

A structure-preserving discretization for the coupled Stokes–Biot equations

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Abstract

We develop and analyze a structure-preserving discretization for the coupled Stokes–Biot equations. The method is based on a fully-mixed formulation and computes directly the fluid and poroelastic stresses and the Darcy velocity. The scheme employs equal-order polynomial approximations for all unknowns, yet exhibits local momentum conservation, strongly symmetric stress fields, and exactly divergence-free velocity in the fluid domain. Transmission conditions enforcing mass conservation, stress balance, and the Beavers–Joseph–Saffman slip with friction law are incorporated directly into the formulation as natural interface conditions through careful balance of finite element spaces across subdomains. Well-posedness of the fully discrete scheme based on the backward Euler time-stepping method is established, followed by optimal convergence error estimates for all variables. The analysis shows that the method is locking-free in the regimes of small storativity and almost incompressible material. Several numerical experiments are carried out to validate the accuracy and robustness of the proposed scheme.

Keywords: discontinuous Galerkin methods; divergence-free velocity; structure-preserving; the Stokes equations; Biot equations.

1 Introduction

Fluid–poroelastic structure interaction (FPSI), which models the coupling between free fluids and deformable porous media, is an important multiphysics paradigm essential for predicting complex phenomena across vital applications, including hydrocarbon extraction in fractured reservoirs, hemodynamics in arterial tissues, and the design of advanced filtration systems. The free fluid domain is governed by the Stokes or Navier–Stokes equations, while flow within the deformable porous matrix is described by the Biot system of poroelasticity. These systems are intricately coupled through dynamic and kinematic transmission conditions: balance of forces, continuity of normal velocity, and the Beavers–Joseph–Saffman (BJS) slip with friction condition. Consequently, the development of accurate and robust numerical schemes for solving this coupled FPSI system is paramount for reliably simulating the underlying physics and directly impacts both the numerical analysis community and practitioners in geoscience, biomechanics, and engineering.

The Stokes–Biot model integrates two well-studied classical couplings: fluid–structure interaction (FSI) with thick structure and the Stokes–Darcy model for coupling free fluid and porous media flows.

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A comprehensive investigation of these two constituent models has been conducted, see e.g. [29, 44] and references therein for FSI and [36, 26, 45, 27, 32, 51, 35, 31, 16, 14, 30, 21] for Stokes–Darcy flows. The theoretical foundations for the Stokes–Biot system established in [50] have prompted substantial research into the design of stable, accurate, and efficient discretizations. Early contributions include monolithic and iterative domain decomposition strategies [6], a non-iterative operator splitting scheme [12] and a method based on a mixed Darcy formulation using Nitsche’s coupling [11]. Subsequent developments include Lagrange multiplier techniques that weakly impose interface conditions via auxiliary variables [5, 2, 52], methods with natural transmission conditions [18, 8], total pressure formulation [46], mixed elasticity formulation [38], multipoint stress-flux mixed finite element methods [17], hybridizable discontinuous Galerkin (HDG) methods [19, 20], augmented fully mixed formulation [37], polytopal discontinuous Galerkin methods [28], dimensional model reduction for flow through fractured poroelastic media [13], coupling of FPSI with transport [3, 15], modeling of fluid-porohyperelastic structure interaction [49], and Robin-Robin splitting methods [25]. However, it should be noted that all of the aforementioned works employ either displacement formulations or mixed elasticity formulations with weakly symmetric stress, and none of these works achieves both local momentum and mass conservation.

In this paper we develop a novel mixed discretization for the Stokes–Biot model that exhibits both local momentum and mass conservation, strongly enforces stress symmetry, and employs equal-order polynomial approximations for all variables. A key innovation is the use of carefully balanced finite element discretizations for each subdomain, capable of seamlessly handling interface conditions. To this end, we introduce the deviatoric operator for the Stokes system, which allows for the elimination of the fluid pressure and enables a unified treatment of the Biot and Stokes equations. We utilize partially discontinuous polynomials and Alfeld splits [1, 47] in both regions to enrich the standard polynomial spaces. The discretization includes Lagrange multipliers modeling the traces of the fluid and structure velocities on the primal and dual edges (faces in 3D), which are used to enforce normal stress continuity. This framework renders the fluid velocity–displacement interface conditions natural. Moreover, it allows us to enforce stress symmetry without the need for stabilization terms. While Alfeld splits for symmetric stress spaces have been previously explored for fluid–structure interaction [22], the present work adapts this approach to the Stokes–Biot model. The Darcy equations are modeled with a velocity–pressure formulation and are discretized using a staggered discontinuous Galerkin method proposed in [23] and further developed in [57, 56, 55]. The proposed scheme has several salient features: accurate approximation of physical variables of interest with $H(\text{div})$ -conforming fluid and poroelastic stresses and local momentum conservation within both the fluid and poroelastic regions, divergence-free fluid velocity, and local mass conservation for the poroelastic fluid in the limit of zero storativity. Furthermore, the analysis establishes robustness with respect to c_0 and λ_p in the limits of vanishing storativity and nearly incompressible poroelasticity, with the remaining physical coefficients fixed. We establish well-posedness and rigorous stability and *a priori* convergence error estimates for the fully discrete method with the backward Euler time discretization, with explicit tracking of parameter dependencies in the error estimates. A key ingredient of the analysis is the development of an interpolant in the strongly symmetric stress finite element space. It is used in the stress error estimate and to establish a discrete inf-sup condition for the fluid and structure velocities in a suitable broken H^1 -norm. Superconvergence is obtained for the discrete velocity error in this norm, as well as for the Darcy velocity discrete divergence error. Several numerical experiments are carried out to verify the scheme’s accuracy and parameter robustness.

The major contributions of this paper can be summarized as follows:

1. We propose a novel mixed finite element method for the Stokes–Biot model that achieves strong stress symmetry while simultaneously ensuring local momentum conservation for both the Biot and Stokes regions, divergence-free fluid velocity, and local mass conservation in the poroelastic

region in the limit of zero storativity.

2. We establish rigorous stability and optimal convergence and superconvergence estimates with explicit parameter dependence, demonstrating uniform performance in the critical limits of nearly incompressible materials and vanishing storativity.

The remainder of this paper is organized as follows. In the next section we present the mathematical model. The discrete formulation for the Stokes–Biot system is derived in Section 3. Section 4 is devoted to the well-posedness of the fully-discrete scheme. Convergence error estimates are established in Section 5. Finally, in Section 6 several numerical experiments are presented to demonstrate the performance of the proposed scheme, including convergence tests with manufactured solutions and a simulation of coupled subsurface and surface flows.

2 Model problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a Lipschitz domain, which is subdivided into two nonoverlapping and possibly non-connected regions: the fluid region Ω_f and the poroelastic region Ω_p . Let $\Gamma := \partial\Omega_f \cap \partial\Omega_p$ denote the interface between these two regions and let $\Gamma_f := \partial\Omega_f \setminus \Gamma$ and $\Gamma_p := \partial\Omega_p \setminus \Gamma$ denote the external parts on the boundary $\partial\Omega_f$ and $\partial\Omega_p$, respectively. We use $\mathbf{n}_f$ and $\mathbf{n}_p$ to represent the unit normal vectors pointing outward from $\partial\Omega_f$ and $\partial\Omega_p$, respectively.

The poroelastic domain Ω_p is governed by the quasi-static Biot system (cf. [7])

$$-\nabla \cdot \underline{\sigma}_p = \mathbf{f}_p \quad \text{in } \Omega_p \times (0, T], \quad (2.1)$$

$$\mathcal{A}_p(\underline{\sigma}_p + \alpha p_p \underline{I}) = \varepsilon(\boldsymbol{\eta}) \quad \text{in } \Omega_p \times (0, T], \quad (2.2)$$

$$\mathbf{u}_p + K\nu_p^{-1} \nabla p_p = \mathbf{0} \quad \text{in } \Omega_p \times (0, T], \quad (2.3)$$

$$\partial_t(c_0 p_p + \alpha \nabla \cdot \boldsymbol{\eta}) + \nabla \cdot \mathbf{u}_p = g_p \quad \text{in } \Omega_p \times (0, T], \quad (2.4)$$

$$\boldsymbol{\eta} = \mathbf{0} \text{ and } p_p = 0 \quad \text{on } \Gamma_p \times (0, T], \quad (2.5)$$

$$p_p(0) = p_{p,0} \quad \text{in } \Omega_p, \quad (2.6)$$

where $\underline{\sigma}_p$ is the poroelastic stress, $\boldsymbol{\eta}$ is the solid displacement, p_p is the pore pressure, $\mathbf{u}_p$ is the Darcy velocity, $\varepsilon(\boldsymbol{\eta}) = \frac{1}{2}(\nabla \boldsymbol{\eta} + \nabla \boldsymbol{\eta}^\top)$, $\mathbf{f}_p$ and g_p are given (sufficiently regular) loading and source terms, respectively, and the fourth-order tensor $\mathcal{A}_p$ is defined for all $\underline{\tau} \in \mathbb{R}^{d \times d}$ such that

$$\mathcal{A}_p(\underline{\tau}) = \frac{1}{2\mu_p} \left(\underline{\tau} - \frac{\lambda_p}{2\mu_p + d\lambda_p} \text{tr}(\underline{\tau}) \underline{I} \right) \quad \text{with } \mathcal{A}_p^{-1}(\underline{\tau}) = 2\mu_p \underline{\tau} + \lambda_p \text{tr}(\underline{\tau}) \underline{I}, \quad (2.7)$$

where $\lambda_p > 0$ and $\mu_p \geq \mu_0 > 0$ are the Lamé constants and $\underline{I}$ is the $d \times d$ identity matrix. In (2.3), $\nu_p > 0$ represents the dynamic viscosity of the fluid and K is a symmetric and positive definite tensor representing the permeability. In (2.2) and (2.4), $0 < \alpha \leq 1$ and $c_0 \geq 0$ denote the Biot–Willis and storativity coefficients, respectively. From (2.2) we can write

$$\nabla \cdot \boldsymbol{\eta} = \text{tr}(\varepsilon(\boldsymbol{\eta})) = \text{tr}(\mathcal{A}_p(\underline{\sigma}_p + \alpha p_p \underline{I})),$$

where $\text{tr}(\underline{\tau}) = \sum_{i=1}^d \tau_{ii}$. Then (2.4) can be written as

$$\partial_t(c_0 p_p + \alpha \text{tr}(\mathcal{A}_p(\underline{\sigma}_p + \alpha p_p \underline{I}))) + \nabla \cdot \mathbf{u}_p = g_p. \quad (2.8)$$

In the fluid domain Ω_f , we consider the following incompressible Stokes problem

$$-\nabla \cdot \underline{\sigma}_f = \mathbf{h} \quad \text{in } \Omega_f \times (0, T], \quad (2.9)$$

$$\underline{\sigma}_f - 2\nu\varepsilon(\mathbf{u}_f) + p_f \underline{I} = \underline{0} \quad \text{in } \Omega_f \times (0, T], \quad (2.10)$$

$$\nabla \cdot \mathbf{u}_f = 0 \quad \text{in } \Omega_f \times (0, T], \quad (2.11)$$

$$\mathbf{u}_f = \mathbf{0} \quad \text{on } \Gamma_f \times (0, T], \quad (2.12)$$

where ν is the fluid viscosity, $\mathbf{u}_f$ represents the fluid velocity and $\underline{\sigma}_f$ is the stress tensor. It follows from (2.10) and (2.11) that the fluid pressure p_f can be recovered from the stress $\underline{\sigma}_f$ according to $p_f = -\frac{1}{d}\text{tr}(\underline{\sigma}_f)$ in view of $\text{tr}(\varepsilon(\mathbf{u}_f)) = \nabla \cdot \mathbf{u}_f = 0$. Let us define

$$\mathcal{A}_f(\underline{\sigma}_f) := \frac{1}{2\nu}\underline{\sigma}_f^D, \quad \text{where } \underline{\sigma}_f^D := \underline{\sigma}_f - \frac{1}{d}\text{tr}(\underline{\sigma}_f)\underline{I} \quad (2.13)$$

is the deviatoric part of $\underline{\sigma}_f$. Thus, the Stokes equations (2.9)–(2.11) can be rewritten in a stress-velocity formulation as

$$-\nabla \cdot \underline{\sigma}_f = \mathbf{h} \quad \text{in } \Omega_f \times (0, T], \quad (2.14)$$

$$\mathcal{A}_f(\underline{\sigma}_f) - \varepsilon(\mathbf{u}_f) = \underline{0} \quad \text{in } \Omega_f \times (0, T]. \quad (2.15)$$

The transmission conditions on the interface Γ can be prescribed by (cf. [40, 50, 6])

$$(\partial_t \boldsymbol{\eta} + \mathbf{u}_p) \cdot \mathbf{n}_p = \mathbf{u}_f \cdot \mathbf{n}_p \quad \text{on } \Gamma \times (0, T], \quad (2.16)$$

$$p_p = -\underline{\sigma}_f \mathbf{n}_p \cdot \mathbf{n}_p \quad \text{on } \Gamma \times (0, T], \quad (2.17)$$

$$\underline{\sigma}_p \mathbf{n}_p \cdot \mathbf{n}_p = \underline{\sigma}_f \mathbf{n}_p \cdot \mathbf{n}_p \quad \text{on } \Gamma \times (0, T], \quad (2.18)$$

$$\underline{\sigma}_p \mathbf{n}_f \cdot \mathbf{t}_{f,j} = \underline{\sigma}_f \mathbf{n}_f \cdot \mathbf{t}_{f,j} = -\delta(\mathbf{u}_f - \partial_t \boldsymbol{\eta}) \cdot \mathbf{t}_{f,j} \quad \text{on } \Gamma \times (0, T], \quad (2.19)$$

where $\mathbf{t}_{f,j}$, $1 \leq j \leq d-1$, is an orthogonal system of unit tangent vectors on Γ . Conditions (2.16)–(2.19) represent continuity of normal velocity, balance of fluid stress, conservation of momentum, and the Beavers–Joseph–Saffman slip with friction condition, where δ is an inverse friction coefficient. The well-posedness of this mathematical model is established in [50].

We introduce some notation that will be used throughout the paper. We will use the Hilbert spaces $H^r(\mathcal{O})$ for non-negative integer r , where $\mathcal{O} \subset \Omega, \Omega \subset \mathbb{R}^d$. The spaces of vector- and matrix-valued functions with all the components in $H^r(\mathcal{O})$ will be respectively denoted as $\mathbf{H}^r(\mathcal{O})$ and $\underline{H}^r(\mathcal{O})$. Also, we use $\underline{H}^r(\mathbb{S}, \mathcal{O})$ to indicate the symmetric matrix-valued functions. The corresponding norm is denoted as $\|\cdot\|_{H^r(\mathcal{O})}$. We use $(\cdot, \cdot)_{\mathcal{O}}$ to represent the standard L^2 -inner product over $\mathcal{O} \subset \mathbb{R}^d$ and when $\mathcal{O}$ coincides with Ω , the subscript will be omitted. The corresponding norm is denoted as $\|\cdot\|_{L^2(\mathcal{O})}$. In addition, we define

$$\mathbf{H}(\text{div}; \mathcal{O}) := \{\mathbf{v} \in \mathbf{L}^2(\mathcal{O}) : \nabla \cdot \mathbf{v} \in L^2(\mathcal{O})\}.$$

The space of matrix-valued functions where each row belongs to $\mathbf{H}(\text{div}; \mathcal{O})$ will be denoted by $\underline{H}(\text{div}; \mathcal{O})$. We will also use the Bochner spaces and norms

$$\|\phi\|_{H^1(0,T;X)}^2 := \int_0^T \left(\|\phi(t)\|_X^2 + \|\partial_t \phi(t)\|_X^2 \right) dt.$$

3 Discrete formulation

Let $\mathcal{T}_{pr}$ denote a family of shape-regular simplicial meshes of the domain Ω , where all meshes are assumed to be aligned with the interface Γ for simplicity. For each mesh cell $M \in \mathcal{T}_{pr}$, we select an interior point and connect it to all vertices of M , thereby subdividing M into $(d+1)$ simplices; see

Figure 3.1 for an illustration in 2D. We remark that this subdivision is called an Alfeld split in the literature [1, 47]. The resulting simplicial meshes, denoted by $\mathcal{T}_h$, are assumed to satisfy the shape-regularity assumption. We define the set of primal faces $\mathcal{F}_{pr}$ as the union of all faces in the initial simplicial meshes $\mathcal{T}_{pr}$ excluding those lying on the interface Γ . The set of dual faces $\mathcal{F}_{dl}$ consists of all faces generated during the subdivision process. The collection of faces coinciding with the interface is denoted by $\mathcal{F}_h^\Gamma$. To distinguish between subdomains, let $\mathcal{T}_h^i := \mathcal{T}_h \cap \Omega_i$ for $i = p, f$. Similarly, define $\mathcal{F}_{pr}^i := \mathcal{F}_{pr} \cap \bar{\Omega}_i$ and $\mathcal{F}_{dl}^i := \mathcal{F}_{dl} \cap \bar{\Omega}_i$, leading to the face sets $\mathcal{F}_h^i := \mathcal{F}_{pr}^i \cup \mathcal{F}_{dl}^i$ for each subdomain. Moreover, we define the interior primal faces of Ω_i as $\mathcal{F}_{pr}^{0,i} := \mathcal{F}_{pr}^i \cap \Omega_i$. We let $\mathcal{F}_h := \mathcal{F}_h^p \cup \mathcal{F}_h^f \cup \mathcal{F}_h^\Gamma$, with $\mathcal{F}_h^0 := \mathcal{F}_h \cap \Omega$ representing its subset of interior faces and $\mathcal{F}_h^{0,i} := \mathcal{F}_h^0 \cap \Omega_i$ representing the interior faces of each subdomain. For each $\tau \in \mathcal{T}_h$, we let h_τ represent its diameter and let $\mathbf{n}$ denote the outward unit normal vector on $\partial\tau$. Moreover, we let $h := \max_{\tau \in \mathcal{T}_h} h_\tau$. For each face e , we use h_e to denote the diameter of e . We also use $\mathbf{n}$ to represent the unit normal vector on e pointing from τ_1 to τ_2 , where τ_1 and τ_2 are the elements sharing the common face e . The meaning of $\mathbf{n}$ will be clear from the context. For each interior face e shared by elements τ_1 and τ_2 , we define the jump and average of a scalar function q over e as

$$[[q]]_e = q_{1|e} - q_{2|e} \quad \text{and} \quad \{q\}_e = \frac{q_{1|e} + q_{2|e}}{2},$$

where $q_i = q|_{\tau_i}$, $i = 1, 2$. For the boundary faces, we simply define $[[q]]_e = q|_e$ and $\{q\}_e = q|_e$. We also use the same notation to indicate the jump and average of the vector and tensor functions. For $k \geq d - 1$, we use $P_k(\tau)$ and $P_k(e)$ to represent the polynomial functions defined on τ and e whose order is less than or equal to k . Similarly, $\mathbf{P}_k(\tau)$ and $\underline{P}_k(\tau)$ represent vector-valued and tensor-valued functions on τ , respectively. In the sequel, we use C to represent a generic positive constant independent of the meshsize, which may have different values at different occurrences.

