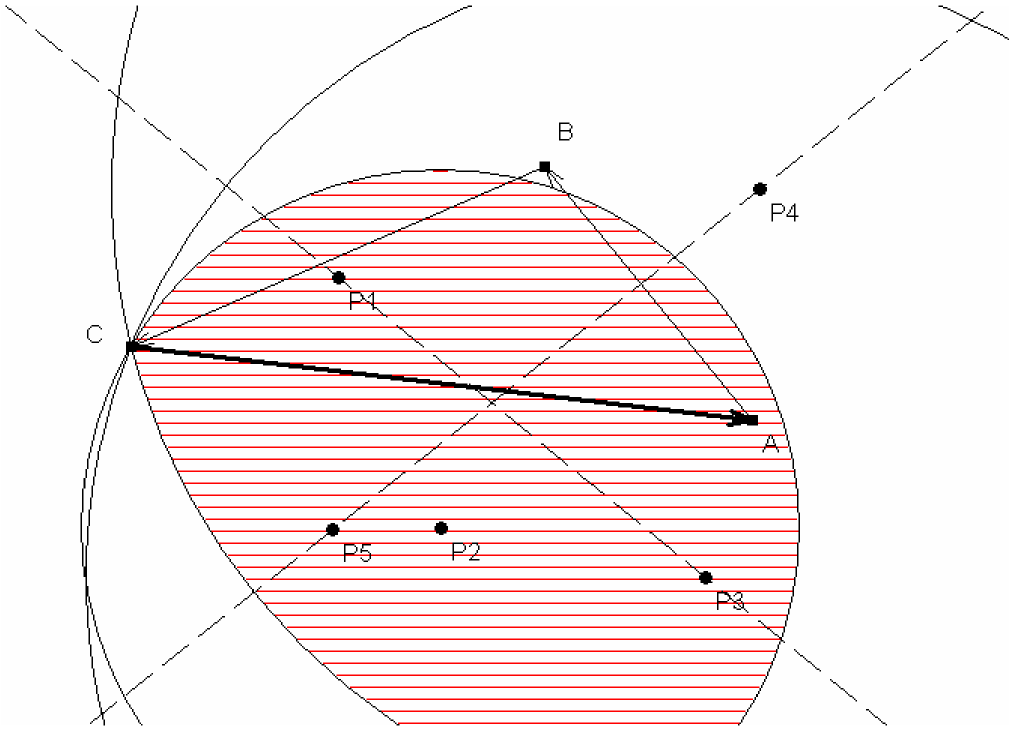
Majority {P1, P4, P5} votes for B over A

Constructing a preference cycle



Majority {P1, P2, P5} votes for C over B

Constructing a preference cycle



Majority {P2, P3, P4} votes for A over C

Top cycle set

Alternatives in the top cycle set

- Defeat all alternatives outside the set
- Preference cycles over the alternatives in the set

Example:

