Proof of Arrow's Theorem for 2 person, 3 alternative case with strict preferences

PS 2703
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In the table below, the rows correspond to all possible strict preference orderings of Person 1, and the columns correspond to all possible strict preference orderings of Person 2. Therefore, each cell represents a preference profile.

	xyz	xzy	yxz	yzx	zxy	zyx
xyz						
xzy						
yxz						
yzx						
zxy						
zyx						

We will assume that a preference aggregation rule is weakly Paretian, transitive, and independent of alternatives and show that this forces us to conclude that the preference rule must also be dictatorial.

Step 1. We first use the weak Pareto property to deduce what any PAR have as the social preference ordering for as many cells as we can. In the next table, we fill in the social preference orderings, where xyz indicates xPyPz and xy, xz indicates xPy and xPz (without specifying the preference between y and z).

	xyz	**xzy**	**yxz**	**yzx**	**zxy**	**zyx**
xyz	xyz	xy, xz	xz, yz	yz	xy	
xzy	xz, xy	xzy	xz		xy, zy	zy
yxz	xz, yz	xz	yxz	yz, yx		yx
yzx	yz		yz, yx	yzx	zx	yx, zx
zxy	xy	xy, zy		zx	zxy	zx, zy
zyx		zy	yx	yx, zx	zx, zy	zyx

Step 2. The next step is take a preference profile where some preference is unspecified, consider each possible remaining possible ordering, then derive its implications using the other properties. So suppose that for the preference profile (xyz, yzx), the PAR tells us that the social preference is xPy. Transitivity then implies xPyPz:

	xyz	**xzy**	**yxz**	**yzx**	**zxy**	**zyx**
xyz	xyz	xy, xz	xz, yz	xyz	xy	
xzy	xz, xy	xzy	xz		xy, zy	zy
yxz	xz, yz	xz	yxz	yz, yx		yx
yzx	yz		yz, yx	yzx	zx	yx, zx
zxy	xy	xy, zy		zx	zxy	zx, zy
zyx		zy	yx	yx, zx	zx, zy	zyx

Step 3. We now use independence of irrelevant alternatives to show that for *any* preference profile where x and y are ranked the same as in the preference profile (xyz, yzx) – that is, when Person 1 has xPy and Person 2 has yPx – then the social preference must also be the same, that is xPy. This then forces us to fill in the following cells (also applying transitivity when applicable):

	xyz	**xzy**	**yxz**	**yzx**	**zxy**	**zyx**
xyz	xyz	xy, xz	xyz	xyz	xy	xy
xzy	xz, xy	xzy	xz, xy	xy	xy, zy	zy, xy
yxz	xz, yz	xz	yxz	yz, yx		yx
yzx	yz		yz, yx	yzx	zx	yx, zx
zxy	xy	xy, zy	xy	zxy	zxy	zxy
zyx		zy	yx	yx, zx	zx, zy	zyx

Step 4. Notice that from the profile (xyz, yzx) that transitivity also implied xPz. We again use the IIA assumption to fill in every other cell where the preferences over x and z are the same (where Person 1 has xPz and Person 2 has zPx):

	xyz	**xzy**	**yxz**	**yzx**	**zxy**	**zyx**
xyz	xyz	xy, xz	xyz	xyz	xy, xz	xy, xz
xzy	xz, xy	xzy	xz, xy	xy, xz	xzy	xzy
yxz	xz, yz	xz	yxz	yxz	xz	yxz
yzx	yz		yz, yx	yzx	zx	yx, zx
zxy	xy	xy, zy	xy	zxy	zxy	zxy
zyx		zy	yx	yx, zx	zx, zy	zyx

Step 5. Then we notice the fact that in the profile (zxy, yzx) that the PAR has zPy, and then again use IIA (with transitivity) to fill in even more cells:

	xyz	xzy	yxz	yzx	zxy	zyx
xyz	xyz	xy, xz	xyz	xyz	xy, xz	xy, xz
xzy	xzy	xzy	xzy	xzy	xzy	xzy
yxz	xz, yz	xz	yxz	yxz	xz	yxz
yzx	yz		yz, yx	yzx	zx	yx, zx
zxy	xy, zy	xy, zy	xy, zy	zxy	zxy	zxy
zyx	zy	zy	zyx	zyx	zx, zy	zyx

Step 6. From the profile (yxz, zyx) we note that the PAR has yPz, so we apply IIA to any cell where Person 1 has yPz and Person 2 has zPy:

	xyz	xzy	yxz	yzx	zxy	zyx
xyz	xyz	xyz	xyz	xyz	xyz	xyz
xzy	xzy	xzy	xzy	xzy	xzy	xzy
yxz	xz, yz	xz, yz	yxz	yxz	xz, yz	yxz
yzx	yz	yz	yz, yx	yzx	yzx	yzx
zxy	xy, zy	xy, zy	xy, zy	zxy	zxy	zxy
zyx	zy	zy	zyx	zyx	zx, zy	zyx

Step 7. Then notice that in the profile (zyx, yxz), the social preference is zPx. So using the same technique find all cells where Person 1 has zPx and Person 2 has xPz and use IIA with transitivity:

	xyz	xzy	yxz	yzx	zxy	zyx
xyz	xyz	xyz	xyz	xyz	xyz	xyz
xzy	xzy	xzy	xzy	xzy	xzy	xzy
yxz	xz, yz	xz, yz	yxz	yxz	xz, yz	yxz
yzx	yzx	yzx	yzx	yzx	yzx	yzx
zxy	zxy	zxy	zxy	zxy	zxy	zxy
zyx	zy, zx	zy, zx	zyx	zyx	zx, zy	zyx

Step 8. One last time, we note that in the profile (yzx, zxy) that the social preference is yPx, applying IIA we get:

	xyz	xzy	yxz	yzx	zxy	zyx
xyz	xyz	xyz	xyz	xyz	xyz	xyz
xzy	xzy	xzy	xzy	xzy	xzy	xzy
yxz	yxz	yxz	yxz	yxz	yxz	yxz
yzx	yzx	yzx	yzx	yzx	yzx	yzx
zxy	zxy	zxy	zxy	zxy	zxy	zxy
zyx	zyx	zyx	zyx	zyx	zyx	zyx

Notice what we have: in every cell, the social preference exactly matches Person 1's preferences, so Person 1 is a dictator!

Notice also that if we go back to Step 2 and assume instead that the social preference is yPx, that we will end up (by symmetry) with Person 2 being the dictator.

Finally, the last possibility is for the social preference relation to be xIy. If this is the case, then transitivity implies that xPz. However, note that in the profile (xzy, zyx) we already have zPy. But then IIA in this cell implies xPz, and by transitivity xPzPy. However, IIA implies that in these two cells, the social preference between x and y should be the same—a contradiction! So the social preference cannot be indifference, and we are done.

	xyz	**xzy**	**yxz**	**yzx**	**zxy**	**zyx**
xyz	xyz	xy, xz	xz, yz	yz, xIy, xz	xy	
xzy	xz, xy	xzy	xz		xy, zy	xzy
yxz	xz, yz	xz	yxz	yz, yx		yx
yzx	yz		yz, yx	yzx	zx	yx, zx
zxy	xy	xy, zy		zx	zxy	zx, zy
zyx		zy	yx	yx, zx	zx, zy	zyx