For a vector function $\mathbf{v}$, we use $\mathbf{v}^n$ and $\mathbf{v}^t$ to represent the normal component and the tangential component, respectively, that is,

$$\mathbf{v}^n = (\mathbf{v} \cdot \mathbf{n})\mathbf{n}, \quad \mathbf{v}^t = \mathbf{v} - \mathbf{v}^n,$$

which satisfy

$$\mathbf{v} = \mathbf{v}^n + \mathbf{v}^t. \quad (3.1)$$

In the numerical method we solve for approximations of the variables $(\underline{\sigma}, \mathbf{u}, \mathbf{u}_p, p_p, \hat{\mathbf{u}}, \lambda)$, where

$$\begin{aligned} \underline{\sigma}|_{\Omega_i} &:= \underline{\sigma}_i, \quad i = p, f, \quad \mathbf{u}|_{\Omega_p} := \mathbf{u}_s := \partial_t \boldsymbol{\eta}, \quad \mathbf{u}|_{\Omega_f} := \mathbf{u}_f, \\ \hat{\mathbf{u}} &:= \mathbf{u}_s \text{ on } \mathcal{F}_{pr}^p, \quad \hat{\mathbf{u}} := \mathbf{u}_f \text{ on } \mathcal{F}_{pr}^f, \quad \hat{\mathbf{u}} := \mathbf{u}_s^t \text{ on } \mathcal{F}_{dl}^p, \quad \hat{\mathbf{u}} := \mathbf{u}_f^t \text{ on } \mathcal{F}_{dl}^f, \\ \hat{\mathbf{u}} &:= \mathbf{u}_f \text{ on } \Gamma, \quad \lambda := \mathbf{u}_p \cdot \mathbf{n}_p \text{ on } \Gamma. \end{aligned} \quad (3.2)$$

We use the following finite element spaces in the construction of our scheme (cf. [57, 22]):

$$\begin{aligned} \underline{\Sigma}_h &:= \{\underline{\theta}|_\tau \in \underline{P}_k(\mathbb{S}, \tau) \mid \forall \tau \in \mathcal{T}_h\}, \\ \mathbf{U}_h &:= \{\mathbf{v}|_\tau \in \mathbf{P}_k(\tau) \mid \forall \tau \in \mathcal{T}_h; [[\mathbf{v} \cdot \mathbf{n}]]_e = 0 \mid \forall e \in \mathcal{F}_{dl}\}, \\ \boldsymbol{\Lambda}_h &:= \{\hat{\mathbf{v}}|_e \in \mathbf{P}_k(e) \mid \forall e \in \mathcal{F}_h; \hat{\mathbf{v}} \cdot \mathbf{n}|_e = 0 \mid \forall e \in \mathcal{F}_{dl}; \hat{\mathbf{v}}|_e = \mathbf{0} \mid \forall e \in \Gamma_f \cup \Gamma_p\}, \\ Q_h^p &:= \{q_p|_\tau \in P_k(\tau) \mid \forall \tau \in \mathcal{T}_h^p, [[q_p]]_e = 0 \mid \forall e \in \mathcal{F}_{pr}^p\}, \\ \Gamma_h &:= \{\mu|_e \in P_k(e) \mid \forall e \in \mathcal{F}_h^\Gamma\}. \end{aligned}$$

In the sequel we use

$$\underline{\Sigma}_h^i := \underline{\Sigma}_h|_{\Omega_i} \quad \text{and} \quad \mathbf{U}_h^i := \mathbf{U}_h|_{\Omega_i}, \quad i = p, f.$$

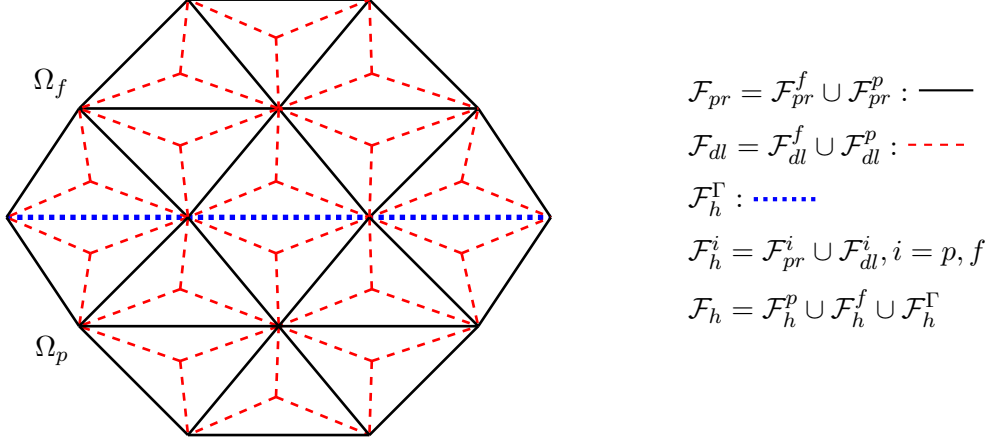


Figure 3.1: Meshes in 2D. $\mathcal{F}_{pr}$: primal edges (solid black lines), $\mathcal{F}_{dl}$: dual edges (dashed red lines), $\mathcal{F}_h^\Gamma$: meshes on the interface Γ (dotted blue lines).

We note that the boundary conditions $\boldsymbol{\eta} = \mathbf{0}$ on Γ_p and $\mathbf{u}_f = \mathbf{0}$ on Γ_f are incorporated in the definition of $\boldsymbol{\Lambda}_h$ and the boundary condition $p_p = 0$ on Γ_p is incorporated in the definition of Q_h^p .

In the numerical method presented in Section 3.1 below, at each time step we solve for $(\underline{\sigma}_h, \mathbf{u}_h, \mathbf{u}_{ph}, p_{ph}, \hat{\mathbf{u}}_h, \lambda_h) \in \underline{\Sigma}_h \times \mathbf{U}_h \times \mathbf{U}_h^p \times Q_h^p \times \boldsymbol{\Lambda}_h \times \Gamma_h$. To derive the method, we first derive the variational formulation satisfied by the exact solution. To this end, we test equations (2.1)–(2.19) with the corresponding test functions. Multiplying (2.1) by a test function $\mathbf{v}_s \in \mathbf{U}_h^p$ and using integration by parts leads to

$$\sum_{\tau \in \mathcal{T}_h^p} (\underline{\sigma}_p, \varepsilon(\mathbf{v}_s))_\tau - \sum_{\tau \in \mathcal{T}_h^p} (\underline{\sigma}_p \mathbf{n}, \mathbf{v}_s)_{\partial\tau \cap (\mathcal{F}_{pr}^p \cup \mathcal{F}_h^\Gamma)} - \sum_{\tau \in \mathcal{T}_h^p} ((\underline{\sigma}_p \mathbf{n})^t, (\mathbf{v}_s)^t)_{\partial\tau \cap \mathcal{F}_{dl}^p} = (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p}, \quad (3.3)$$

where we used that $\mathbf{v}_s \in \mathbf{U}_h^p$ and thus $[\![\mathbf{v}_s \cdot \mathbf{n}]\!]_e = 0$ for $e \in \mathcal{F}_{dl}^p$.

Recalling the definition of $\mathbf{u}_s$ and λ , cf. (3.2), it follows from (2.16) and (2.19) that for $\underline{\tau}_p \in \underline{\Sigma}_h^p$

$$\begin{aligned} \sum_{e \in \mathcal{F}_h^\Gamma} (\mathbf{u}_s, \underline{\tau}_p \mathbf{n}_p)_e &= \sum_{e \in \mathcal{F}_h^\Gamma} (\mathbf{u}_f \cdot \mathbf{n}_p - \lambda, \underline{\tau}_p \mathbf{n}_p \cdot \mathbf{n}_p)_e + \sum_{e \in \mathcal{F}_h^\Gamma} (\mathbf{u}_f^{t_p} - \delta^{-1}(\underline{\sigma}_p \mathbf{n}_p)^{t_p}, (\underline{\tau}_p \mathbf{n}_p)^{t_p})_e \\ &= \sum_{e \in \mathcal{F}_h^\Gamma} (\mathbf{u}_f, \underline{\tau}_p \mathbf{n}_p)_e - \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda, \underline{\tau}_p \mathbf{n}_p \cdot \mathbf{n}_p)_e - \sum_{e \in \mathcal{F}_h^\Gamma} (\delta^{-1}(\underline{\sigma}_p \mathbf{n}_p)^{t_p}, (\underline{\tau}_p \mathbf{n}_p)^{t_p})_e, \end{aligned} \quad (3.4)$$

where we used $\mathbf{v} = \mathbf{v}^{n_p} + \mathbf{v}^{t_p}$ for any vector function $\mathbf{v}$ (cf. (3.1)).

Taking the time derivative of (2.2), multiplying by the test function $\underline{\tau}_p \in \underline{\Sigma}_h^p$, and using the definition of $\hat{\mathbf{u}}$, cf. (3.2) and the boundary condition for $\hat{\mathbf{u}}$, we obtain via integration by parts and (3.4)

$$\begin{aligned} (\partial_t \mathcal{A}_p(\underline{\sigma}_p + \alpha p_p \mathbf{I}), \underline{\tau}_p)_{\Omega_p} &= - \sum_{\tau \in \mathcal{T}_h^p} (\mathbf{u}_s, \nabla \cdot \underline{\tau}_p)_\tau + \sum_{e \in \mathcal{F}_{dl}^p} (\mathbf{u}_s \cdot \mathbf{n}, [\![\underline{\tau}_p \mathbf{n} \cdot \mathbf{n}]\!])_e \\ &\quad + \sum_{e \in \mathcal{F}_h^{0,p}} (\hat{\mathbf{u}}, [\![\underline{\tau}_p \mathbf{n}]\!])_e + \sum_{e \in \mathcal{F}_h^\Gamma} (\hat{\mathbf{u}}, \underline{\tau}_p \mathbf{n}_p)_e \\ &\quad - \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda, \underline{\tau}_p \mathbf{n}_p \cdot \mathbf{n}_p)_e - \sum_{e \in \mathcal{F}_h^\Gamma} (\delta^{-1}(\underline{\sigma}_p \mathbf{n}_p)^{t_p}, (\underline{\tau}_p \mathbf{n}_p)^{t_p})_e. \end{aligned} \quad (3.5)$$

Multiplying (2.3) by a test function $\mathbf{v}_p \in \mathbf{U}_h^p$, integrating by parts, and using $p_p = 0$ on Γ_p yields

$$(K^{-1}\nu_p \mathbf{u}_p, \mathbf{v}_p)_{\Omega_p} + \sum_{e \in \mathcal{F}_h^\Gamma \cup \mathcal{F}_{pr}^{0,p}} (p_p, \llbracket \mathbf{v}_p \cdot \mathbf{n} \rrbracket)_e - \sum_{\tau \in \mathcal{T}_h^p} (p_p, \nabla \cdot \mathbf{v}_p)_\tau = 0. \quad (3.6)$$

Multiplying (2.8) by a test function $q_p \in Q_h^p$ and applying integration by parts yields

$$\begin{aligned} & (c_0 \partial_t p_p, q_p)_{\Omega_p} + (\alpha \partial_t \mathcal{A}_p(\underline{\sigma}_p + \alpha p_p \underline{I}), q_p \underline{I})_{\Omega_p} + \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda, q_p)_e \\ & + \sum_{e \in \mathcal{F}_{dl}^p} (\mathbf{u}_p \cdot \mathbf{n}, \llbracket q_p \rrbracket)_e - \sum_{\tau \in \mathcal{T}_h^p} (\mathbf{u}_p, \nabla q_p)_\tau = (g_p, q_p)_{\Omega_p}. \end{aligned} \quad (3.7)$$

Similarly, we have from (2.14)

$$\begin{aligned} & - \sum_{\tau \in \mathcal{T}_h^f} (\underline{\sigma}_f \mathbf{n}, \mathbf{v}_f)_{\partial \tau \cap \mathcal{F}_{pr}^f} - \sum_{\tau \in \mathcal{T}_h^f} ((\underline{\sigma}_f \mathbf{n})^t, \mathbf{v}_f^t)_{\partial \tau \cap \mathcal{F}_{dl}^f} \\ & - \sum_{e \in \mathcal{F}_h^\Gamma} (\underline{\sigma}_f \mathbf{n}_f, \mathbf{v}_f)_e + \sum_{\tau \in \mathcal{T}_h^f} (\underline{\sigma}_f, \varepsilon(\mathbf{v}_f))_\tau = (\mathbf{h}, \mathbf{v}_f)_{\Omega_f} \quad \forall \mathbf{v}_f \in \mathbf{U}_h^f. \end{aligned}$$

Finally, we infer from (2.15)

$$\begin{aligned} & (\mathcal{A}_f \underline{\sigma}_f, \underline{\tau}_f)_{\Omega_f} + \sum_{\tau \in \mathcal{T}_h^f} (\mathbf{u}_f, \nabla \cdot \underline{\tau}_f)_\tau - \sum_{e \in \mathcal{F}_{dl}^f} (\mathbf{u}_f \cdot \mathbf{n}, \llbracket \underline{\tau}_f \mathbf{n} \cdot \mathbf{n} \rrbracket)_e \\ & - \sum_{e \in \mathcal{F}_h^{0,f}} (\hat{\mathbf{u}}, \llbracket \underline{\tau}_f \mathbf{n} \rrbracket)_e - \sum_{e \in \mathcal{F}_h^\Gamma} (\hat{\mathbf{u}}, \underline{\tau}_f \mathbf{n}_f)_e = 0. \end{aligned} \quad (3.8)$$

We define the following bilinear forms for $i = p, f$

$$a_i(\mathbf{v}, \underline{\tau}) := \sum_{\tau \in \mathcal{T}_h^i} (\mathbf{v}, \nabla \cdot \underline{\tau})_\tau - \sum_{e \in \mathcal{F}_{dl}^i} (\mathbf{v} \cdot \mathbf{n}, \llbracket \underline{\tau} \mathbf{n} \cdot \mathbf{n} \rrbracket)_e, \quad (3.9)$$

$$b_p(\mathbf{v}_p, q_p) := - \sum_{\tau \in \mathcal{T}_h^p} (\mathbf{v}_p, \nabla q_p)_\tau + \sum_{e \in \mathcal{F}_{dl}^p} (\mathbf{v}_p \cdot \mathbf{n}, \llbracket q_p \rrbracket)_e. \quad (3.10)$$

Integration by parts implies the following equivalent formulation

$$a_i(\mathbf{v}, \underline{\tau}) = - \sum_{\tau \in \mathcal{T}_h^i} (\varepsilon(\mathbf{v}), \underline{\tau})_\tau + \sum_{\tau \in \mathcal{T}_h^i} (\mathbf{v}, \underline{\tau} \mathbf{n})_{\partial \tau \cap (\mathcal{F}_{pr} \cup \mathcal{F}_h^\Gamma)} + \sum_{\tau \in \mathcal{T}_h^i} (\mathbf{v}^t, (\underline{\tau} \mathbf{n})^t)_{\partial \tau \cap \mathcal{F}_{dl}}, \quad (3.11)$$

$$b_p(\mathbf{v}_p, q_p) = \sum_{\tau \in \mathcal{T}_h^p} (q_p, \nabla \cdot \mathbf{v}_p)_\tau - \sum_{e \in \mathcal{F}_h^\Gamma \cup \mathcal{F}_{pr}^{0,p}} (q_p, \llbracket \mathbf{v}_p \cdot \mathbf{n} \rrbracket)_e.$$

In addition, an application of (2.18)–(2.19) yields

$$\sum_{e \in \mathcal{F}_h^\Gamma} (\llbracket \underline{\sigma} \mathbf{n} \rrbracket, \hat{\mathbf{v}})_e = 0 \quad \forall \hat{\mathbf{v}} \in \mathbf{\Lambda}_h, \quad (3.12)$$

where for $e \in \mathcal{F}_h^\Gamma$, $\llbracket \underline{\tau} \mathbf{n} \rrbracket|_e = \underline{\tau}_p \mathbf{n}_p + \underline{\tau}_f \mathbf{n}_f$. An application of (2.17) gives

$$\sum_{e \in \mathcal{F}_h^\Gamma} (p_p + \underline{\sigma}_p \mathbf{n}_p \cdot \mathbf{n}_p, \mu)_e = 0 \quad \forall \mu \in \Gamma_h. \quad (3.13)$$

In view of (3.3)–(3.12), the solution for (2.1)–(2.19) satisfies for all $(\underline{\tau}, \mathbf{v}, \mathbf{v}_p, q_p, \hat{\mathbf{v}}, \mu) \in \underline{\Sigma}_h \times \mathbf{U}_h \times \mathbf{U}_h^p \times Q_h^p \times \mathbf{\Lambda}_h \times \Gamma_h$:

$$\begin{aligned} & (\partial_t \mathcal{A}_p(\underline{\sigma}_p + \alpha p_p \underline{I}), \underline{\tau}_p)_{\Omega_p} + a_p(\mathbf{u}_s, \underline{\tau}_p) + \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda, \underline{\tau}_p \mathbf{n}_p \cdot \mathbf{n}_p)_e \\ & + \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} ((\underline{\sigma}_p \mathbf{n}_p)^{t_p}, (\underline{\tau}_p \mathbf{n}_p)^{t_p})_e + (\mathcal{A}_f \underline{\sigma}_f, \underline{\tau}_f)_{\Omega_f} + a_f(\mathbf{u}_f, \underline{\tau}_f) - \sum_{e \in \mathcal{F}_h^0} (\hat{\mathbf{u}}, \llbracket \underline{\tau} \mathbf{n} \rrbracket)_e = 0, \end{aligned} \quad (3.14a)$$

$$- a_p(\mathbf{v}_s, \underline{\sigma}_p) - a_f(\mathbf{v}_f, \underline{\sigma}_f) = (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p} + (\mathbf{h}, \mathbf{v}_f)_{\Omega_f}, \quad (3.14b)$$

$$\sum_{e \in \mathcal{F}_h^0} (\llbracket \underline{\sigma} \mathbf{n} \rrbracket, \hat{\mathbf{v}})_e = 0, \quad (3.14c)$$

$$(K^{-1} \nu_p \mathbf{u}_p, \mathbf{v}_p)_{\Omega_p} - b_p(p_p, \mathbf{v}_p) = 0, \quad (3.14d)$$

$$(c_0 \partial_t p_p, q_p)_{\Omega_p} + (\alpha \partial_t \mathcal{A}(\underline{\sigma}_p + \alpha p_p \underline{I}), q_p \underline{I})_{\Omega_p} + \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda, q_p)_e + b_p(q_p, \mathbf{u}_p) = (g_p, q_p)_{\Omega_p}, \quad (3.14e)$$

$$\sum_{e \in \mathcal{F}_h^\Gamma} (\underline{\sigma}_p \mathbf{n}_p \cdot \mathbf{n}_p + p_p, \mu)_e = 0, \quad (3.14f)$$

where $\underline{\tau}|_{\Omega_i} := \underline{\tau}_i$, $\mathbf{v}|_{\Omega_p} := \mathbf{v}_s$ and $\mathbf{v}|_{\Omega_f} := \mathbf{v}_f$.