aPb, bPc, cPa, aPd, bPd, cPd

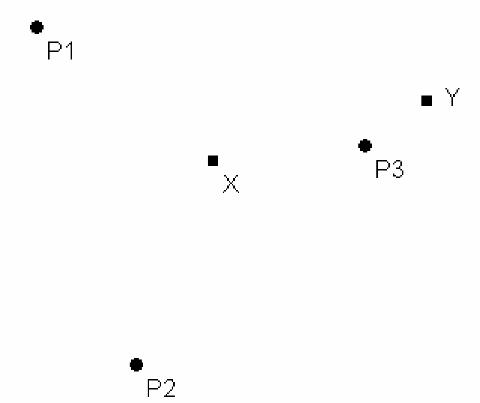
T = {a,b,c}

McKelvey's Theorem

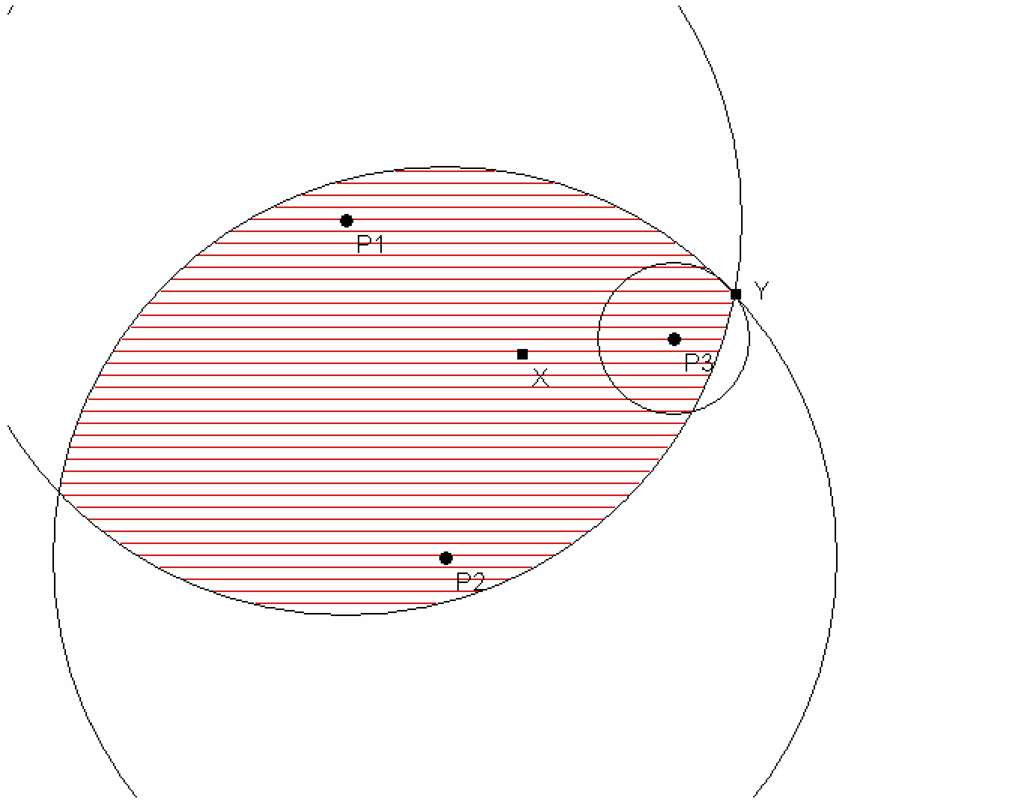
Given the spatial model, the majority rule core is either **non-empty or the top cycle set is T = X**.

McKelvey's Theorem (corollary)

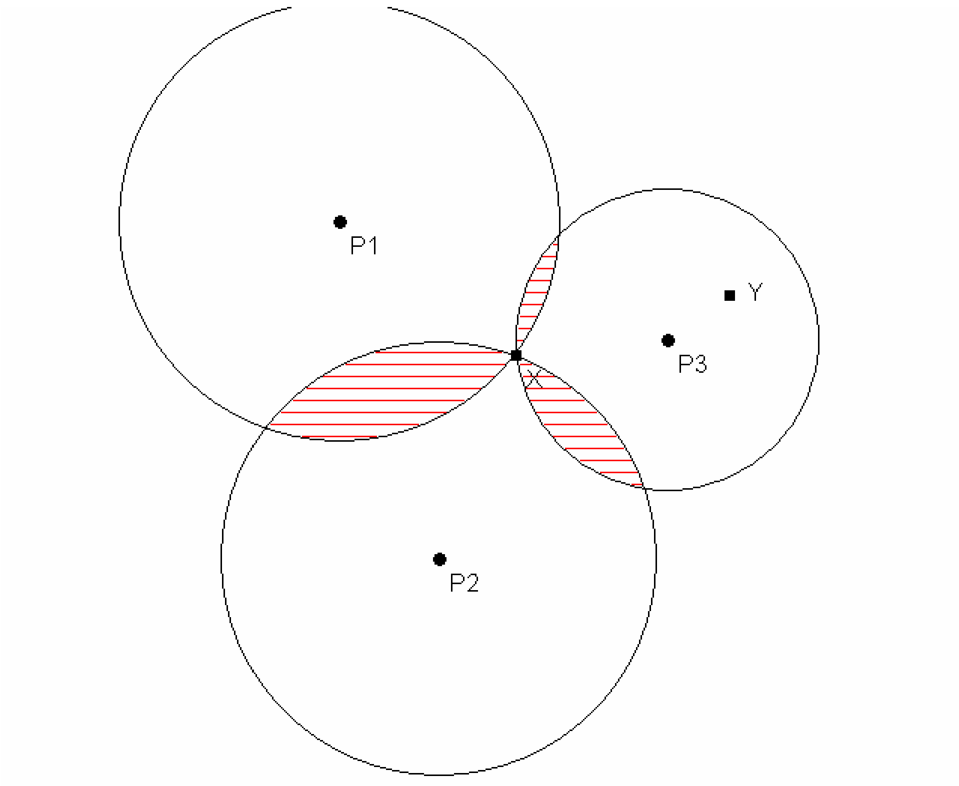
If the Plott conditions are not satisfied, then for any two points x and y, there exists a finite chain of policies $\{a_1,a_2,\ldots,a_n\}$ such that $xPa_1Pa_2\ldots Pa_nPy$



