(An Introduction to) Formal Political Theory

PS 2703, Fall 2007

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Overview

- What is formal theory?
- Why learn formal theory?
- Learning goals
- Assumptions and expectations

Logic and Proofs

Deduction

The process of drawing valid conclusions from a set of premises using rules of inference.

Deduction

A process of reasoning in which a conclusion follows necessarily from the premises presented, so that the conclusion cannot be false if the premises are true.

Valid arguments

Example 1

– It will either rain or snow today.

– It's too warm for snow.

– Therefore, it will rain.

premises

conclusion

Valid arguments

Example 2

- If today is Monday, then I have to teach today.
- Today is Monday.
- Therefore, I have to teach today.

Valid arguments

Example 3

- I will go to work either today or tomorrow.
- I will stay home today.
- Therefore, I will go to work tomorrow.

Valid arguments

- Truth of the premises force us to accept the truth of the conclusion
- For the purposes of deduction, assume premises are true
- Whether premises are actually true in the “real world” is an empirical question

An invalid argument

Either the butler is guilty or the maid is guilty

Either the maid is guilty or the cook is guilty

Therefore, either the butler is guilty or the cook is guilty.

What if the maid is guilty?

- Both premises satisfied (i.e. true)
- Conclusion false

Logic Primer

- Statements
- Connectives
- Conditionals
- Quantifiers
- Equivalences

Statements

- Represented by letters
- Either true or false, but not both

P = It will rain today

Q = It will snow today

R = Today is Monday

S = I have to teach today

T = I will go to work today

U = I will go to work tomorrow

Connectives

Logical operators allow us to modify and connect statements

Operator	Symbol/Usage	Meaning
not	$\neg P$ (or $\sim P$)	"it is not the case that P" (negation)
and	$P \wedge Q$	"both P and Q" (conjunction)
or	$P \vee Q$	"either P or Q, or both" (disjunction)

Example 1

P = It will rain today

Q = It will snow today

It will either rain or snow today

It's too warm for snow

Therefore, it will rain

Example 1

P = It will rain today

Q = It will snow today

$P \vee Q$

$\neg Q$

Therefore, P

Conditional connective

$$P \Rightarrow Q$$

- "If P then Q"
- "P implies Q"
- "P only if Q"
- "Q, if P"
- "P is a sufficient condition for Q"
- "Q is a necessary condition for P"

Conditional connective

$$P \Rightarrow Q$$

- "the truth of P guarantees the truth of Q"
- P is the antecedent
- Q is the consequent

equivalent to $\neg P \vee Q$

Example 2

R = Today is Monday

S = I have to teach today

If today is Monday, then I have to teach today

Today is Monday

Therefore, I have to teach today

Example 2

R = Today is Monday

S = I have to teach today

$R \Rightarrow S$

R

Therefore, S

Example 2

R = Today is Monday

S = I have to teach today

$\neg R \vee S$

R

Therefore, S

Example 2

R = Today is Monday

S = I have to teach today

Either today is not Monday or I have to teach today

Today is Monday

Therefore, I have to teach today

Exercise

Which of the following is a valid deductive argument?

(Hint: Use an equivalence of the form $\neg P \vee Q$)

1) $S \Rightarrow R$

   R

   Therefore: S

2) $\neg S \Rightarrow \neg R$

   R

   Therefore: S

Solution to 1

Valid deduction: conclusion cannot be false if the premises are true.

Invalid

$S \Rightarrow R \;\equiv\; \neg S \vee R$

R

Therefore: S

(But $\neg$ S is also consistent with premises)

Converse

$S \Rightarrow R$ is the converse of $R \Rightarrow S$

They are not equivalent:

$$\begin{array}{ccc} S \Rightarrow R & & R \Rightarrow S \\ \neg S \vee R & \neq & \neg R \vee S \end{array}$$

Solution to 2

Valid deduction: conclusion cannot be false if the premises are true.