3.1 Fully discrete numerical method

For an arbitrary function $\gamma \in L^2(\Omega)$, we introduce the following notation, $d_t \gamma^{n+1} := \frac{\gamma^{n+1} - \gamma^n}{\Delta t}$. For the given time interval $[0, T]$, we divide it into subintervals $[t_n, t_{n+1}]$ for $0 \leq n \leq N-1$ (N is an integer) and denote the time step size as $\Delta t := T/N$. With the backward Euler scheme in time, the fully discrete scheme reads as follows: Given $(\underline{\sigma}_{ph}^0, p_{ph}^0)$, for $n = 0, \dots, N-1$, find $(\underline{\sigma}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{u}_{ph}^{n+1}, p_{ph}^{n+1}, \hat{\mathbf{u}}_h^{n+1}, \lambda_h^{n+1}) \in \underline{\Sigma}_h \times \mathbf{U}_h \times \mathbf{U}_h^p \times Q_h^p \times \mathbf{\Lambda}_h \times \Gamma_h$ such that for all $(\underline{\tau}, \mathbf{v}, \mathbf{v}_p, q_p, \hat{\mathbf{v}}, \mu) \in \underline{\Sigma}_h \times \mathbf{U}_h \times \mathbf{U}_h^p \times Q_h^p \times \mathbf{\Lambda}_h \times \Gamma_h$:

$$\begin{aligned} & (d_t \mathcal{A}_p(\underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I}), \underline{\tau}_p)_{\Omega_p} + a_p(\mathbf{u}_{sh}^{n+1}, \underline{\tau}_p) + \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda_h^{n+1}, \underline{\tau}_p \mathbf{n}_p \cdot \mathbf{n}_p)_e \\ & + \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} ((\underline{\sigma}_{ph}^{n+1} \mathbf{n}_p)^{t_p}, (\underline{\tau}_p \mathbf{n}_p)^{t_p})_e + (\mathcal{A}_f \underline{\sigma}_{fh}^{n+1}, \underline{\tau}_f)_{\Omega_f} \\ & + a_f(\mathbf{u}_{fh}^{n+1}, \underline{\tau}_f) - \sum_{e \in \mathcal{F}_h^0} (\hat{\mathbf{u}}_h^{n+1}, \llbracket \underline{\tau} \mathbf{n} \rrbracket)_e = 0, \end{aligned} \quad (3.15a)$$

$$- a_p(\mathbf{v}_s, \underline{\sigma}_{ph}^{n+1}) - a_f(\mathbf{v}_f, \underline{\sigma}_{fh}^{n+1}) = (\mathbf{f}_p^{n+1}, \mathbf{v}_s)_{\Omega_p} + (\mathbf{h}^{n+1}, \mathbf{v}_f)_{\Omega_f}, \quad (3.15b)$$

$$(K^{-1} \nu_p \mathbf{u}_{ph}^{n+1}, \mathbf{v}_p)_{\Omega_p} - b_p(p_{ph}^{n+1}, \mathbf{v}_p) = 0, \quad (3.15c)$$

$$\begin{aligned} & (c_0 d_t p_{ph}^{n+1}, q_p)_{\Omega_p} + (\alpha d_t \mathcal{A}_p(\underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I}), q_p \underline{I})_{\Omega_p} \\ & + b_p(q_p, \mathbf{u}_{ph}^{n+1}) + \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda_h^{n+1}, q_p)_e = (g_p^{n+1}, q_p)_{\Omega_p}, \end{aligned} \quad (3.15d)$$

$$\sum_{e \in \mathcal{F}_h^0} (\llbracket \underline{\sigma}_h^{n+1} \mathbf{n} \rrbracket, \hat{\mathbf{v}})_e = 0, \quad (3.15e)$$

$$\sum_{e \in \mathcal{F}_h^\Gamma} (p_{ph}^{n+1} + \underline{\sigma}_{ph}^{n+1} \mathbf{n}_p \cdot \mathbf{n}_p, \mu)_e = 0. \quad (3.15f)$$

3.2 Structure-preserving properties

In this subsection we omit the index $(\cdot)^{n+1}$ to simplify the presentation.

Following [23, Lemma 2.3] and [57, Lemma 3.2], we can define the degrees of freedom for $\mathbf{U}_h$ as follows.

(UD1) For $e \in \mathcal{F}_{dl}$, we have

$$P_e(\mathbf{q}_h) := (\mathbf{q}_h \cdot \mathbf{n}, \phi)_e \quad \forall \phi \in P_k(e). \quad (3.16)$$

(UD2) For each $\tau \in \mathcal{T}_h$ and $k \geq 1$, we define

$$P_\tau(\mathbf{q}_h) := (\mathbf{q}_h, \phi)_\tau \quad \forall \phi \in \mathbf{P}_{k-1}(\tau). \quad (3.17)$$

Lemma 3.1 (Divergence-free fluid velocity). *The velocity field $\mathbf{u}_h$ obtained from (3.15) satisfies $\mathbf{u}_{fh} \in \mathbf{H}(\text{div}; \Omega_f)$ with $\nabla \cdot \mathbf{u}_{fh} = 0$ in Ω_f .*

Proof. Let $Q_h^f := \{q_f|_\tau \in P_k(\tau) \mid \forall \tau \in \mathcal{T}_h^f; \llbracket q_f \rrbracket|_e = 0 \mid \forall e \in \mathcal{F}_{pr}^{0,f}\}$. Let $\omega_f \in Q_h^f$ and take $\underline{\tau}_f = \omega_f \underline{I}$ and $\underline{\tau}_p = \underline{0}$ in (3.15a). Using that

$$(\mathcal{A}_f \underline{\sigma}_{fh}, \omega_f \underline{I})_{\Omega_f} = \frac{1}{2\nu} (\underline{\sigma}_{fh}^D, \omega_f \underline{I})_{\Omega_f} = 0, \quad (3.18a)$$

$$- \sum_{e \in \mathcal{F}_h^f} (\hat{\mathbf{u}}_h, \llbracket (\omega_f \underline{I}) \mathbf{n} \rrbracket)_e = - \sum_{e \in \mathcal{F}_{dl}^f} (\hat{\mathbf{u}}_h \cdot \mathbf{n}, \llbracket \omega_f \rrbracket)_e = 0, \quad (3.18b)$$

and the definition of $a_f(\cdot, \cdot)$ (cf. (3.9)), we obtain

$$- \sum_{e \in \mathcal{F}_{dl}^f} (\mathbf{u}_{fh} \cdot \mathbf{n}, \llbracket \omega_f \rrbracket)_e + \sum_{\tau \in \mathcal{T}_h^f} (\mathbf{u}_{fh}, \nabla \omega_f)_\tau - \sum_{e \in \mathcal{F}_h^\Gamma} (\hat{\mathbf{u}}_h \cdot \mathbf{n}_f, \omega_f)_e = 0,$$

which can be recast into the following equivalent form via integration by parts

$$\sum_{e \in \mathcal{F}_{pr}^f} (\llbracket \mathbf{u}_{fh} \cdot \mathbf{n} \rrbracket, \omega_f)_e - \sum_{\tau \in \mathcal{T}_h^f} (\nabla \cdot \mathbf{u}_{fh}, \omega_f)_\tau + \sum_{e \in \mathcal{F}_h^\Gamma} ((\mathbf{u}_{fh} - \hat{\mathbf{u}}_h) \cdot \mathbf{n}_f, \omega_f)_e = 0.$$

Then in view of the degrees of freedom defined for Q_h^f in [23, Lemma 2.2], we can define ω_f such that

$$\begin{aligned} (\omega_f, q)_\tau &= -(\nabla \cdot \mathbf{u}_{fh}, q)_\tau \quad \forall q \in P_{k-1}(\tau), \tau \in \mathcal{T}_h^f, \\ (\omega_f, \chi)_e &= (\llbracket \mathbf{u}_{fh} \cdot \mathbf{n} \rrbracket, \chi)_e \quad \forall \chi \in P_k(e), e \in \mathcal{F}_{pr}^f, \\ (\omega_f, \chi)_e &= ((\mathbf{u}_{fh} - \hat{\mathbf{u}}_h) \cdot \mathbf{n}_f, \chi)_e \quad \forall \chi \in P_k(e), e \in \mathcal{F}_h^\Gamma, \end{aligned}$$

obtaining

$$\sum_{\tau \in \mathcal{T}_h^f} \|\nabla \cdot \mathbf{u}_{fh}\|_{L^2(\tau)}^2 + \sum_{e \in \mathcal{F}_{pr}^f} \|\llbracket \mathbf{u}_{fh} \cdot \mathbf{n} \rrbracket\|_{L^2(e)}^2 + \sum_{e \in \mathcal{F}_h^\Gamma} \|(\mathbf{u}_{fh} - \hat{\mathbf{u}}_h) \cdot \mathbf{n}_f\|_{L^2(e)}^2 = 0.$$

As such, it holds that $\llbracket \mathbf{u}_{fh} \cdot \mathbf{n} \rrbracket|_e = 0$ for all $e \in \mathcal{F}_{pr}^f$, hence $\mathbf{u}_{fh} \in \mathbf{H}(\text{div}; \Omega_f)$, and $\nabla \cdot \mathbf{u}_{fh} = 0$ in Ω_f . Therefore, the proof is completed. $\square$

Lemma 3.2 (mass conservation in Ω_p). *For the solution of (3.15), if $c_0 = 0$ and g_p is a polynomial of order $k-1$ over each $\tau \in \mathcal{T}_h$, it holds that*

$$\alpha \mathbf{u}_{sh} + \mathbf{u}_{ph} \in \mathbf{H}(\text{div}; \Omega_p), \quad \nabla \cdot (\alpha \mathbf{u}_{sh} + \mathbf{u}_{ph}) = g_p \quad \text{in } \Omega_p.$$

Proof. Letting $\underline{\tau}_p = q_p \underline{I}$, $q_p \in Q_h^p$, and $\underline{\tau}_f = \underline{0}$ in (3.15a), we obtain

$$(d_t \mathcal{A}_p(\underline{\sigma}_{ph} + \alpha p_{ph} \underline{I}), q_p \underline{I})_{\Omega_p} + a_p(\mathbf{u}_{sh}, q_p \underline{I}) + \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda_h, q_p)_e - \sum_{e \in \mathcal{F}_h^\Gamma} (\hat{\mathbf{u}}_h \cdot \mathbf{n}_p, q_p)_e = 0,$$

where we used that $((q_p \underline{I} \mathbf{n}_p)^{t_p})|_\Gamma = 0$ and $\hat{\mathbf{u}}_h \cdot \mathbf{n}|_e = 0$ for any $e \in \mathcal{F}_{dl}^p$. This, combined with (3.15d), yields

$$-a_p(\alpha \mathbf{u}_{sh}, q_p \underline{I}) + b_p(q_p, \mathbf{u}_{ph}) + \sum_{e \in \mathcal{F}_h^\Gamma} (\hat{\mathbf{u}}_h \cdot \mathbf{n}_p, q_p)_e = (g_p, q_p)_{\Omega_p}.$$

Using the definitions of $a_p(\cdot, \cdot)$ and $b_p(\cdot, \cdot)$ given in (3.9)–(3.10), we obtain from integration by parts

$$\begin{aligned} & \sum_{\tau \in \mathcal{T}_h^p} (\alpha \nabla \cdot \mathbf{u}_{sh}, q_p)_\tau - \sum_{e \in \mathcal{F}_{pr}^{0,p}} ([\![\alpha \mathbf{u}_{sh} \cdot \mathbf{n}]\!] , q_p)_e + \sum_{\tau \in \mathcal{T}_h^p} (q_p, \nabla \cdot \mathbf{u}_{ph})_\tau \\ & - \sum_{e \in \mathcal{F}_{pr}^{0,p}} (q_p, [\![\mathbf{u}_{ph} \cdot \mathbf{n}]\!])_e - \sum_{e \in \mathcal{F}_h^\Gamma} (q_p, \alpha \mathbf{u}_{sh} \cdot \mathbf{n}_p + \mathbf{u}_{ph} \cdot \mathbf{n}_p - \hat{\mathbf{u}}_h \cdot \mathbf{n}_p)_e = (g_p, q_p)_{\Omega_p}. \end{aligned}$$

Then using the degrees of freedom for Q_h^p defined in [23, Lemma 2.2], we can define q_p such that

$$\begin{aligned} (q_p, q)_\tau &= (\nabla \cdot (\alpha \mathbf{u}_{sh} + \mathbf{u}_{ph}) - g_p, q)_\tau \quad \forall q \in P_{k-1}(\tau), \tau \in \mathcal{T}_h^p, \\ (q_p, \chi)_e &= -([\![\alpha \mathbf{u}_{sh} + \mathbf{u}_{ph}] \cdot \mathbf{n}]\!] , \chi)_e \quad \forall \chi \in P_k(e), e \in \mathcal{F}_{pr}^{0,p}, \\ (q_p, \chi)_e &= -(\alpha \mathbf{u}_{sh} \cdot \mathbf{n}_p + \mathbf{u}_{ph} \cdot \mathbf{n}_p - \hat{\mathbf{u}}_h \cdot \mathbf{n}_p, \chi)_e \quad \forall \chi \in P_k(e), e \in \mathcal{F}_h^\Gamma, \end{aligned}$$

giving

$$\begin{aligned} & \sum_{\tau \in \mathcal{T}_h^p} \|\nabla \cdot (\alpha \mathbf{u}_{sh} + \mathbf{u}_{ph}) - g_p\|_{L^2(\tau)}^2 + \sum_{e \in \mathcal{F}_{pr}^{0,p}} \|[\![\alpha \mathbf{u}_{sh} + \mathbf{u}_{ph}] \cdot \mathbf{n}]\!] \|_{L^2(e)}^2 \\ & + \sum_{e \in \mathcal{F}_h^\Gamma} \|\alpha \mathbf{u}_{sh} \cdot \mathbf{n}_p + \mathbf{u}_{ph} \cdot \mathbf{n}_p - \hat{\mathbf{u}}_h \cdot \mathbf{n}_p\|_{L^2(e)}^2 = 0. \end{aligned}$$

Therefore, $[\![\alpha \mathbf{u}_{sh} + \mathbf{u}_{ph}] \cdot \mathbf{n}]\!]_e = 0$ for all $e \in \mathcal{F}_{pr}^{0,p}$, implying that $\alpha \mathbf{u}_{sh} + \mathbf{u}_{ph} \in \mathbf{H}(\text{div}; \Omega_p)$, and $\nabla \cdot (\alpha \mathbf{u}_{sh} + \mathbf{u}_{ph}) - g_p = 0$. Thus, the proof is completed. $\square$

Lemma 3.3 (Momentum conservation). *For the solution of (3.15), if $\mathbf{f}_p$ and $\mathbf{h}$ are polynomial functions of order $k-1$ over each $\tau \in \mathcal{T}_h$, it holds that*

$$\underline{\sigma}_h \in \underline{H}(\text{div}; \Omega), \quad \nabla \cdot \underline{\sigma}_{ph} = -\mathbf{f}_p \quad \text{in } \Omega_p, \quad \nabla \cdot \underline{\sigma}_{fh} = -\mathbf{h} \quad \text{in } \Omega_f. \quad (3.19)$$