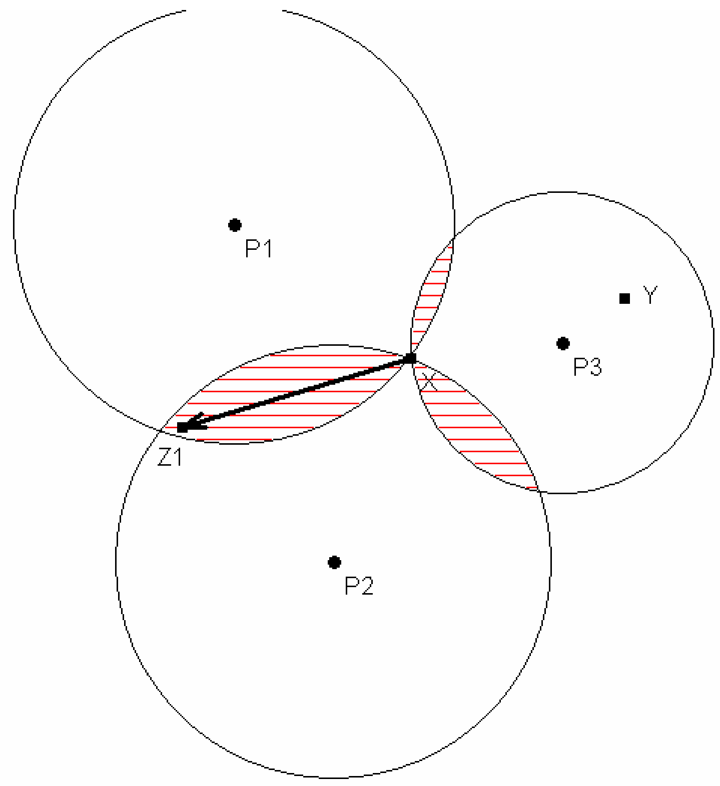
Construct a chain from y to x



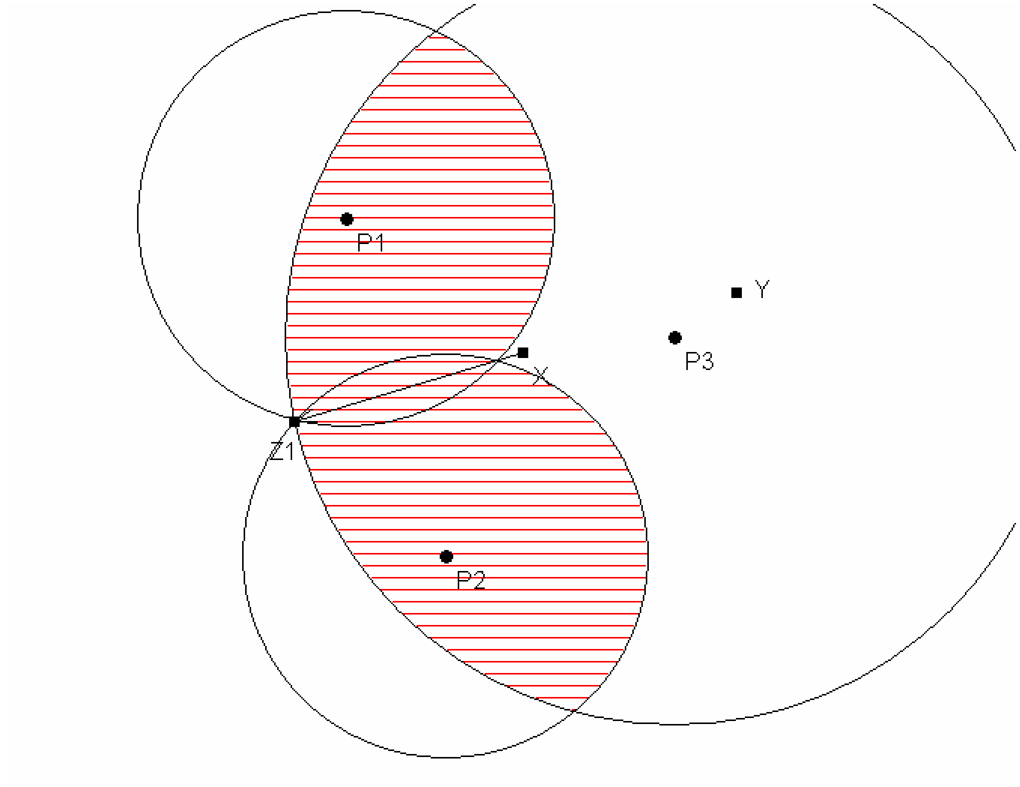
Note that x is majority preferred to y!



