Jury Voting Example
PS 2703

The purpose of the jury voting application is to introduce the idea of "information aggregation" in a strategic context and to demonstrate how to use Bayes' Rule and how to check whether a strategy profile is a Bayesian Nash equilibrium. A number of simplifying assumptions were made for the purpose of keeping the in-class analysis tractable...but we still ended up with a mess!

The Bayesian Game	
Players	$N = \{1, 2, 3\}$
States	$\omega \in \{G, I\}$
Types	$\theta_i \in \{0, 1\}$
Nature's randomization	$\Pr(\omega = G) = \pi > 1/2$
	$\Pr(\theta_i = 0 \mid \omega = I) = \Pr(\theta_i = 1 \mid \omega = G) = p > 1/2$
Actions	$A_i = \{a, c\}$
Preferences	$u_i(a_1, a_2, a_3; \theta_i) = \begin{cases} 1 & \text{if "correct" decision} \\ 0 & \text{if "incorrect" decision} \end{cases}$

A *sincere* (or *truthful*) strategy is one where Player i's vote follows her signal: a if $\theta_i = 0$ and c if $\theta_i = 1$. The strategy profile where each player uses the sincere strategy is a Bayesian Nash equilibrium if the following two inequalities hold:

$$EU_i(a, s_i; \theta_i = 0) \geq EU_i(c, s_i; \theta_i = 0) \tag{1}$$

$$EU_i(c, s_i; \theta_i = 1) \geq EU_i(a, s_i; \theta_i = 1) \tag{2}$$

The expressions for the expected utilities are a function of a player's beliefs and Bernoulli utility *given* the strategies of the other players and that a player knows his own signal.

The first expected utility in the first inequality is:

$$\begin{aligned} EU_1(a, s_i; \theta_i = 0) \;=\; & \Pr(\omega = G \wedge \theta_2 = 0 \wedge \theta_3 = 0 \mid \theta_1 = 0) u_1(a, a, a, \theta_1 = 0) \\ & + \Pr(\omega = G \wedge \theta_2 = 0 \wedge \theta_3 = 1 \mid \theta_1 = 0) u_1(a, a, c, \omega = G) \\ & + \Pr(\omega = G \wedge \theta_2 = 1 \wedge \theta_3 = 0 \mid \theta_1 = 0) u_1(a, c, a, \omega = G) \\ & + \Pr(\omega = G \wedge \theta_2 = 1 \wedge \theta_3 = 1 \mid \theta_1 = 0) u_1(a, c, c, \omega = G) \\ & + \Pr(\omega = I \wedge \theta_2 = 0 \wedge \theta_3 = 0 \mid \theta_1 = 0) u_1(a, a, a, \omega = I) \end{aligned}$$

$$
\begin{aligned}
&+\Pr(\omega = I \wedge \theta_2 = 0 \wedge \theta_3 = 1|\theta_1 = 0)u_1(a,a,c,\omega = I)\\
&+\Pr(\omega = I \wedge \theta_2 = 1 \wedge \theta_3 = 0|\theta_1 = 0)u_1(a,c,a,\omega = I)\\
&+\Pr(\omega = I \wedge \theta_2 = 1 \wedge \theta_3 = 1|\theta_1 = 0)u_1(a,c,c,\omega = I)
\end{aligned}
$$

Plugging in the appropriate utility values (0 or 1) gives us

$$
\begin{aligned}
EU_1(a, s_i; \theta_i = 0) \quad = \quad & \Pr(\omega = G \wedge \theta_2 = 1 \wedge \theta_3 = 1|\theta_1 = 0)\\
&+\Pr(\omega = I \wedge \theta_2 = 0 \wedge \theta_3 = 0|\theta_1 = 0)\\
&+\Pr(\omega = I \wedge \theta_2 = 0 \wedge \theta_3 = 1|\theta_1 = 0)\\
&+\Pr(\omega = I \wedge \theta_2 = 1 \wedge \theta_3 = 0|\theta_1 = 0)
\end{aligned}
$$