Proof. First, (3.15e) implies that $[\![\underline{\sigma}_h \mathbf{n}]\!]_e = \mathbf{0} \quad \forall e \in \mathcal{F}_{pr}^{0,f} \cup \mathcal{F}_{pr}^{0,p} \cup \mathcal{F}_h^\Gamma$ and $[\![\underline{\sigma}_h \mathbf{n}]\!]^t|_e = \mathbf{0} \quad \forall e \in \mathcal{F}_{dl}$. In view of (3.16)–(3.17), we can define $\mathbf{v}_s$ and $\mathbf{v}_f$ such that

$$\begin{aligned} (\mathbf{v}_s \cdot \mathbf{n}, \mu)_e &= ([\![\underline{\sigma}_{ph} \mathbf{n} \cdot \mathbf{n}]\!] , \mu)_e \quad \forall \mu \in P_k(e), e \in \mathcal{F}_{dl}^p, \\ -(\mathbf{v}_s, \phi)_\tau &= (\nabla \cdot \underline{\sigma}_{ph} + \mathbf{f}_p, \phi)_\tau \quad \forall \phi \in \mathbf{P}_{k-1}(\tau), \tau \in \mathcal{T}_h^p \end{aligned}$$

and

$$\begin{aligned} (\mathbf{v}_f \cdot \mathbf{n}, \mu)_e &= ([\![\underline{\sigma}_{fh} \mathbf{n} \cdot \mathbf{n}]\!] , \mu)_e \quad \forall \mu \in P_k(e), e \in \mathcal{F}_{dl}^f, \\ -(\mathbf{v}_f, \phi)_\tau &= (\nabla \cdot \underline{\sigma}_{fh} + \mathbf{h}, \phi)_\tau \quad \forall \phi \in \mathbf{P}_{k-1}(\tau), \tau \in \mathcal{T}_h^f. \end{aligned}$$

These and (3.15b) yield

$$\begin{aligned} & \sum_{e \in \mathcal{F}_{dl}^f} \|[\![\underline{\sigma}_{fh} \mathbf{n} \cdot \mathbf{n}]\!]\|_{L^2(e)}^2 + \sum_{e \in \mathcal{F}_{dl}^p} \|[\![\underline{\sigma}_{ph} \mathbf{n} \cdot \mathbf{n}]\!]\|_{L^2(e)}^2 \\ & + \sum_{\tau \in \mathcal{T}_h^p} \|\nabla \cdot \underline{\sigma}_{ph} + \mathbf{f}_p\|_{L^2(\tau)}^2 + \sum_{\tau \in \mathcal{T}_h^f} \|\nabla \cdot \underline{\sigma}_{fh} + \mathbf{h}\|_{L^2(\tau)}^2 = 0, \end{aligned}$$

which implies $[\![\underline{\sigma}_h \mathbf{n} \cdot \mathbf{n}]\!]_e = 0 \ \forall e \in \mathcal{F}_{dl}$ and thus $\underline{\sigma}_h \in \underline{H}(\text{div}; \Omega)$. Moreover, $\nabla \cdot \underline{\sigma}_{fh} = -\mathbf{h}$ in Ω_f and $\nabla \cdot \underline{\sigma}_{ph} = -\mathbf{f}_p$ in Ω_p . $\square$

4 Well-posedness of the fully-discrete system

4.1 Preliminaries

We introduce several auxiliary norms that will be used in the analysis. We use $\mathbf{\Lambda}_h^p$ to represent the restriction of $\mathbf{\Lambda}_h$ to $\overline{\Omega}_p$, which includes the edges $e \in \mathcal{F}_h^\Gamma$. Let

$$\begin{aligned} \|\mathbf{v}\|_{1,h}^2 &:= \sum_{\tau \in \mathcal{T}_h} \|\nabla \mathbf{v}\|_{L^2(\tau)}^2 + \sum_{e \in \mathcal{F}_h} h_e^{-1} \|[\![\mathbf{v}]\!]\|_{L^2(e)}^2 \quad \forall \mathbf{v} \in \mathbf{U}_h, \\ \|(\mathbf{v}, \widehat{\mathbf{v}})\|_{1,h}^2 &:= \sum_{\tau \in \mathcal{T}_h} \|\nabla \mathbf{v}\|_{L^2(\tau)}^2 + \sum_{\tau \in \mathcal{T}_h^f} h_\tau^{-1} \|\mathbf{v}_f - \widehat{\mathbf{v}}\|_{L^2(\partial\tau)}^2 \\ &+ \sum_{\tau \in \mathcal{T}_h^p} h_\tau^{-1} \|\mathbf{v}_s - \widehat{\mathbf{v}}\|_{L^2(\partial\tau \setminus \mathcal{F}_h^\Gamma)}^2 \quad \forall (\mathbf{v}, \widehat{\mathbf{v}}) \in \mathbf{U}_h \times \mathbf{\Lambda}_h, \end{aligned} \quad (4.1)$$

$$\|(\mathbf{v}_s, \widehat{\mathbf{v}}_s)\|_{1,h,p}^2 := \sum_{\tau \in \mathcal{T}_h^p} \|\nabla \mathbf{v}_s\|_{L^2(\tau)}^2 + \sum_{\tau \in \mathcal{T}_h^p} h_\tau^{-1} \|\mathbf{v}_s - \widehat{\mathbf{v}}_s\|_{L^2(\partial\tau)}^2 \quad \forall (\mathbf{v}_s, \widehat{\mathbf{v}}_s) \in \mathbf{U}_h^p \times \mathbf{\Lambda}_h^p. \quad (4.2)$$

Note that $\|\mathbf{v}\|_{1,h}$ is a norm on $\mathbf{U}_h$, due to the discrete Poincaré inequality (cf. [9])

$$\|\mathbf{v}\|_{L^2(\Omega)} \leq C \|\mathbf{v}\|_{1,h}. \quad (4.3)$$

Lemma 4.1 (Velocity norm). *$\|(\mathbf{v}, \widehat{\mathbf{v}})\|_{1,h}$ is a norm on $\mathbf{U}_h \times \mathbf{\Lambda}_h$ such that*

$$\|\mathbf{v}\|_{1,h} + \sum_{e \in \mathcal{F}_h} h_e \|[\![\widehat{\mathbf{v}}]\!]\|_{L^2(e)}^2 \leq C \|(\mathbf{v}, \widehat{\mathbf{v}})\|_{1,h} \quad \forall (\mathbf{v}, \widehat{\mathbf{v}}) \in \mathbf{U}_h \times \mathbf{\Lambda}_h. \quad (4.4)$$

Proof. It is straightforward to see that

$$\|\mathbf{v}\|_{1,h} \leq C \|(\mathbf{v}, \widehat{\mathbf{v}})\|_{1,h} \quad \forall (\mathbf{v}, \widehat{\mathbf{v}}) \in \mathbf{U}_h \times \mathbf{\Lambda}_h. \quad (4.5)$$

To control the other term, using the triangle inequality, the discrete trace inequality, (4.3), and (4.5),

we have, for all $(\mathbf{v}, \widehat{\mathbf{v}}) \in \mathbf{U}_h \times \mathbf{\Lambda}_h$,

$$\begin{aligned}
\sum_{e \in \mathcal{F}_h} h_e \|\widehat{\mathbf{v}}\|_{L^2(e)}^2 &\leq \sum_{\tau \in \mathcal{T}_h^f} (h_\tau \|\widehat{\mathbf{v}} - \mathbf{v}\|_{L^2(\partial\tau)}^2 + h_\tau \|\mathbf{v}\|_{L^2(\partial\tau)}^2) \\
&\quad + \sum_{\tau \in \mathcal{T}_h^p} (h_\tau \|\widehat{\mathbf{v}} - \mathbf{v}\|_{L^2(\partial\tau \setminus \mathcal{F}_h^\Gamma)}^2 + h_\tau \|\mathbf{v}\|_{L^2(\partial\tau \setminus \mathcal{F}_h^\Gamma)}^2) \\
&\leq \sum_{\tau \in \mathcal{T}_h^f} h_\tau \|\widehat{\mathbf{v}} - \mathbf{v}\|_{L^2(\partial\tau)}^2 + \sum_{\tau \in \mathcal{T}_h^p} h_\tau \|\widehat{\mathbf{v}} - \mathbf{v}\|_{L^2(\partial\tau \setminus \mathcal{F}_h^\Gamma)}^2 \\
&\quad + C \sum_{\tau \in \mathcal{T}_h} (h_e^2 \|\nabla \mathbf{v}\|_{L^2(\tau)}^2 + \|\mathbf{v}\|_{L^2(\tau)}^2) \\
&\leq C \|(\mathbf{v}, \widehat{\mathbf{v}})\|_{1,h}^2.
\end{aligned} \tag{4.6}$$

A combination of (4.5) and (4.6) completes the proof. $\square$

Lemma 4.2 (Velocity inf-sup condition). *There exists a positive constant C independent of the mesh size such that for any $(\mathbf{v}, \widehat{\mathbf{v}}) \in \mathbf{U}_h \times \mathbf{\Lambda}_h$,*

$$\|(\mathbf{v}, \widehat{\mathbf{v}})\|_{1,h} \leq C \sup_{\mathbf{0} \neq \mathcal{I} \in \Sigma_h : \mathcal{I}_p \mathbf{n}|_{\mathcal{F}_h^\Gamma} = \mathbf{0}} \frac{a_p(\mathbf{v}_s, \mathcal{I}_p) + a_f(\mathbf{v}_f, \mathcal{I}_f) - \sum_{e \in \mathcal{F}_h^0} (\widehat{\mathbf{v}}, \llbracket \mathcal{I} \mathbf{n} \rrbracket)_e}{\|\mathcal{I}\|_{L^2(\Omega)}}. \tag{4.7}$$

We postpone the proof of the above lemma until Section 5.1.

An adaptation of Lemma 4.2 that gives for any $(\mathbf{v}_s, \widehat{\mathbf{v}}_s) \in \mathbf{U}_h^p \times \mathbf{\Lambda}_h^p$,

$$\|(\mathbf{v}_s, \widehat{\mathbf{v}}_s)\|_{1,h,p} \leq C \sup_{\mathbf{0} \neq \mathcal{I}_p \in \Sigma_h^p} \frac{a_p(\mathbf{v}_s, \mathcal{I}_p) - \sum_{e \in \mathcal{F}_h^{0,p}} (\widehat{\mathbf{v}}_s, \llbracket \mathcal{I}_p \mathbf{n} \rrbracket)_e - \sum_{e \in \mathcal{F}_h^\Gamma} (\widehat{\mathbf{v}}_s, \mathcal{I}_p \mathbf{n}_p)_e}{\|\mathcal{I}_p\|_{L^2(\Omega_p)}}. \tag{4.8}$$

We emphasize that in (4.7) we impose the restriction $\mathcal{I}_p \mathbf{n}|_{\mathcal{F}_h^\Gamma} = \mathbf{0}$, which is needed to decouple the influence of $(\sigma_{ph}^{n+1} \mathbf{n}_p)|_{\mathcal{F}_h^\Gamma}$ and λ_h^{n+1} in (3.15a), while $\mathcal{I}_p$ is unrestricted on Γ in (4.8).

Next, for $\mathbf{v}_p \in \mathbf{U}_h^p$, $\mu \in \Gamma_h$, and $q_p \in Q_h^p$ we define

$$\begin{aligned}
\|(\mathbf{v}_p, \mu)\|_{\text{div}_p}^2 &:= \sum_{\tau \in \mathcal{T}_h^p} \|\nabla \cdot \mathbf{v}_p\|_{L^2(\tau)}^2 + \sum_{e \in \mathcal{F}_{pr}^{0,p}} h_e^{-1} \|\llbracket \mathbf{v}_p \cdot \mathbf{n} \rrbracket\|_{L^2(e)}^2 + \sum_{e \in \mathcal{F}_h^\Gamma} h_e^{-1} \|\mu - \mathbf{v}_p \cdot \mathbf{n}\|_{L^2(e)}^2, \\
\|q_p\|_Z^2 &:= \sum_{\tau \in \mathcal{T}_h^p} \|\nabla q_p\|_{L^2(\tau)}^2 + \sum_{e \in \mathcal{F}_{dl}^p} h_e^{-1} \|\llbracket q_p \rrbracket\|_{L^2(e)}^2.
\end{aligned}$$

We note that $\|q_p\|_Z$ is a norm on Q_h^p , since the discrete Poincaré inequality (cf. [9]) implies

$$\|q_p\|_{L^2(\Omega_p)} \leq C \|q_p\|_Z. \tag{4.9}$$

Clearly $\|(\mathbf{v}_p, \mu)\|_{\text{div}_p}$ controls the discrete divergence of $\mathbf{v}_p$. The following result shows that, combined with $\|\mathbf{v}_p\|_{L^2(\Omega_p)}$, it also provides control on μ .

Lemma 4.3 (Lagrange multiplier norm). *For all $(\mathbf{v}_p, \mu) \in \mathbf{U}_h^p \times \Gamma_h$, it holds*

$$\sum_{e \in \mathcal{F}_h^\Gamma} h_e \|\mu\|_{L^2(e)}^2 \leq C (\|(\mathbf{v}_p, \mu)\|_{\text{div}_p} + \|\mathbf{v}_p\|_{L^2(\Omega_p)}). \tag{4.10}$$

Proof. Using the triangle inequality and the discrete trace inequality, we obtain, for any $(\mathbf{v}_p, \mu) \in \mathbf{U}_h^p \times \Gamma_h$,

$$\begin{aligned} \sum_{e \in \mathcal{F}_h^\Gamma} h_e \|\mu\|_{L^2(e)}^2 &\leq \sum_{e \in \mathcal{F}_h^\Gamma} (h_e \|\mu - \mathbf{v}_p \cdot \mathbf{n}\|_{L^2(e)}^2 + h_e \|\mathbf{v}_p \cdot \mathbf{n}\|_{L^2(e)}^2) \\ &\leq \sum_{e \in \mathcal{F}_h^\Gamma} h_e \|\mu - \mathbf{v}_p \cdot \mathbf{n}\|_{L^2(e)}^2 + C \sum_{\tau \in \mathcal{T}_h^p} \|\mathbf{v}_p\|_{L^2(\tau)}^2 \leq C (\|(\mathbf{v}_p, \mu)\|_{\text{div}_p}^2 + \|\mathbf{v}_p\|_{L^2(\Omega_p)}^2). \end{aligned}$$

□

The proof of the following inf-sup conditions follows from a slight adaptation of the proof of [57, Theorem 3.1].

Lemma 4.4 (Darcy velocity and pressure inf-sup conditions). *For all $(\mathbf{v}_p, \mu) \in \mathbf{U}_h^p \times \Gamma_h$, it holds*

$$\|(\mathbf{v}_p, \mu)\|_{\text{div}_p} \leq C \sup_{0 \neq q_p \in Q_h^p} \frac{b_p(q_p, \mathbf{v}_p) + \sum_{e \in \mathcal{F}_h^\Gamma} (\mu, q_p)_e}{\|q_p\|_{L^2(\Omega_p)}}. \quad (4.11)$$

For all $q_p \in Q_h^p$, it holds

$$\|q_p\|_Z \leq C \sup_{0 \neq \mathbf{v}_p \in \mathbf{U}_h^p} \frac{b_p(q_p, \mathbf{v}_p)}{\|\mathbf{v}_p\|_{L^2(\Omega_p)}}. \quad (4.12)$$

Next, we introduce a norm of the discrete divergence of the stress:

$$\|\underline{w}\|_{\text{div},h,i}^2 := \sum_{\tau \in \mathcal{T}_h^i} \|\nabla \cdot \underline{w}\|_{L^2(\tau)}^2 + \sum_{e \in \mathcal{F}_{di}^i} h_e^{-1} \|\llbracket \underline{w} \mathbf{n} \cdot \mathbf{n} \rrbracket\|_{L^2(e)}^2 \quad \forall \underline{w} \in \underline{\Sigma}_h^i.$$

Let $\|\underline{w}\|_{\mathcal{A}_i(\Omega_i)}^2 = (\mathcal{A}_i \underline{w}, \underline{w})_{\Omega_i}$ for $i = f, p$. The following result can be found in [22, Lemma 5.7].

Lemma 4.5 (Control on the trace of the stress). *For all $\underline{w}_h \in \underline{\Sigma}_h$, it holds that*

$$\|\text{tr}(\underline{w}_{fh})\|_{L^2(\Omega_f)} \leq C \left(\nu^{\frac{1}{2}} \|\underline{w}_{fh}\|_{\mathcal{A}_f(\Omega_f)} + \|\underline{w}_{fh}\|_{\text{div},h,f} \right), \quad (4.13)$$

$$\|\text{tr}(\underline{w}_{ph})\|_{L^2(\Omega_p)} \leq C \left(\mu_p^{\frac{1}{2}} \|\underline{w}_{ph}\|_{\mathcal{A}_p(\Omega_p)} + \|\underline{w}_{ph}\|_{\text{div},h,p} \right). \quad (4.14)$$

The following bounds follow from Lemma 4.5 and the definitions of the operators $\mathcal{A}_p$ and $\mathcal{A}_f$, cf. (2.7) and (2.13).