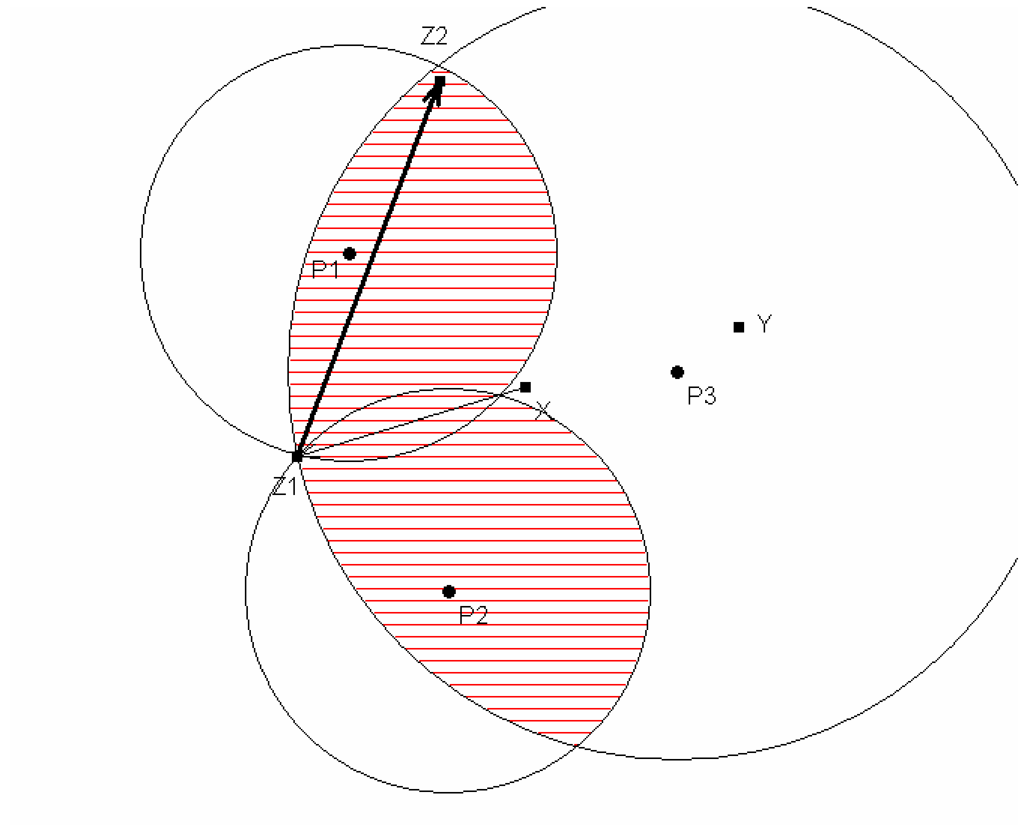
W(x)