<u>Valid</u>

$\neg S \Rightarrow \neg R \equiv \neg(\neg S) \vee (\neg R) \equiv S \vee (\neg R)$

R

Therefore: S

($\neg$ S cannot be true given the premises)

Contrapositive

$\neg S \Rightarrow \neg R$ is the contrapositive of $R \Rightarrow S$

They are equivalent:

$\neg S \Rightarrow \neg R$	$R \Rightarrow S$
$\neg\neg S \vee \neg R$	$\neg R \vee S$
$S \vee \neg R$	$S \vee \neg R$

Biconditional connective

$$P \Leftrightarrow Q$$

"P if and only if Q"

"P is necessary and sufficient for Q"

"P implies Q and Q implies P"

equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Quantifiers

Used for statements involving variables

Existential	$\exists\ x$ s.t. P	"there exists a value of x such that P is true"
Universal	$\forall\ x$, P	"for all values of x, P is true"

Equivalences

- Double negation law
- Commutative laws
- Associative laws
- Idempotent laws
- Distributive laws
- DeMorgan’s laws
- Conditional laws

Summary

- Valid arguments: conclusion cannot be false if premises are true
- Statements (P, Q,…)
- Connectives ($\neg$, $\wedge$, $\vee$)
- Conditionals ($\Rightarrow$, $\Leftrightarrow$)
- Qualifiers ($\exists$, $\forall$)
- Equivalences

Proofs

A proof is a rigorous mathematical argument which unequivocally demonstrates the truth of a given proposition.

Counterexamples

Used to demonstrate that a (universal) statement is false

Conjecture: All formal theorists majored in math

Counterexample: Your instructor majored in political science

Direct proof

Starts with premises and arrives at a conclusion directly, usually by way of several intermediate deductions

To prove $P \Rightarrow R$

- Assume P is true
- Show that P implies Q
- Show that Q implies R

Example

Prop. If (1) $\neg Q \vee P$

(2) $\neg R \Rightarrow \neg P$

then $Q \Rightarrow R$.

Proof. First assume Q

(1) implies P must be true

The contrapositive of (2) is $P \Rightarrow R$

So it follows that R must be true.

Example

Prop. If $0 < a < b$, then $a^2 < b^2$

Proof. Suppose $0 < a < b$

Multiplying $a < b$ by a yields $a^2 < ab$

Multiplying $a < b$ by b yields $ab < b^2$

Therefore, $a^2 < ab < b^2$

So $a^2 < b^2$

Indirect proof

Prove the contrapositive, which is logically equivalent

To prove $P \Rightarrow R$, instead prove $\neg R \Rightarrow \neg P$

- Assume $\neg R$
- Show $\neg R \Rightarrow Q$
- Show that $Q \Rightarrow \neg P$

Example

Prop. Suppose $a > b$. If $ac \leq$ bc then $c \leq 0$.

Proof. We will prove the contrapositive

(if $c > 0$ then $ac > bc$)

So suppose $c > 0$

Multiplying $a > b$ by c gives $ac > bc$

Therefore, if $ac \leq$ bc then $c \leq 0$

Proof by contradiction

To prove P is true

– Assume P is false	$\neg P$
– Deduce a contradiction	$\neg P \Rightarrow P$
– Thus, $\neg$ P is false	$\neg \neg P$
– Therefore, P must be true	P

Example

Prop. If (1) $P \vee Q$
(2) $\neg(P \wedge R)$ and
(3) R
then Q.

Proof. Suppose the conclusion is false: $\neg Q$
(1) implies P must be true
(2) implies R must be false
This contradicts (3)
So the assumption $\neg Q$ must be false
Therefore, Q must be true

Named results

Conjectures:	Guesses that have not been proven
Lemmas:	Intermediate results (used to prove other results)
Propositions:	Main results
Theorem:	Really important results
Corollary:	Follow closely from main results

Advice for writing proofs

- Use definitions and equivalences
- Trial and error
- Work through examples
- Make simplifying assumptions to figure out the main technique, then prove the original result
- Try different types of proofs
- Explain your reasoning clearly and precisely (unambiguously)

Summary

- Proofs demonstrate truth of a result
- Counterexamples show a result is false
- Direct proof
- Indirect proof
- Proof by contradiction

Sets

Basics

- A set is a collection of objects (denoted by uppercase letters)
- Objects of a set are called elements (denoted by lower case letters)
- A set is defined by
  - Enumeration (listing its elements)
  - Describing a property unique to the set

Examples

A = {1,2,3,4,5}

= $\{n \mid n$ is an integer and $1 \leq n \leq 5\}$

B = $\{x \mid x \geq 0\}$

C = {Canada, U.S., Mexico}

= {s | s is a member of NAFTA}

Notation

$x \in A$	"x is an element of A"
$x \notin A$	"x is not an element of A"
$\varnothing$	"empty set" or "null set" (contains no elements)