Corollary 4.1. *For all $\underline{w}_h \in \underline{\Sigma}_h$, it holds that*

$$\|\underline{w}_{fh}\|_{L^2(\Omega_f)} \leq C \left(\nu^{\frac{1}{2}} \|\underline{w}_{fh}\|_{\mathcal{A}_f(\Omega_f)} + \|\underline{w}_{fh}\|_{\text{div},h,f} \right), \quad (4.15)$$

$$\|\underline{w}_{ph}\|_{L^2(\Omega_p)} \leq C \left(\mu_p^{\frac{1}{2}} \|\underline{w}_{ph}\|_{\mathcal{A}_p(\Omega_p)} + \|\underline{w}_{ph}\|_{\text{div},h,p} \right). \quad (4.16)$$

Lemma 4.6 (Stress inf-sup condition). *It holds that*

$$\|\underline{\theta}\|_{\text{div},h,i} \leq C \sup_{0 \neq \mathbf{v} \in \mathbf{U}_h^i} \frac{a_i(\mathbf{v}, \underline{\theta})}{\|\mathbf{v}\|_{L^2(\Omega_i)}} \quad \forall \underline{\theta} \in \underline{\Sigma}_h^i. \quad (4.17)$$

Proof. Given $\underline{\theta} \in \underline{\Sigma}_h^i$, it suffices to find $\mathbf{v} \in \mathbf{U}_h^i$ such that

$$a_i(\mathbf{v}, \underline{\theta}) = \|\underline{\theta}\|_{\text{div}, h, i}^2 \quad \text{and} \quad \|\mathbf{v}\|_{L^2(\Omega_i)} \leq C \|\underline{\theta}\|_{\text{div}, h, i}.$$

In view of the degrees of freedom defined for $\mathbf{U}_h$ (cf. (3.16)–(3.17)), we can define $\mathbf{v}$ such that

$$\begin{aligned} (\mathbf{v} \cdot \mathbf{n}, \phi)_e &= -h_e^{-1}(\llbracket \underline{\theta} \mathbf{n} \cdot \mathbf{n} \rrbracket, \phi)_e \quad \forall \phi \in P_k(e), e \in \mathcal{F}_{dl}^i, \\ (\mathbf{v}, \phi)_\tau &= (\nabla \cdot \underline{\theta}, \phi)_\tau \quad \forall \phi \in \mathbf{P}_{k-1}(\tau), \tau \in \mathcal{T}_h^i. \end{aligned}$$

A scaling argument implies

$$\|\mathbf{v}\|_{L^2(\Omega_i)} \leq C \|\underline{\theta}\|_{\text{div}, h, i}.$$

The definition (3.9) of $a_i(\cdot, \cdot)$ yields

$$a_i(\mathbf{v}, \underline{\theta}) = \|\underline{\theta}\|_{\text{div}, h, i}^2.$$

Therefore, the proof is completed. $\square$

The following trace inequality is proved in [32, Theorem 4.4] and [55, Lemma 3.6].

Lemma 4.7 (Trace inequality). *For all $q_p \in Q_h^p$, there exists a positive constant C independent of h such that*

$$\|q_p\|_{L^2(\Gamma)} \leq C \|q_p\|_Z. \quad (4.18)$$

4.2 Discrete initial data

Next, we construct discrete initial data for all variables that satisfy the equations in the numerical method (3.15) without time derivatives, i.e. (3.15b), (3.15c), (3.15e), and (3.15f). Let $(\underline{\sigma}_p^0, \underline{\sigma}_f^0, \mathbf{u}^0, \mathbf{u}_p^0, p_p^0, \hat{\mathbf{u}}^0, \lambda^0)$ be the values of the solution to the model problem (2.1)–(2.19), where the definition (3.2) of $\mathbf{u}^0$ and $\hat{\mathbf{u}}^0$ is modified such that $\mathbf{u}^0|_{\Omega_p} := \boldsymbol{\eta}^0$ and $\hat{\mathbf{u}}^0|_{\Omega_p} := \hat{\boldsymbol{\eta}}^0$ with $\hat{\boldsymbol{\eta}}^0 := \boldsymbol{\eta}^0$ on $\mathcal{F}_{pr}^p$ and $\hat{\boldsymbol{\eta}}^0 := (\boldsymbol{\eta}^0)^t$ on $\mathcal{F}_{dl}^p$. Recalling that the true solution satisfies the weak formulation (3.14), we define the discrete initial data as the elliptic projection of the true solution at the initial time $t = 0$. Let $(\underline{\sigma}_h^0, \mathbf{u}_h^0, \mathbf{u}_{ph}^0, p_{ph}^0, \hat{\mathbf{u}}_h^0, \lambda_h^0) \in \underline{\Sigma}_h \times \mathbf{U}_h \times \mathbf{U}_h^p \times Q_h^p \times \boldsymbol{\Lambda}_h \times \Gamma_h$ be such that for all $(\underline{\tau}, \mathbf{v}, \mathbf{v}_p, q_p, \hat{\mathbf{v}}, \mu) \in \underline{\Sigma}_h \times \mathbf{U}_h \times \mathbf{U}_h^p \times Q_h^p \times \boldsymbol{\Lambda}_h \times \Gamma_h$:

$$\begin{aligned} & (\mathcal{A}_p(\underline{\sigma}_{ph}^0 + \alpha p_{ph}^0 \underline{I}), \underline{\tau}_p)_{\Omega_p} + a_p(\boldsymbol{\eta}_h^0, \underline{\tau}_p) + \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda_h^0, \underline{\tau}_p \mathbf{n}_p \cdot \mathbf{n}_p)_e \\ & + \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} ((\underline{\sigma}_{ph}^0 \mathbf{n}_p)^{t_p}, (\underline{\tau}_p \mathbf{n}_p)^{t_p})_e + (\mathcal{A}_f \underline{\sigma}_{fh}^0, \underline{\tau}_f)_{\Omega_f} + a_f(\mathbf{u}_{fh}^0, \underline{\tau}_f) - \sum_{e \in \mathcal{F}_h^0} (\hat{\mathbf{u}}_h^0, \llbracket \underline{\tau} \mathbf{n} \rrbracket)_e \\ & = (\mathcal{A}_p(\underline{\sigma}_p^0 + \alpha p_p^0 \underline{I}), \underline{\tau}_p)_{\Omega_p} + a_p(\boldsymbol{\eta}^0, \underline{\tau}_p) + \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda^0, \underline{\tau}_p \mathbf{n}_p \cdot \mathbf{n}_p)_e \\ & + \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} ((\underline{\sigma}_p^0 \mathbf{n}_p)^{t_p}, (\underline{\tau}_p \mathbf{n}_p)^{t_p})_e + (\mathcal{A}_f \underline{\sigma}_f^0, \underline{\tau}_f)_{\Omega_f} + a_f(\mathbf{u}_f^0, \underline{\tau}_f) - \sum_{e \in \mathcal{F}_h^0} (\hat{\mathbf{u}}^0, \llbracket \underline{\tau} \mathbf{n} \rrbracket)_e \\ & = \sum_{e \in \mathcal{F}_h^\Gamma} (\boldsymbol{\eta}^0 \cdot \mathbf{n}_p + \lambda^0 - \mathbf{u}_f^0 \cdot \mathbf{n}_p, \underline{\tau}_p \mathbf{n}_p \cdot \mathbf{n}_p)_e + \sum_{e \in \mathcal{F}_h^\Gamma} (\delta^{-1} (\underline{\sigma}_p^0 \mathbf{n}_p)^{t_p} + (\boldsymbol{\eta}^0)^{t_p} - (\mathbf{u}_f^0)^{t_p}, (\underline{\tau}_p \mathbf{n}_p)^{t_p})_e, \quad (4.19a) \\ & - a_p(\mathbf{v}_s, \underline{\sigma}_{ph}^0) - a_f(\mathbf{v}_f, \underline{\sigma}_{fh}^0) = -a_p(\mathbf{v}_s, \underline{\sigma}_p^0) - a_f(\mathbf{v}_f, \underline{\sigma}_f^0) = (\mathbf{f}_p^0, \mathbf{v}_s)_{\Omega_p} + (\mathbf{h}^0, \mathbf{v}_f)_{\Omega_f}, \quad (4.19b) \end{aligned}$$

$$(K^{-1}\nu_p \mathbf{u}_{ph}^0, \mathbf{v}_p)_{\Omega_p} - b_p(p_{ph}^0, \mathbf{v}_p) = (K^{-1}\nu_p \mathbf{u}_p^0, \mathbf{v}_p)_{\Omega_p} - b_p(p_p^0, \mathbf{v}_p) = 0, \quad (4.19c)$$

$$\begin{aligned} & (\alpha \mathcal{A}_p(\underline{\sigma}_{ph}^0 + \alpha p_{ph}^0 \underline{I}), q_p \underline{I})_{\Omega_p} + b_p(q_p, \mathbf{u}_{ph}^0) + \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda_h^0, q_p)_e \\ &= (\alpha \mathcal{A}_p(\underline{\sigma}_p^0 + \alpha p_p^0 \underline{I}), q_p \underline{I})_{\Omega_p} + b_p(q_p, \mathbf{u}_p^0) + \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda^0, q_p)_e \\ &= (\alpha \nabla \cdot \boldsymbol{\eta}^0, q_p)_{\Omega_p} + (\nabla \cdot \mathbf{u}_p^0, q_p)_{\Omega_p}, \end{aligned} \quad (4.19d)$$

$$\sum_{e \in \mathcal{F}_h^{0,f}} ([\underline{\sigma}_{fh}^0 \mathbf{n}], \hat{\mathbf{v}})_e + \sum_{e \in \mathcal{F}_h^{0,p}} ([\underline{\sigma}_{ph}^0 \mathbf{n}], \hat{\mathbf{v}})_e + \sum_{e \in \mathcal{F}_h^\Gamma} (\underline{\sigma}_{ph}^0 \mathbf{n}_p + \underline{\sigma}_{fh}^0 \mathbf{n}_f, \hat{\mathbf{v}})_e = 0, \quad (4.19e)$$

$$\sum_{e \in \mathcal{F}_h^\Gamma} (p_{ph}^0 + \underline{\sigma}_{ph}^0 \mathbf{n}_p \cdot \mathbf{n}_p, \mu)_e = \sum_{e \in \mathcal{F}_h^\Gamma} (p_p^0 + \underline{\sigma}_p^0 \mathbf{n}_p \cdot \mathbf{n}_p, \mu)_e = 0. \quad (4.19f)$$

Lemma 4.8 (existence, uniqueness and stability bound for the initial data). *There exists a unique solution to (4.19) and there exists a positive constant C independent of c_0 , λ_p and h such that*

$$\begin{aligned} & \|\underline{\sigma}_{fh}^0\|_{L^2(\Omega_f)} + \|\mathbf{u}_{ph}^0\|_{L^2(\Omega_p)} + \|\underline{\sigma}_{ph}^0\|_{L^2(\Omega_p)} + \|p_{ph}^0\|_{L^2(\Omega_p)} + \|(\mathbf{u}_h^0, \hat{\mathbf{u}}_h^0)\|_{1,h} + \|(\mathbf{u}_{ph}^0, \lambda_h^0)\|_{\text{div}_p} \\ &+ \delta^{-\frac{1}{2}} \|(\underline{\sigma}_{ph}^0 \mathbf{n})^t\|_{L^2(\Gamma)} \leq C \left(\|\boldsymbol{\eta}^0\|_{H^1(\Omega_p)} + \|\mathbf{u}_f^0\|_{H^1(\Omega_f)} + \|\mathbf{h}^0\|_{L^2(\Omega_f)} + \|\mathbf{f}_p^0\|_{L^2(\Omega_p)} \right. \\ & \left. + \|\nabla \cdot \boldsymbol{\eta}^0\|_{L^2(\Omega_p)} + \|\nabla \cdot \mathbf{u}_p^0\|_{L^2(\Omega_p)} + \|\lambda^0\|_{L^2(\Gamma)} + \|(\underline{\sigma}_p^0 \mathbf{n}_p)^{t_p}\|_{L^2(\Gamma)} \right). \end{aligned} \quad (4.20)$$

Proof. Since (4.19) is a finite dimensional linear system, existence of a solution follows from uniqueness, which follows by taking the right-hand side of the stability bound (4.20) to be zero. Thus, it remains to establish (4.20). Taking $\underline{\tau} = \underline{\sigma}_h^0$, $\mathbf{v} = \mathbf{u}_h^0$, $\mathbf{v}_p = \mathbf{u}_{ph}^0$, $q_p = p_{ph}^0$, $\hat{\mathbf{v}} = \hat{\mathbf{u}}_h^0$ and $\mu = \lambda_h^0$ in (4.19) and summing up the resulting equations yields

$$\begin{aligned} & \|\underline{\sigma}_{ph}^0 + \alpha p_{ph}^0 \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 + \delta^{-1} \|(\underline{\sigma}_{ph}^0 \mathbf{n})^t\|_{L^2(\Gamma)}^2 + \|\underline{\sigma}_{fh}^0\|_{\mathcal{A}_f(\Omega_f)}^2 + \|(K^{-1}\nu_p)^{\frac{1}{2}} \mathbf{u}_{ph}^0\|_{L^2(\Omega_p)}^2 + c_0 \|p_{ph}^0\|_{L^2(\Omega_p)}^2 \\ &= \sum_{e \in \mathcal{F}_h^\Gamma} (\boldsymbol{\eta}^0 \cdot \mathbf{n}_p + \lambda^0 - \mathbf{u}_f^0 \cdot \mathbf{n}_p, \underline{\sigma}_{ph}^0 \mathbf{n}_p \cdot \mathbf{n}_p)_e + \sum_{e \in \mathcal{F}_h^\Gamma} (\delta^{-1} (\underline{\sigma}_p^0 \mathbf{n}_p)^{t_p} + (\boldsymbol{\eta}^0)^{t_p} - (\mathbf{u}_f^0)^{t_p}, (\underline{\sigma}_{ph}^0 \mathbf{n}_p)^{t_p})_e \\ &+ (\mathbf{f}_p^0, \boldsymbol{\eta}_h^0)_{\Omega_p} + (\mathbf{h}^0, \mathbf{u}_{fh}^0)_{\Omega_f} + (\alpha \nabla \cdot \boldsymbol{\eta}^0, p_{ph}^0)_{\Omega_p} - (\nabla \cdot \mathbf{u}_p^0, p_{ph}^0)_{\Omega_p}. \end{aligned} \quad (4.21)$$

Using the discrete Poincaré inequality (4.9) and the trace inequality (4.18), along with the inf-sup condition (4.12) and (4.19c), we can infer that

$$\|p_{ph}^0\|_{L^2(\Omega_p)} + \|p_{ph}^0\|_{L^2(\Gamma)} \leq C \|p_{ph}^0\|_Z \leq C \sup_{\mathbf{0} \neq \mathbf{v}_p \in \mathbf{U}_h^p} \frac{b_p(p_{ph}^0, \mathbf{v}_p)}{\|\mathbf{v}_p\|_{L^2(\Omega_p)}} \leq C \|\mathbf{u}_{ph}^0\|_{L^2(\Omega_p)}. \quad (4.22)$$

For the first term on the right-hand side of (4.21), using (4.19f), the Cauchy-Schwarz inequality, and (4.22), we obtain

$$\begin{aligned} & \sum_{e \in \mathcal{F}_h^\Gamma} (\boldsymbol{\eta}^0 \cdot \mathbf{n}_p + \lambda^0 - \mathbf{u}_f^0 \cdot \mathbf{n}_p, \underline{\sigma}_{ph}^0 \mathbf{n}_p \cdot \mathbf{n}_p)_e = - \sum_{e \in \mathcal{F}_h^\Gamma} (\boldsymbol{\eta}^0 \cdot \mathbf{n}_p + \lambda^0 - \mathbf{u}_f^0 \cdot \mathbf{n}_p, p_{ph}^0)_e \\ & \leq C \|\boldsymbol{\eta}^0 \cdot \mathbf{n}_p + \lambda^0 - \mathbf{u}_f^0 \cdot \mathbf{n}_p\|_{L^2(\Gamma)} \|\mathbf{u}_{ph}^0\|_{L^2(\Omega_p)}. \end{aligned} \quad (4.23)$$

The second term on the right-hand side of (4.21) can be bounded by the Cauchy-Schwarz inequality

$$\begin{aligned} & \sum_{e \in \mathcal{F}_h^\Gamma} (\delta^{-1} (\underline{\sigma}_p^0 \mathbf{n}_p)^{t_p} + (\boldsymbol{\eta}^0)^{t_p} - (\mathbf{u}_f^0)^{t_p}, (\underline{\sigma}_{ph}^0 \mathbf{n}_p)^{t_p})_e \\ & \leq \left(\delta^{-\frac{1}{2}} \|(\underline{\sigma}_p^0 \mathbf{n}_p)^{t_p}\|_{L^2(\Gamma)} + \delta^{\frac{1}{2}} \|(\boldsymbol{\eta}^0)^{t_p} - (\mathbf{u}_f^0)^{t_p}\|_{L^2(\Gamma)} \right) \delta^{-\frac{1}{2}} \|(\underline{\sigma}_{ph}^0 \mathbf{n}_p)^{t_p}\|_{L^2(\Gamma)}. \end{aligned} \quad (4.24)$$

Next, the inf-sup condition (4.7) and (4.19a) yield

$$\|(\mathbf{u}_h^0, \hat{\mathbf{u}}_h^0)\|_{1,h} \leq C \left(\|\underline{\sigma}_{ph}^0 + \alpha p_{ph}^0 \underline{I}\|_{\mathcal{A}_p(\Omega_p)} + \|\boldsymbol{\sigma}_{fh}^0\|_{\mathcal{A}_f(\Omega_f)} \right). \quad (4.25)$$

The third and fourth terms on the right-hand side of (4.21) can be bounded by the Cauchy-Schwarz inequality, (4.3), (4.4) and (4.25):

$$\begin{aligned} (\mathbf{f}_p^0, \boldsymbol{\eta}_h^0)_{\Omega_p} + (\mathbf{h}^0, \mathbf{u}_{fh}^0)_{\Omega_f} &\leq \left(\|\mathbf{f}_p^0\|_{L^2(\Omega_p)} + \|\mathbf{h}^0\|_{L^2(\Omega_f)} \right) \|\mathbf{u}_h^0\|_{L^2(\Omega)} \\ &\leq C \left(\|\mathbf{f}_p^0\|_{L^2(\Omega_p)} + \|\mathbf{h}^0\|_{L^2(\Omega_f)} \right) \left(\|\underline{\sigma}_{ph}^0 + \alpha p_{ph}^0 \underline{I}\|_{\mathcal{A}_p(\Omega_p)} + \|\boldsymbol{\sigma}_{fh}^0\|_{\mathcal{A}_f(\Omega_f)} \right). \end{aligned} \quad (4.26)$$

The last two terms on the right-hand side of (4.21) can be bounded by the Cauchy-Schwarz inequality and (4.22):

$$(\nabla \cdot \mathbf{u}_p^0, p_{ph}^0)_{\Omega_p} + (\alpha \nabla \cdot \boldsymbol{\eta}^0, p_{ph}^0)_{\Omega_p} \leq C \left(\|\nabla \cdot \mathbf{u}_p^0\|_{L^2(\Omega_p)} + \|\nabla \cdot \boldsymbol{\eta}^0\|_{L^2(\Omega_p)} \right) \|\mathbf{u}_{ph}^0\|_{L^2(\Omega_p)}. \quad (4.27)$$