Doing the same for the expression on the right hand side of the inequality yields

$$
\begin{aligned}
EU_1(c, s_i; \theta_i = 0) \quad = \quad & \Pr(\omega = G \wedge \theta_2 = 1 \wedge \theta_3 = 1|\theta_1 = 0)\\
&+\Pr(\omega = G \wedge \theta_2 = 0 \wedge \theta_3 = 1|\theta_1 = 0)\\
&+\Pr(\omega = G \wedge \theta_2 = 1 \wedge \theta_3 = 0|\theta_1 = 0)\\
&+\Pr(\omega = I \wedge \theta_2 = 0 \wedge \theta_3 = 0|\theta_1 = 0)
\end{aligned}
$$

Next, we can reduce the inequality by cancelling terms that appear on both sides to get

$$
\begin{array}{lcl}
\Pr(\omega = I \wedge \theta_2 = 0 \wedge \theta_3 = 1|\theta_1 = 0) & & \Pr(\omega = G \wedge \theta_2 = 0 \wedge \theta_3 = 1|\theta_1 = 0)\\
+\Pr(\omega = I \wedge \theta_2 = 1 \wedge \theta_3 = 0|\theta_1 = 0) & \geq & +\Pr(\omega = G \wedge \theta_2 = 1 \wedge \theta_3 = 0|\theta_1 = 0)
\end{array}
$$

We need to figure out what the posterior probabilities are in the last expression. This is where Bayes' Rule comes into play. To see how Bayes' Rule works, we can first write the posterior probability using the definition of conditional probability. (This is the part I messed up in class. After writing it correctly on the board, I ended up replacing it with an incorrect expression.)

$$
\Pr(\omega = I \wedge \theta_2 = 0 \wedge \theta_3 = 1|\theta_1 = 0) = \frac{Pr(\omega = I \wedge \theta_2 = 0 \wedge \theta_3 = 1 \wedge \theta_1 = 0)}{Pr(\theta_1 = 0)}
$$

The numerator can be written as

$$
\Pr(\theta_1 = 0 \wedge \theta_2 = 0 \wedge \theta_3 = 1|\omega = I)\Pr(\omega = I)
$$

Then applying the independence of signals this becomes

$$\Pr(\theta_1 = 0|\omega = I)\Pr(\theta_2 = 0|\omega = I)\Pr(\theta_3 = 1|\omega = I)\Pr(\omega = I)$$

The denominator of the expression (this is where I messed up in class) is

$$\Pr(\theta_1 = 0) = \Pr(\theta_1 = 1|\omega = I)\Pr(\omega = I) + \Pr(\theta_1 = 0|\omega = G)\Pr(\omega = G)$$

Then substituting the appropriate parameters gives us

$$\Pr(\omega = I \wedge \theta_2 = 0 \wedge \theta_3 = 1|\theta_1 = 0) = \frac{p^2(1-p)(1-\pi)}{p(1-\pi) + (1-p)\pi}$$

After applying Bayes' Rule to the other expression in the inequality, we find that voting to acquit based on the signal $\theta_1 = 0$ is optimal only if

$$2\frac{p^2(1-p)(1-\pi)}{p(1-\pi) + (1-p)\pi} \geq 2\frac{p(1-p)^2\pi}{p(1-\pi) + (1-p)\pi}$$

Fortunately, this still reduces to

$$p \geq \pi$$

Repeating the steps for the second inequality gives us

$$p \geq 1 - \pi$$

which is always true since $p > 1/2 > 1 - \pi$. Thus, we can conclude that sincere voting by each player is a Bayesian Nash equilibrium if

$$p \geq \pi$$

McCarty and Meirowitz also note that these inequalities are equivalent to the probabilities of voting correctly given Player i's signal and being pivotal being greater than or equal to 1/2.