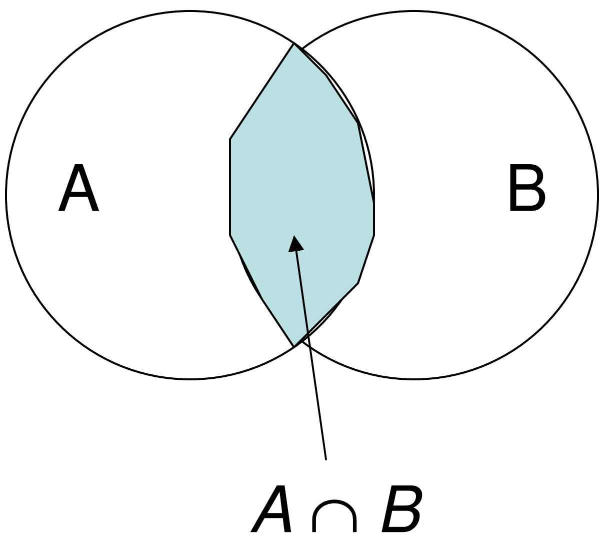
Relationships

$A \subseteq B$	"A is a subset of B" $x \in A \Rightarrow x \in B$
$A \subset B$	"A is a proper (or strict) subset of B" $A \subseteq B \wedge A \neq B$
$A = B$	"A is equal to B" $A \subseteq B \wedge B \subseteq A$
$A \neq B$	"A is not equal to B"
$A \not\subset B$	"A is not a subset of B"

Intersection

$A \cap B$ "the intersection of A and B"

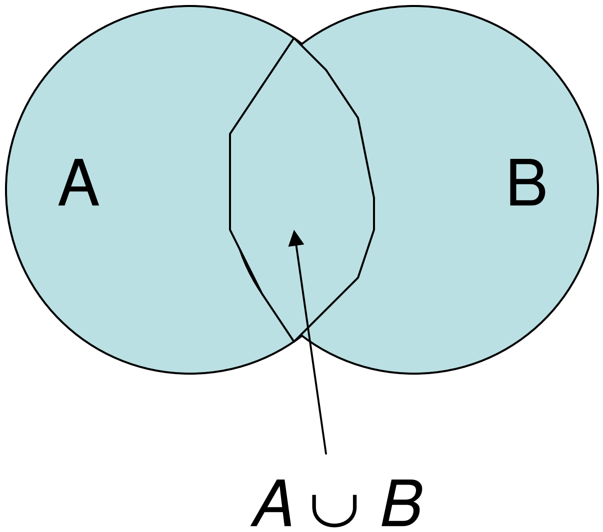
$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$



Union

$A \cup B$ "the union of A and B"

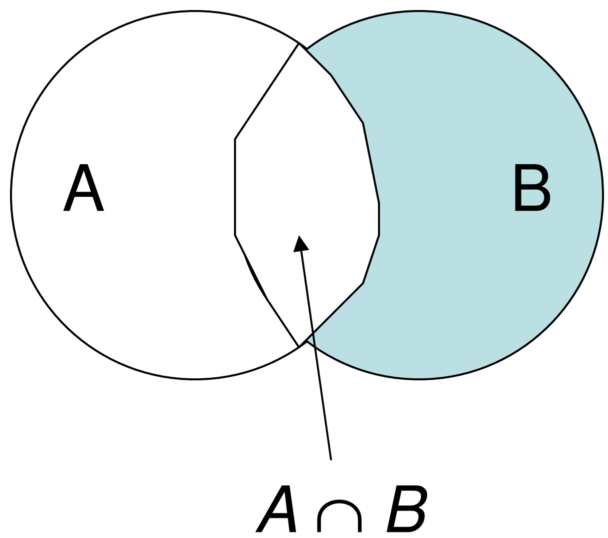
$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



Complement (Subtraction)

$B \setminus A$ "the complement of A relative to B"

$$B \setminus A = \{x \mid x \in B \wedge x \notin A\}$$



Laws for sets

Commutative

$$A \cap B \equiv B \cap A$$

$$A \cup B \equiv B \cup A$$

Associative

$$(A \cap B) \cap C \equiv A \cap (B \cap C)$$

$$(A \cup B) \cup C \equiv A \cup (B \cup C)$$

Distributive

$$A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$$

Cartesian products

The product of two sets is a set of ordered pairs:

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Coming up

Wednesday: **Class starts at 10 am**

- 8/29: Rationality and individual choice

Next week

- 9/3: Labor day, no class
- 9/5: Social choice