Combining (4.21)–(4.27) yields

$$\begin{aligned} &\|\underline{\sigma}_{ph}^0 + \alpha p_{ph}^0 \underline{I}\|_{\mathcal{A}_p(\Omega_p)} + \delta^{-\frac{1}{2}} \|(\underline{\sigma}_{ph}^0 \mathbf{n})^t\|_{L^2(\Gamma)} + \|\underline{\sigma}_{fh}^0\|_{\mathcal{A}_f(\Omega_f)} + \|\mathbf{u}_{ph}^0\|_{L^2(\Omega_p)} \\ &\quad + \|p_{ph}^0\|_{L^2(\Omega_p)} + \|(\mathbf{u}_h^0, \hat{\mathbf{u}}_h^0)\|_{1,h} \leq C \left(\|\boldsymbol{\eta}^0\|_{H^1(\Omega_p)} + \|\mathbf{u}_f^0\|_{H^1(\Omega_f)} + \|\mathbf{h}^0\|_{L^2(\Omega_f)} + \|\mathbf{f}_p^0\|_{L^2(\Omega_p)} \right. \\ &\quad \left. + \|\nabla \cdot \boldsymbol{\eta}^0\|_{L^2(\Omega_p)} + \|\nabla \cdot \mathbf{u}_p^0\|_{L^2(\Omega_p)} + \|\lambda^0\|_{L^2(\Gamma)} + \|(\underline{\sigma}_p^0 \mathbf{n}_p)^{t_p}\|_{L^2(\Gamma)} \right). \end{aligned} \quad (4.28)$$

The inf-sup condition (4.11) and (4.19d) give

$$\|(\mathbf{u}_{ph}^0, \lambda_h^0)\|_{\text{div}_p} \leq C \|\underline{\sigma}_{ph}^0 + \alpha p_{ph}^0 \underline{I}\|_{\mathcal{A}_p(\Omega_p)}. \quad (4.29)$$

Similarly, the inf-sup condition (4.17) and (4.19b) yield

$$\|\underline{\sigma}_{fh}^0\|_{\text{div},h,f} + \|\underline{\sigma}_{ph}^0\|_{\text{div},h,p} \leq C \left(\|\mathbf{f}_p^0\|_{L^2(\Omega_p)} + \|\mathbf{h}^0\|_{L^2(\Omega_f)} \right). \quad (4.30)$$

The triangle inequality gives

$$\|\underline{\sigma}_{ph}^0\|_{\mathcal{A}_p(\Omega_p)} \leq \|\underline{\sigma}_{ph}^0 + \alpha p_{ph}^0 \underline{I}\|_{\mathcal{A}_p(\Omega_p)} + \|p_{ph}^0\|_{L^2(\Omega_p)}. \quad (4.31)$$

A combination of (4.28)–(4.31), (4.15) and (4.16) yields (4.20). $\square$

4.3 Existence, uniqueness, and stability

Next, we show the well-posedness and stability of the discrete system (3.15). The dependence on the physical parameters will be carefully tracked. In particular, the stability constant is shown to be independent of c_0 and λ_p .

We recall the following result, which can be proved using the Taylor series theorem (see also [34, Lemma 7.67]). For any $\mathbf{v}$ with $\partial_{tt}\mathbf{v} \in \mathbf{L}^2(\Omega)$, it holds that

$$\|(\partial_t - d_t)\mathbf{v}^j\|_{L^2(\Omega)}^2 \leq \frac{\Delta t}{3} \int_{t_{j-1}}^{t_j} \|\partial_{tt}\mathbf{v}\|_{L^2(\Omega)}^2 dt, \quad (4.32)$$

$$\|d_t\mathbf{v}^j\|_{L^2(\Omega)}^2 \leq \frac{1}{\Delta t} \int_{t_{j-1}}^{t_j} \|\partial_t\mathbf{v}\|_{L^2(\Omega)}^2 dt. \quad (4.33)$$

Theorem 4.1 (Existence, uniqueness, and stability). *Given $(\underline{\sigma}_{ph}^0, p_{ph}^0)$, there exists a unique solution to (3.15). In addition, there exists a positive constant C independent of c_0 , λ_p , h and Δt such that*

$$\begin{aligned}
& \|\underline{\sigma}_{fh}^N\|_{L^2(\Omega_f)}^2 + \sum_{n=0}^{N-1} \Delta t \|\underline{\sigma}_{fh}^{n+1}\|_{L^2(\Omega_f)}^2 + \|\underline{\sigma}_{ph}^N\|_{L^2(\Omega_p)}^2 + \delta^{-1} \sum_{n=0}^{N-1} \Delta t \|(\underline{\sigma}_{ph}^{n+1} \mathbf{n})^t\|_{L^2(\Gamma)}^2 \\
& + \|\mathbf{u}_{ph}^N\|_{L^2(\Omega_p)}^2 + \sum_{n=0}^{N-1} \Delta t \|\mathbf{u}_{ph}^{n+1}\|_{L^2(\Omega_p)}^2 + \|p_{ph}^N\|_{L^2(\Omega_p)}^2 + \Delta t \sum_{n=0}^{N-1} \|p_{ph}^{n+1}\|_{L^2(\Omega_p)}^2 \\
& + \sum_{n=0}^{N-1} \Delta t \|(\mathbf{u}_h^{n+1}, \widehat{\mathbf{u}}_h^{n+1})\|_{1,h}^2 + \sum_{n=0}^{N-1} \Delta t \|\mathbf{u}_h^{n+1}\|_{L^2(\Omega)}^2 \\
& + \sum_{n=0}^{N-1} \Delta t \left(\|\underline{\sigma}_{ph}^{n+1}\|_{\text{div},h,p}^2 + \|\underline{\sigma}_{fh}^{n+1}\|_{\text{div},h,f}^2 \right) + \sum_{n=0}^{N-1} \Delta t \|(\mathbf{u}_{ph}^{n+1}, \lambda_h^{n+1})\|_{\text{div}_p}^2 \leq C \text{Res}_1,
\end{aligned} \tag{4.34}$$

where

$$\begin{aligned}
\text{Res}_1 &= \|\mathbf{f}_p\|_{H^1(0,T;L^2(\Omega_p))}^2 + \|\mathbf{h}\|_{H^1(0,T;L^2(\Omega_f))}^2 + \|g_p\|_{H^1(0,T;L^2(\Omega_p))}^2 \\
& + \|p_{ph}^0\|_{L^2(\Omega_p)}^2 + \|\underline{\sigma}_{ph}^0\|_{L^2(\Omega_p)}^2 + \|\underline{\sigma}_{fh}^0\|_{L^2(\Omega_f)}^2 + \|\mathbf{u}_{ph}^0\|_{L^2(\Omega_p)}^2 \\
& + \sum_{n=0}^{N-1} \Delta t \left(\|\mathbf{f}_p^{n+1}\|_{L^2(\Omega_p)}^2 + \|\mathbf{h}^{n+1}\|_{L^2(\Omega_f)}^2 + \|g_p^{n+1}\|_{L^2(\Omega_p)}^2 \right).
\end{aligned}$$

Proof. We first show the existence and uniqueness. Since the system at each time step (3.15a)–(3.15f) is linear, it is sufficient to show the uniqueness. Assume there are two different solutions of (3.15a)–(3.15f) with the same initial condition and we let $(\underline{\sigma}_h^{*,n+1}, \mathbf{u}_h^{*,n+1}, \mathbf{u}_{ph}^{*,n+1}, p_{ph}^{*,n+1}, \widehat{\mathbf{u}}_h^{*,n+1}, \lambda_h^{*,n+1})$ be their differences. Taking $\underline{\tau}_p = \underline{\sigma}_{ph}^{*,n+1}$, $\underline{\tau}_f = \underline{\sigma}_{fh}^{*,n+1}$, $\mathbf{v}_s = \mathbf{u}_{sh}^{*,n+1}$, $\mathbf{v}_f = \mathbf{u}_{fh}^{*,n+1}$, $\mathbf{v}_p = \mathbf{u}_{ph}^{*,n+1}$, $q_p = p_{ph}^{*,n+1}$, $\widehat{\mathbf{v}} = \widehat{\mathbf{u}}_h^{*,n+1}$ and $\mu = \lambda_h^{*,n+1}$ in (3.15a)–(3.15f), we obtain

$$\begin{aligned}
& (d_t \mathcal{A}_p(\underline{\sigma}_{ph}^{*,n+1} + \alpha p_{ph}^{*,n+1} \underline{I}), \underline{\sigma}_{ph}^{*,n+1} + \alpha p_{ph}^{*,n+1} \underline{I})_{\Omega_p} + \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} ((\underline{\sigma}_{ph}^{*,n+1} \mathbf{n})^t, (\underline{\sigma}_{ph}^{*,n+1} \mathbf{n})^t)_e \\
& + (\mathcal{A}_f \underline{\sigma}_{fh}^{*,n+1}, \underline{\sigma}_{fh}^{*,n+1})_{\Omega_f} + (K^{-1} \nu_p \mathbf{u}_{ph}^{*,n+1}, \mathbf{u}_{ph}^{*,n+1})_{\Omega_p} + (c_0 d_t p_{ph}^{*,n+1}, p_{ph}^{*,n+1})_{\Omega_p} = 0.
\end{aligned}$$

Summing over n ($n = 0, \dots, N-1$) and using $(a-b)a = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$, we conclude that $\mathbf{u}_{ph}^{*,n+1} = \mathbf{0}$, $\mathcal{A}_f \underline{\sigma}_{fh}^{*,n+1} = \underline{0}$, $\mathcal{A}_p(\underline{\sigma}_{ph}^{*,n+1} + \alpha p_{ph}^{*,n+1} \underline{I}) = \underline{0}$ and $d_t \mathcal{A}_p(\underline{\sigma}_{ph}^{*,n+1} + \alpha p_{ph}^{*,n+1} \underline{I}) = \underline{0}$. Then an application of (4.12) and (3.15c) implies $p_{ph}^{*,n+1} = 0$ and thus $\mathcal{A}_p(\underline{\sigma}_{ph}^{*,n+1}) = \underline{0}$. An appeal to (4.10), (4.11), (3.15d) and the fact that $\mathbf{u}_{ph}^{*,n+1} = \mathbf{0}$ gives $\lambda_h^{*,n+1} = 0$.

We infer from (4.7) and (3.15a) that $\|(\mathbf{u}_h^{*,n+1}, \widehat{\mathbf{u}}_h^{*,n+1})\|_{1,h} = 0$, which, combined with Lemma 4.1, gives $\mathbf{u}_h^{*,n+1} = \mathbf{0}$ and $\widehat{\mathbf{u}}_h^{*,n+1} = \mathbf{0}$. Then it follows from the inf-sup condition (4.17) and (3.15b) that

$$\|\underline{\sigma}_{ph}^{*,n+1}\|_{\text{div},h,p} + \|\underline{\sigma}_{fh}^{*,n+1}\|_{\text{div},h,f} = 0,$$

which, coupled with (4.15), (4.16), and the fact that $\mathcal{A}_f \underline{\sigma}_{fh}^{*,n+1} = \mathcal{A}_p(\underline{\sigma}_{ph}^{*,n+1}) = \underline{0}$, yields $\underline{\sigma}_{fh}^{*,n+1} = \underline{\sigma}_{ph}^{*,n+1} = \underline{0}$. Therefore, the uniqueness proof is completed.

Next, we show the stability estimates. Taking $\underline{\tau}_p = \underline{\sigma}_{ph}^{n+1}$, $\underline{\tau}_f = \underline{\sigma}_{fh}^{n+1}$, $\mathbf{v}_s = \mathbf{u}_{sh}^{n+1}$, $\mathbf{v}_f = \mathbf{u}_{fh}^{n+1}$, $\mathbf{v}_p = \mathbf{u}_{ph}^{n+1}$, $q_p = p_{ph}^{n+1}$, $\widehat{\mathbf{v}} = \widehat{\mathbf{u}}_h^{n+1}$ and $\mu = \lambda_h^{n+1}$ in (3.15a)–(3.15f) and summing up the resulting equations, we obtain

$$(d_t \mathcal{A}_p(\underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I}), \underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I})_{\Omega_p} + \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} ((\underline{\sigma}_{ph}^{n+1} \mathbf{n})^t, (\underline{\sigma}_{ph}^{n+1} \mathbf{n})^t)_e$$

$$\begin{aligned}
& + (\mathcal{A}_f \underline{\sigma}_{fh}^{n+1}, \underline{\sigma}_{fh}^{n+1})_{\Omega_f} + (K^{-1} \nu_p \mathbf{u}_{ph}^{n+1}, \mathbf{u}_{ph}^{n+1})_{\Omega_p} + (c_0 d_t p_{ph}^{n+1}, p_{ph}^{n+1})_{\Omega_p} \\
& = (\mathbf{f}_p^{n+1}, \mathbf{u}_{sh}^{n+1})_{\Omega_p} + (\mathbf{h}^{n+1}, \mathbf{u}_{fh}^{n+1})_{\Omega_f} + (g_p^{n+1}, p_{ph}^{n+1})_{\Omega_p}.
\end{aligned}$$

Using the identity $(a - b)a = \frac{1}{2}(a^2 - b^2 + (a - b)^2)$, we have

$$\begin{aligned}
& \frac{1}{2\Delta t} \left(\|\underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 - \|\underline{\sigma}_{ph}^n + \alpha p_{ph}^n \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 + (\Delta t)^2 \|d_t(\underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 \right) \\
& + \|\underline{\sigma}_{fh}^{n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 + \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} \|(\underline{\sigma}_{ph}^{n+1} \mathbf{n})^t\|_{L^2(e)}^2 + \|(K^{-1} \nu_p)^{\frac{1}{2}} \mathbf{u}_{ph}^{n+1}\|_{L^2(\Omega_p)}^2 \\
& + \frac{1}{2\Delta t} c_0 \left(\|p_{ph}^{n+1}\|_{L^2(\Omega_p)}^2 - \|p_{ph}^n\|_{L^2(\Omega_p)}^2 + \|p_{ph}^{n+1} - p_{ph}^n\|_{L^2(\Omega_p)}^2 \right) \\
& = (\mathbf{f}_p^{n+1}, \mathbf{u}_{sh}^{n+1})_{\Omega_p} + (\mathbf{h}^{n+1}, \mathbf{u}_{fh}^{n+1})_{\Omega_f} + (g_p^{n+1}, p_{ph}^{n+1})_{\Omega_p}.
\end{aligned}$$

Thus, summing over $n = 0, \dots, N-1$ yields

$$\begin{aligned}
& \frac{1}{\Delta t} \|\underline{\sigma}_{ph}^N + \alpha p_{ph}^N \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 + \sum_{n=0}^{N-1} \|\underline{\sigma}_{fh}^{n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 + \sum_{n=0}^{N-1} \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} \|(\underline{\sigma}_{ph}^{n+1} \mathbf{n})^t\|_{L^2(e)}^2 \\
& + \sum_{n=0}^{N-1} \|(K^{-1} \nu_p)^{\frac{1}{2}} \mathbf{u}_{ph}^{n+1}\|_{L^2(\Omega_p)}^2 + \frac{1}{\Delta t} c_0 \|p_{ph}^N\|_{L^2(\Omega_p)}^2 \\
& \leq C \left(\sum_{n=0}^{N-1} \left(\|\mathbf{f}_p^{n+1}\|_{L^2(\Omega_p)} \|\mathbf{u}_{sh}^{n+1}\|_{L^2(\Omega_p)} + \|\mathbf{h}^{n+1}\|_{L^2(\Omega_f)} \|\mathbf{u}_{fh}^{n+1}\|_{L^2(\Omega_f)} \right. \right. \\
& \quad \left. \left. + \|g_p^{n+1}\|_{L^2(\Omega_p)} \|p_{ph}^{n+1}\|_{L^2(\Omega_p)} \right) + \frac{1}{\Delta t} \|\underline{\sigma}_{ph}^0 + \alpha p_{ph}^0 \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 + \frac{1}{\Delta t} c_0 \|p_{ph}^0\|_{L^2(\Omega_p)}^2 \right).
\end{aligned} \tag{4.35}$$

We can deduce from (4.9), (4.12) and (3.15c) that

$$\|p_{ph}^{n+1}\|_{L^2(\Omega_p)} \leq C \|p_{ph}^{n+1}\|_Z \leq C \sup_{\mathbf{0} \neq \mathbf{v}_p \in \mathbf{U}_h^p} \frac{b_p(p_{ph}^{n+1}, \mathbf{v}_p)}{\|\mathbf{v}_p\|_{L^2(\Omega_p)}} \leq C \|\mathbf{u}_{ph}^{n+1}\|_{L^2(\Omega_p)}, \quad \forall 0 \leq n \leq N-1. \tag{4.36}$$

It follows from (4.7) and (3.15a) that

$$\begin{aligned}
\|(\mathbf{u}_h^{n+1}, \widehat{\mathbf{u}}_h^{n+1})\|_{1,h} & \leq C \sup_{\mathbf{0} \neq \underline{\tau} \in \Sigma_h: \underline{\tau}_p \mathbf{n}|_{\mathcal{F}_h^\Gamma} = \mathbf{0}} \frac{a_f(\mathbf{u}_{fh}^{n+1}, \underline{\tau}_f) + a_p(\mathbf{u}_{sh}^{n+1}, \underline{\tau}_p) - \sum_{e \in \mathcal{F}_h^0} (\widehat{\mathbf{u}}_h^{n+1}, [\underline{\tau} \mathbf{n}])_e}{\|\underline{\tau}\|_{L^2(\Omega)}} \\
& \leq C (\|\underline{\sigma}_h^{n+1}\|_{\mathcal{A}_f(\Omega_f)} + \|d_t(\underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}).
\end{aligned} \tag{4.37}$$