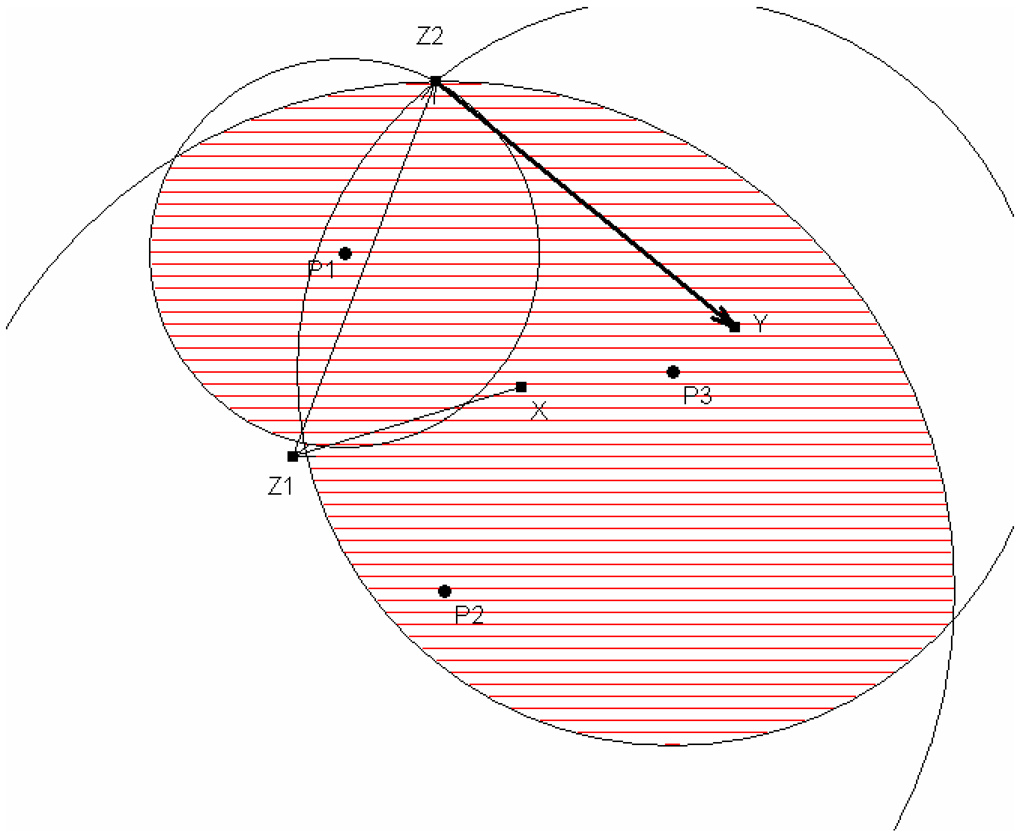
$z_1 \, P \, x$



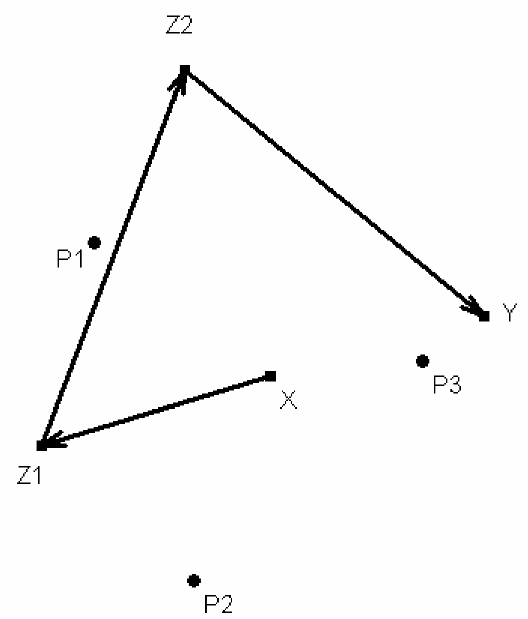
$W(z_1)$



z_2 P z_1 P x



$y \; P \; z_2 \; P \; z_1 \; P \; x$



Although x P y, we have the chain: y P z_2 P z_1 P x

Implications

- Plott conditions are very rarely satisfied
- In two dimensions, we can cycle over every policy
- McKelvey's Theorem does not predict "chaos"
- All preference aggregation rules are problematic, including majority rule
- Preference aggregation alone is insufficient to understand political outcomes