The following estimate is obtained by combining (4.35)–(4.37) and (4.3)–(4.4):

$$\begin{aligned}
& \|\underline{\sigma}_{ph}^N + \alpha p_{ph}^N \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 + \sum_{n=0}^{N-1} \Delta t \|\underline{\sigma}_{fh}^{n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 + \sum_{n=0}^{N-1} \delta^{-1} \Delta t \sum_{e \in \mathcal{F}_h^\Gamma} \|(\underline{\sigma}_{ph}^{n+1} \mathbf{n})^t\|_{L^2(e)}^2 \\
& + \sum_{i=0}^{N-1} \Delta t \|\mathbf{u}_{ph}^{i+1}\|_{L^2(\Omega_p)}^2 + \sum_{i=0}^{N-1} \Delta t \|p_{ph}^{i+1}\|_{L^2(\Omega_p)}^2 + \sum_{i=0}^{N-1} \Delta t \|(\mathbf{u}_h^{i+1}, \widehat{\mathbf{u}}_h^{i+1})\|_{1,h}^2 \\
& + \sum_{i=0}^{N-1} \Delta t \|\mathbf{u}_h^{i+1}\|_{L^2(\Omega)}^2 + c_0 \|p_{ph}^N\|_{L^2(\Omega_p)}^2 \\
& \leq C \left(\sum_{i=0}^{N-1} \Delta t \left(\|\mathbf{f}_p^{i+1}\|_{L^2(\Omega_p)}^2 + \|\mathbf{h}^{i+1}\|_{L^2(\Omega_f)}^2 + \|g_p^{i+1}\|_{L^2(\Omega_p)}^2 \right) + \|\underline{\sigma}_{ph}^0 + \alpha p_{ph}^0 \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 \right)
\end{aligned} \tag{4.38}$$

$$+ c_0 \|p_{ph}^0\|_{L^2(\Omega_p)}^2 + \sum_{n=0}^{N-1} \Delta t \|d_t(\underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2.$$

It remains to estimate $\sum_{n=0}^{N-1} \Delta t \|d_t(\underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2$. To this end, we take the discrete time derivative of (3.15b), (3.15c), (3.15e), and (3.15f), where for $n = 0$ we utilize (4.19b), (4.19c), (4.19e) and (4.19f). We then set $\underline{\tau}_p = d_t \underline{\sigma}_{ph}^{n+1}$, $\underline{\tau}_f = d_t \underline{\sigma}_{fh}^{n+1}$, $\mathbf{v}_s = \mathbf{u}_{sh}^{n+1}$, $q_p = d_t p_{ph}^{n+1}$, $\mathbf{v}_f = \mathbf{u}_{fh}^{n+1}$, $\widehat{\mathbf{v}} = \widehat{\mathbf{u}}_h^{n+1}$, $\mathbf{v}_p = \mathbf{u}_{ph}^{n+1}$, and $\mu^{n+1} = \lambda_h^{n+1}$, sum up the equations, and sum over all n , resulting in

$$\begin{aligned} & \sum_{n=0}^{N-1} \|d_t(\underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 + \delta^{-1} \frac{1}{\Delta t} (\|\underline{\sigma}_{ph}^N\|_{L^2(\Gamma)}^2 - \|\underline{\sigma}_{ph}^0\|_{L^2(\Gamma)}^2 + \sum_{n=0}^{N-1} \|\underline{\sigma}_{ph}^{n+1} - \underline{\sigma}_{ph}^n\|_{L^2(\Gamma)}^2) \\ & + \frac{1}{2\Delta t} \left(\|\underline{\sigma}_{fh}^N\|_{\mathcal{A}_f(\Omega_f)}^2 - \|\underline{\sigma}_{fh}^0\|_{\mathcal{A}_f(\Omega_f)}^2 + \sum_{n=0}^{N-1} \|\underline{\sigma}_{fh}^{n+1} - \underline{\sigma}_{fh}^n\|_{\mathcal{A}_f(\Omega_f)}^2 \right) + \sum_{n=0}^{N-1} c_0 \|d_t p_{ph}^{n+1}\|_{L^2(\Omega_p)}^2 \\ & + \frac{1}{2\Delta t} \left(\|(K^{-1}\nu_p)^{\frac{1}{2}} \mathbf{u}_{ph}^N\|_{L^2(\Omega_p)}^2 - \|(K^{-1}\nu_p)^{\frac{1}{2}} \mathbf{u}_{ph}^0\|_{L^2(\Omega_p)}^2 + \sum_{n=0}^{N-1} \|(K^{-1}\nu_p)^{\frac{1}{2}} (\mathbf{u}_{ph}^{n+1} - \mathbf{u}_{ph}^n)\|_{L^2(\Omega_p)}^2 \right) \\ & = \sum_{n=0}^{N-1} (d_t \mathbf{f}_p^{n+1}, \mathbf{u}_{sh}^{n+1})_{\Omega_p} + \sum_{n=0}^{N-1} (d_t \mathbf{h}^{n+1}, \mathbf{u}_{fh}^{n+1})_{\Omega_f} + \frac{1}{\Delta t} (g_p^N, p_{ph}^N)_{\Omega_p} - \sum_{n=1}^{N-1} (d_t g_p^{n+1}, p_{ph}^n)_{\Omega_p} - \frac{1}{\Delta t} (g_p^1, p_{ph}^0)_{\Omega_p} \\ & \leq \sum_{n=0}^{N-1} \|d_t \mathbf{f}_p^{n+1}\|_{L^2(\Omega_p)} \|\mathbf{u}_{sh}^{n+1}\|_{L^2(\Omega_p)} + \sum_{n=0}^{N-1} \|d_t \mathbf{h}^{n+1}\|_{L^2(\Omega_f)} \|\mathbf{u}_{fh}^{n+1}\|_{L^2(\Omega_f)} \\ & + \frac{1}{\Delta t} \|g_p^N\|_{L^2(\Omega_p)} \|p_{ph}^N\|_{L^2(\Omega_p)} + \sum_{n=0}^{N-1} \|d_t g_p^{n+1}\|_{L^2(\Omega_p)} \|p_{ph}^n\|_{L^2(\Omega_p)} + \frac{1}{\Delta t} \|g_p^1\|_{L^2(\Omega_p)} \|p_{ph}^0\|_{L^2(\Omega_p)}, \end{aligned} \quad (4.39)$$

where summation by parts is used for $\sum_{n=0}^{N-1} (g_p^{n+1}, d_t p_{ph}^{n+1})_{\Omega_p}$ in the first equality. We infer from (4.33) that

$$\begin{aligned} \|d_t \mathbf{f}_p^{n+1}\|_{L^2(\Omega_p)}^2 & \leq \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \|\partial_t \mathbf{f}_p\|_{L^2(\Omega_p)}^2 dt, \\ \|d_t \mathbf{h}^{n+1}\|_{L^2(\Omega_f)}^2 & \leq \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \|\partial_t \mathbf{h}\|_{L^2(\Omega_f)}^2 dt, \\ \|d_t g_p^{n+1}\|_{L^2(\Omega_p)}^2 & \leq \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \|\partial_t g_p\|_{L^2(\Omega_p)}^2 dt. \end{aligned}$$

Multiplying (4.39) by Δt and applying the bound (4.38) to the first two terms, and (4.36) to the pressure terms $\|p_{ph}^N\|_{L^2(\Omega_p)}$ and $\sum_{n=0}^{N-1} \Delta t \|p_{ph}^n\|_{L^2(\Omega_p)}$, we infer

$$\begin{aligned} & \sum_{n=0}^{N-1} \Delta t \|d_t(\underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 + \|\underline{\sigma}_{fh}^N\|_{\mathcal{A}_f(\Omega_f)}^2 + \sum_{n=0}^{N-1} c_0 \Delta t \|d_t p_{ph}^{n+1}\|_{L^2(\Omega_p)}^2 \\ & + \|\mathbf{u}_{ph}^N\|_{L^2(\Omega_p)}^2 \leq C \text{Res}_1. \end{aligned} \quad (4.40)$$

An application of (4.11) and (3.15d) gives for $n = 0, \dots, N-1$

$$\begin{aligned} \|(\mathbf{u}_{ph}^{n+1}, \lambda_h^{n+1})\|_{\text{div}_p} & \leq C \sup_{0 \neq q_p \in Q_h^p} \frac{b_p(q_p, \mathbf{u}_{ph}^{n+1}) + \sum_{e \in \mathcal{F}_h^\Gamma} (\lambda_h^{n+1}, q_p)}{\|q_p\|_{L^2(\Omega_p)}} \\ & \leq C \left(c_0 \|d_t p_{ph}^{n+1}\|_{L^2(\Omega_p)} + \alpha \|d_t(\underline{\sigma}_{ph}^{n+1} + \alpha p_{ph}^{n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)} + \|g_p^{n+1}\|_{L^2(\Omega_p)} \right). \end{aligned} \quad (4.41)$$

Moreover, it follows from the inf-sup condition (4.17) and (3.15b) that for $n = 0, \dots, N-1$

$$\begin{aligned} \|\underline{\sigma}_{ph}^{n+1}\|_{\text{div},h,p} + \|\underline{\sigma}_{fh}^{n+1}\|_{\text{div},h,f} &\leq C \left(\sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{U}_h^p} \frac{a_p(\mathbf{v}, \underline{\sigma}_{ph}^{n+1})}{\|\mathbf{v}\|_{L^2(\Omega_p)}} + \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{U}_h^f} \frac{a_f(\mathbf{v}, \underline{\sigma}_{fh}^{n+1})}{\|\mathbf{v}\|_{L^2(\Omega_f)}} \right) \\ &\leq C \left(\|\mathbf{f}_p^{n+1}\|_{L^2(\Omega_p)} + \|\mathbf{h}^{n+1}\|_{L^2(\Omega_f)} \right). \end{aligned} \quad (4.42)$$

Finally, the triangle inequality gives the estimate

$$\|\underline{\sigma}_{ph}^{n+1}\|_{\mathcal{A}_p(\Omega_p)} \leq \|\underline{\sigma}_{ph}^{n+1}\|_{\mathcal{A}_p(\Omega_p)} + \alpha p_{ph}^{n+1} \underline{I} \|p_{ph}^{n+1}\|_{L^2(\Omega_p)} + \|p_{ph}^{n+1}\|_{L^2(\Omega_p)} \quad \forall 0 \leq n \leq N-1. \quad (4.43)$$

The stability estimate (4.34) follows from a combination of (4.38), (4.40)–(4.43), and (4.15)–(4.16). $\square$

Remark 4.1. *The stability constant in Theorem 4.1 is independent of c_0 and λ_p . Thus, the stability bound remains uniform in the limits of vanishing storativity, $c_0 \rightarrow 0$, and nearly incompressible elasticity, $\lambda_p \rightarrow \infty$, with the remaining physical coefficients fixed. The constant may depend on ν , μ_p , ν_p , and the spectral bounds of the permeability tensor K .*

5 Error analysis

This section establishes the convergence estimates for the fully discrete scheme, explicitly tracking their dependence on the physical parameters λ_p and c_0 . The main result is given in Theorem 5.1.

5.1 Projection operators

We construct a projection operator for $\underline{\Sigma}_h$ that is needed in the analysis. We follow the approach from [22, Section 5]. We first consider an intermediate matrix space without the symmetry constraint:

$$\tilde{\Sigma}_h := \{\underline{w}|_\tau \in \underline{P}_k(\tau), \forall \tau \in \mathcal{T}_h\}.$$

Lemma 5.1. *Any function $\underline{w} \in \tilde{\Sigma}_h$ is uniquely determined by the following degrees of freedom on each element $\tau \in \mathcal{T}_h$:*

$$\ell_\tau(\underline{w}) := (\underline{w}, \underline{q})_\tau \quad \forall \underline{q} \in \underline{P}_{k-1}(\tau), \quad (5.1)$$

$$\ell_{pr}(\underline{w}) := (\underline{w}\mathbf{n}, \underline{\chi})_e \quad \forall \underline{\chi} \in \mathbf{P}_k(e), e \in (\mathcal{F}_{pr}^{0,f} \cup \mathcal{F}_{pr}^{0,p} \cup \mathcal{F}_h^\Gamma) \cap \partial\tau, \quad (5.2)$$

$$\ell_{dl}(\underline{w}) := ((\underline{w}\mathbf{n})^t, \hat{\mathbf{v}}^t)_e \quad \forall \hat{\mathbf{v}} \in \mathbf{P}_k(e), e \in \mathcal{F}_{dl} \cap \partial\tau. \quad (5.3)$$

Proof. The proof in 2D can be found in [54, Lemma 2.1]. For the three-dimensional case, see [39, Lemma 4.2]. $\square$

Let $\tilde{\Pi}_\Sigma : \underline{H}(\text{div}; \Omega) \cap \underline{L}^p(\Omega) \rightarrow \tilde{\Sigma}_h$, $p > 2$, be the projection operator onto $\tilde{\Sigma}_h$ based on the above degrees of freedom, i.e., $\ell(\underline{w} - \tilde{\Pi}_\Sigma \underline{w}) = 0$, where $\ell \in \{\ell_\tau, \ell_{pr}, \ell_{dl}\}$.

Lemma 5.2. *The following approximation property holds:*

$$\|\tilde{\Pi}_\Sigma \underline{w} - \underline{w}\|_{L^2(\tau)} \leq C h_\tau^{k+1} |\underline{w}|_{H^{k+1}(\tau)} \quad \forall \underline{w} \in \underline{H}^{k+1}(\tau), \tau \in \mathcal{T}_h.$$

Proof. The proof in 2D can be found in [22, Lemma 5.3]. The proof in 3D is similar. $\square$

Note that, even if $\underline{G}$ is symmetric, $\tilde{\Pi}_\Sigma \underline{G}$ may be non-symmetric. In the following we construct a projection operator in the space of symmetric matrices $\underline{\Sigma}_h$ by adding a correction to $\tilde{\Pi}_\Sigma$.

Lemma 5.3. *For $p > 2$, there exists a projection operator $\Pi_\Sigma : \underline{H}(\mathbb{S}, \text{div}; \Omega) \cap \underline{L}^p(\mathbb{S}, \Omega) \rightarrow \underline{\Sigma}_h$ such that for any $\underline{G} \in \underline{H}(\mathbb{S}, \text{div}; \Omega) \cap \underline{L}^p(\mathbb{S}, \Omega)$,*

$$a_i(\underline{v}, \underline{G} - \Pi_\Sigma \underline{G}) = 0 \quad \forall \underline{v} \in \underline{U}_h^i, \quad (5.4)$$

$$(\llbracket (\underline{G} - \Pi_\Sigma \underline{G}) \mathbf{n} \rrbracket, \hat{\underline{v}})_e = 0 \quad \forall \hat{\underline{v}} \in \underline{\Lambda}_h \quad \forall e \in \mathcal{F}_h^0, \quad (5.5)$$

$$((\underline{G} - \Pi_\Sigma \underline{G}) \mathbf{n}, \underline{\mu})_e = 0 \quad \forall \underline{\mu} \in \underline{P}_k(e) \quad \forall e \in \mathcal{F}_{pr} \cup \mathcal{F}_h^\Gamma, \quad (5.6)$$

and on each macroelement $M \in \mathcal{T}_{pr}$,

$$\|\underline{G} - \Pi_\Sigma \underline{G}\|_{L^2(M)} \leq Ch_M^{k+1} |\underline{G}|_{H^{k+1}(M)} \quad \forall \underline{G} \in \underline{H}^{k+1}(\mathbb{S}, M). \quad (5.7)$$

Proof. The proof follows the argument in the proof of [22, Theorem 4.1], which is given in the 2D case. We will define $\Pi_\Sigma \underline{G} := \tilde{\Pi}_\Sigma \underline{G} + \underline{\theta}_h$ such that $\Pi_\Sigma \underline{G} \in \underline{\Sigma}_h$. We take $\underline{\theta}_h := \text{curl}(\underline{\rho}_h)$ when $d = 2$ and $\underline{\theta}_h := \text{curl}(\underline{\rho}_h)$ when $d = 3$, where $\underline{\rho}_h \in \underline{H}_0^1(\Omega)$ and $\underline{\rho}_h \in \underline{H}_0^1(\Omega)$ are suitable discrete vector and matrix functions in 2D and 3D, respectively. Here $\text{curl}(\underline{\psi}) := \begin{pmatrix} -\frac{\partial \psi_1}{\partial y}, \frac{\partial \psi_1}{\partial x} \\ -\frac{\partial \psi_2}{\partial y}, \frac{\partial \psi_2}{\partial x} \end{pmatrix}$ for $d = 2$. The construction of $\underline{\theta}_h$ is given below. We consider the following spaces over each macro-element $M \in \mathcal{T}_{pr}$:

$$\underline{U}_M := \{\underline{v} \in \underline{H}^1(M), M \in \mathcal{T}_{pr} : \underline{v}|_\tau \in \underline{P}_{k+1}(\tau), \tau \subset M, \underline{v}|_{\partial M} = \mathbf{0}\},$$

$$\underline{Q}_M := \left\{ q \in L^2(M), M \in \mathcal{T}_{pr} : q|_\tau \in P_k(\tau), \tau \subset M, \int_M q = 0 \right\}.$$

Given $\Psi_h \in L_0^2(M)$, we consider the local Stokes problem for $(\underline{\psi}_h^*, p_h^*) \in \underline{U}_M \times \underline{Q}_M$ such that

$$(\nabla \underline{\psi}_h^*, \nabla \underline{v}_h)_M + (p_h^*, \nabla \cdot \underline{v}_h)_M = 0 \quad \forall \underline{v}_h \in \underline{U}_M, \quad (5.8)$$

$$(q_h, \nabla \cdot \underline{\psi}_h^*)_M = (\Psi_h, q_h)_M \quad \forall q_h \in \underline{Q}_M. \quad (5.9)$$

The unique solvability of (5.8)–(5.9) results from the inf-sup condition

$$\inf_{0 \neq q_h \in \underline{Q}_M} \sup_{0 \neq \underline{v}_h \in \underline{U}_M} \frac{(q_h, \nabla \cdot \underline{v}_h)_M}{\|q_h\|_{L^2(M)} \|\underline{v}_h\|_{H^1(M)}} \geq C.$$

For $d = 2$, this is proved in [48, 33] for $k \geq 3$ and in [42] for $k = 1, 2$. For $d = 3$, it is proved in [53, 41]. Owing to the inf-sup condition, the following stability bound holds:

$$\|\underline{\psi}_h^*\|_{H^1(M)} \leq C \|\Psi_h\|_{L^2(M)}. \quad (5.10)$$

Define $\text{as}_2(\underline{F}) := \underline{F}_{12} - \underline{F}_{21}$ for $\underline{F} \in \mathbb{R}^{2 \times 2}$ and $\text{as}_3(\underline{F}) := (\underline{F}_{32} - \underline{F}_{23}, \underline{F}_{31} - \underline{F}_{13}, \underline{F}_{21} - \underline{F}_{12})^\top$ for $\underline{F} \in \mathbb{R}^{3 \times 3}$, $\underline{F} := (\underline{F}_{ij})$, $i, j = 1, 2, 3$. Let $\mathbb{E} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ be the algebraic operator defined as $\mathbb{E} \underline{\rho}_h := \text{tr}(\underline{\rho}_h) \underline{I} - \underline{\rho}_h^\top$. A direct calculation shows that

$$\text{as}_2(\text{curl}(\underline{\rho}_h)) = -\nabla \cdot \underline{\rho}_h \quad \text{and} \quad \text{as}_3(\text{curl}(\underline{\rho}_h)) = -\nabla \cdot \mathbb{E} \underline{\rho}_h. \quad (5.11)$$

For $d = 2$, we define $\underline{\rho}_h$ restricted to each macroelement M as the first component of the solution to (5.8)–(5.9) with $\Psi_h := \text{as}_2(\tilde{\Pi}_\Sigma \underline{G})|_M$. We note that $\text{as}_2(\tilde{\Pi}_\Sigma \underline{G})|_M \in \underline{Q}_M$, which follows from $\ell_\tau(\underline{w} - \tilde{\Pi}_\Sigma \underline{w}) = 0$ and taking q to be a constant skew-symmetric matrix in (5.1). Therefore $\underline{\rho}_h$ satisfies

$$\nabla \cdot \underline{\rho}_h = \text{as}_2(\tilde{\Pi}_\Sigma \underline{G}).$$

Then $\underline{\theta}_h = \text{curl}(\underline{\rho}_h)$ satisfies, using (5.11),

$$\text{as}_2(\underline{\theta}_h) = \text{as}_2(\text{curl}(\underline{\rho}_h)) = -\nabla \cdot \underline{\rho}_h = -\text{as}_2(\tilde{\Pi}_\Sigma \underline{G}). \quad (5.12)$$

Invoking (5.10), we have

$$\|\underline{\theta}_h\|_{L^2(M)} \leq \|\underline{\rho}_h\|_{H^1(M)} \leq C \|\text{as}_2(\tilde{\Pi}_\Sigma \underline{G})\|_{L^2(M)}. \quad (5.13)$$

For $d = 3$, we take $\underline{\rho}_{h|M} := \mathbb{E}^{-1} \underline{\eta}_{h|M} = \underline{\eta}_h^\top - \frac{1}{2} \text{tr}(\underline{\eta}_h) \underline{I}$, where each row of $\underline{\eta}_h$ is the solution obtained from solving (5.8)–(5.9) with Ψ_h being each element of $\text{as}_3(\tilde{\Pi}_\Sigma \underline{G})$ restricted to M . This implies that

$$\nabla \cdot \mathbb{E} \underline{\rho}_h = \text{as}_3(\tilde{\Pi}_\Sigma \underline{G}).$$

Taking $\underline{\theta}_h = \text{curl}(\underline{\rho}_h)$, we have, using (5.11),

$$\text{as}_3(\underline{\theta}_h) = \text{as}_3(\text{curl}(\underline{\rho}_h)) = -\nabla \cdot \mathbb{E} \underline{\rho}_h = -\text{as}_3(\tilde{\Pi}_\Sigma \underline{G}). \quad (5.14)$$

In view of (5.10), we have

$$\|\underline{\theta}_h\|_{L^2(M)} \leq \|\underline{\rho}_h\|_{H^1(M)} \leq C \|\text{as}_3(\tilde{\Pi}_\Sigma \underline{G})\|_{L^2(M)}. \quad (5.15)$$

The identities (5.12) and (5.14) imply that $\underline{\Pi}_\Sigma \underline{G} := \tilde{\Pi}_\Sigma \underline{G} + \underline{\theta}_h$ is symmetric, hence $\underline{\Pi}_\Sigma \underline{G} \in \underline{\Sigma}_h$.

We next show (5.4). In view of the definition of $\tilde{\Pi}_\Sigma$ given in Lemma 5.1 and the definition of $a_i(\cdot, \cdot)$ (cf. (3.11)), we have

$$\begin{aligned} a_i(\underline{v}, \underline{G} - \tilde{\Pi}_\Sigma \underline{G}) &= - \sum_{\tau \in \mathcal{T}_h^i} (\varepsilon(\underline{v}), \underline{G} - \tilde{\Pi}_\Sigma \underline{G})_\tau + \sum_{\tau \in \mathcal{T}_h^i} (\underline{v}, (\underline{G} - \tilde{\Pi}_\Sigma \underline{G}) \underline{n})_{\partial\tau \cap (\mathcal{F}_{pr} \cup \mathcal{F}_h^\Gamma)} \\ &\quad + \sum_{\tau \in \mathcal{T}_h^i} (\underline{v}^t, ((\underline{G} - \tilde{\Pi}_\Sigma \underline{G}) \underline{n})^t)_{\partial\tau \cap \mathcal{F}_{dl}} = 0, \quad \forall \underline{v} \in \underline{U}_h^i. \end{aligned}$$

The definition of $a_i(\cdot, \cdot)$ given in (3.9) yields

$$a_i(\underline{v}, \underline{\theta}_h) = \sum_{\tau \in \mathcal{T}_h^i} (\underline{v}, \nabla \cdot \underline{\theta}_h)_\tau - \sum_{e \in \mathcal{F}_{dl}^i} (\underline{v} \cdot \underline{n}, [\underline{\theta}_h \underline{n}])_e = 0, \quad \forall \underline{v} \in \underline{U}_h^i, \quad (5.16)$$

where in the last step we use $\nabla \cdot \underline{\theta}_h = \mathbf{0}$ and $\underline{\theta}_h \in \underline{H}(\text{div}; \Omega)$. Then (5.4) follows from

$$a_i(\underline{v}, \underline{G} - \underline{\Pi}_\Sigma \underline{G}) = a_i(\underline{v}, \underline{G} - \tilde{\Pi}_\Sigma \underline{G}) - a_i(\underline{v}, \underline{\theta}_h) = 0.$$

Next, property (5.5) follows from the fact that, due to (5.2) and (5.3), it is satisfied by $\tilde{\Pi}_\Sigma$, along with $[\underline{\theta}_h \underline{n}]_e = 0$ on each $e \in \mathcal{F}_h^0$, since $\underline{\theta}_h \in \underline{H}(\text{div}; \Omega)$.

For property (5.6), we first note that, due to (5.2), it holds for $\tilde{\Pi}_\Sigma$. On the other hand, since on each macroelement M , $\underline{\theta}_h = \text{curl}(\underline{\rho}_h)$ when $d = 2$ and $\underline{\theta}_h = \text{curl}(\underline{\rho}_h)$ when $d = 3$, where $\underline{\rho}_h \in \underline{H}_0^1(M)$ and $\underline{\rho}_h \in \underline{H}_0^1(M)$, it follows from [4, Lemma 4.4] that $\underline{\theta}_h \underline{n} = \mathbf{0}$ on ∂M , which implies that $\underline{\theta}_h \underline{n}|_e = \mathbf{0}$ for $e \in \mathcal{F}_{pr} \cup \mathcal{F}_h^\Gamma$. Therefore (5.6) holds.

It remains to establish (5.7). The triangle inequality, (5.13), (5.15) and Lemma 5.2 yield for any $M \in \mathcal{T}_{pr}$ and for all $\underline{G} \in \underline{H}^{k+1}(\mathbb{S}, M)$

$$\begin{aligned} \|\underline{G} - \underline{\Pi}_\Sigma \underline{G}\|_{L^2(M)} &= \|\underline{G} - \tilde{\Pi}_\Sigma \underline{G} - \underline{\theta}_h\|_{L^2(M)} \leq \|\underline{G} - \tilde{\Pi}_\Sigma \underline{G}\|_{L^2(M)} + \|\underline{\theta}_h\|_{L^2(M)} \\ &\leq \|\underline{G} - \tilde{\Pi}_\Sigma \underline{G}\|_{L^2(M)} + C \|\text{as}_d(\tilde{\Pi}_\Sigma \underline{G})\|_{L^2(M)} = \|\underline{G} - \tilde{\Pi}_\Sigma \underline{G}\|_{L^2(M)} + C \|\text{as}_d(\tilde{\Pi}_\Sigma \underline{G} - \underline{G})\|_{L^2(M)} \\ &\leq C \|\underline{G} - \tilde{\Pi}_\Sigma \underline{G}\|_{L^2(M)} \leq Ch_M^{k+1} |\underline{G}|_{H^{k+1}(M)}. \end{aligned}$$

Therefore, the proof is completed. $\square$

We are now in position to establish the inf-sup condition (4.7).

Proof of Lemma 4.2. Using the operator $\tilde{\Pi}_\Sigma$ and following the proof of [22, Lemma 5.6], we can show that for all $(\mathbf{v}, \hat{\mathbf{v}}) \in \mathbf{U}_h \times \mathbf{\Lambda}_h$

$$\|(\mathbf{v}, \hat{\mathbf{v}})\|_{1,h} \leq C \sup_{\mathbf{0} \neq \underline{\tau} \in \tilde{\Sigma}_h : \underline{\tau}_p \mathbf{n}_{|\mathcal{F}_h^\Gamma} = \mathbf{0}} \frac{a_p(\mathbf{v}_p, \underline{\tau}_p) + a_f(\mathbf{v}_f, \underline{\tau}_f) - \sum_{e \in \mathcal{F}_h^0} (\hat{\mathbf{v}}, \llbracket \underline{\tau} \mathbf{n} \rrbracket)_e}{\|\underline{\tau}\|_{L^2(\Omega)}}. \quad (5.17)$$

We note that in our case we have the restriction $\underline{\tau}_p \mathbf{n}_{|\mathcal{F}_h^\Gamma} = \mathbf{0}$. This is handled by taking the moments (5.2) on $\mathcal{F}_h^\Gamma \cap \partial\tau$, $\tau \in \mathcal{T}_h^p$, to be equal to zero, which is possible since the norm $\|(\mathbf{v}, \hat{\mathbf{v}})\|_{1,h}$ does not involve the term $\|\mathbf{v}_p - \hat{\mathbf{v}}\|_{L^2(e)}$ for $e \in \mathcal{F}_h^\Gamma$, cf. (4.1). The inf-sup property (5.17) implies that, given $(\mathbf{v}, \hat{\mathbf{v}}) \in \mathbf{U}_h \times \mathbf{\Lambda}_h$, there exists $\tilde{\underline{\tau}} \in \tilde{\Sigma}_h : \tilde{\underline{\tau}}_p \mathbf{n}_{|\mathcal{F}_h^\Gamma} = \mathbf{0}$ such that

$$a_p(\mathbf{v}_p, \tilde{\underline{\tau}}_p) + a_f(\mathbf{v}_f, \tilde{\underline{\tau}}_f) - \sum_{e \in \mathcal{F}_h^0} (\hat{\mathbf{v}}, \llbracket \tilde{\underline{\tau}} \mathbf{n} \rrbracket)_e \geq C \|(\mathbf{v}, \hat{\mathbf{v}})\|_{1,h}^2 \text{ and } \|\tilde{\underline{\tau}}\|_{L^2(\Omega)} \leq C \|(\mathbf{v}, \hat{\mathbf{v}})\|_{1,h}. \quad (5.18)$$

Here $\tilde{\underline{\tau}}$ restricted to each element $\tau \in \mathcal{T}_h$ satisfies $(\tilde{\underline{\tau}}, \underline{w})_\tau = (\varepsilon(\mathbf{v}), \underline{w})_\tau$ for any $\underline{w} \in \underline{P}_k(\tau)$. Then using the discrete Korn's inequality (cf. [10, Theorem 4.2]), we can show (5.18) holds for such a $\tilde{\underline{\tau}}$. Note that the condition $(\tilde{\underline{\tau}}, \underline{w})_\tau = (\varepsilon(\mathbf{v}), \underline{w})_\tau$ for any $\underline{w} \in \underline{P}_k(\tau)$ is needed to ensure that $\text{as}_d(\tilde{\underline{\tau}}) \in Q_M$ as required for $\tilde{\Pi}_\Sigma \underline{G}$ in the proof of Lemma 5.3. Since $\tilde{\underline{\tau}}$ may be non-symmetric, we symmetrize it by defining $\underline{\tau} := \tilde{\underline{\tau}} + \underline{\theta} \in \Sigma_h$, where $\underline{\theta} \in \tilde{\Sigma}_h$ is constructed in a way similar to the proof of Lemma 5.3. We have that $a_i(\mathbf{v}, \underline{\theta}) = 0$, $i = f, p$, cf. (5.16). In addition, as shown in the proof of Lemma 5.3, $\underline{\theta} \mathbf{n}_{|\mathcal{F}_h^\Gamma} = \mathbf{0}$. Furthermore, (5.13) and (5.15) imply that $\|\underline{\theta}\|_{L^2(\Omega)} \leq C \|\tilde{\underline{\tau}}\|_{L^2(\Omega)}$. Therefore, (5.18) implies that $\underline{\tau} = \tilde{\underline{\tau}} + \underline{\theta} \in \Sigma_h : \underline{\tau} \mathbf{n}_{|\mathcal{F}_h^\Gamma} = \mathbf{0}$ satisfies

$$a_p(\mathbf{v}_p, \underline{\tau}_p) + a_f(\mathbf{v}_f, \underline{\tau}_f) - \sum_{e \in \mathcal{F}_h^0} (\hat{\mathbf{v}}, \llbracket \underline{\tau} \mathbf{n} \rrbracket)_e \geq C \|(\mathbf{v}, \hat{\mathbf{v}})\|_{1,h}^2 \text{ and } \|\underline{\tau}\|_{L^2(\Omega)} \leq C \|(\mathbf{v}, \hat{\mathbf{v}})\|_{1,h},$$

hence the inf-sup condition (4.7) holds. $\square$

Next, let $H_{0,\Gamma_p}^1(\Omega_p) := \{q \in H^1(\Omega_p) : v|_{\Gamma_p} = 0\}$ and define $\Pi_{Q_p} : H_{0,\Gamma_p}^1(\Omega_p) \rightarrow Q_p^p$ as follows (cf. [57])

$$\begin{aligned} (\Pi_{Q_p} p, p_k)_e &= (p, p_k)_e \quad \forall p_k \in P_k(e), e \in \mathcal{F}_{pr}^{0,f} \cup \mathcal{F}_{pr}^{0,p} \cup \mathcal{F}_h^\Gamma, \\ (\Pi_{Q_p} p, p_{k-1})_\tau &= (p, p_{k-1})_\tau \quad \forall p_{k-1} \in P_{k-1}(\tau), \tau \in \mathcal{T}_h^p. \end{aligned} \quad (5.19)$$

The following approximation property holds for any function $p \in H^{k+1}(\Omega_p)$ (cf. [57])

$$\|p - \Pi_{Q_p} p\|_{L^2(\Omega_p)} \leq Ch^{k+1} |p|_{H^{k+1}(\Omega_p)}. \quad (5.20)$$

We define the projection operator Π_U in line with the degrees of freedom specified in (3.16)–(3.17). Indeed, $\Pi_U : H^1(\Omega) \rightarrow \mathbf{U}_h$ is defined by

$$\begin{aligned} ((\Pi_U \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}, \varphi)_e &= 0 \quad \forall \varphi \in P_k(e), \forall e \in \mathcal{F}_{dl}, \\ (\Pi_U \mathbf{v} - \mathbf{v}, \phi)_\tau &= 0 \quad \forall \phi \in \mathbf{P}_{k-1}(\tau), \forall \tau \in \mathcal{T}_h. \end{aligned} \quad (5.21)$$

It follows from the definition of (3.9) that for $i = p, f$,

$$a_i(\mathbf{v} - \Pi_U \mathbf{v}, \underline{\tau}) = 0 \quad \forall \underline{\tau} \in \Sigma_h. \quad (5.22)$$

Furthermore, for any function $\mathbf{v} \in \mathbf{H}^{k+1}(\Omega)$, this projection admits the following approximation property (cf. [24, 23]):

$$\|\mathbf{v} - \mathbf{\Pi}_U \mathbf{v}\|_{L^2(\Omega)} \leq Ch^{k+1} |\mathbf{v}|_{H^{k+1}(\Omega)}. \quad (5.23)$$

We also define $\mathbf{\Pi}_\Lambda$ and $\mathbf{\Pi}_\Gamma$ as the L^2 -orthogonal projection operators onto Λ_h and Γ_h , respectively, satisfying

$$(\mathbf{\Pi}_\Lambda \widehat{\mathbf{v}} - \widehat{\mathbf{v}}, \boldsymbol{\varphi})_e = 0 \quad \forall \boldsymbol{\varphi} \in \mathbf{P}_k(e), \quad \forall e \in \mathcal{F}_h, \quad (5.24)$$

$$(\mathbf{\Pi}_\Gamma \mu - \mu, \varphi)_e = 0 \quad \forall \varphi \in P_k(e), \quad \forall e \in \mathcal{F}_h^\Gamma. \quad (5.25)$$

Finally, $\mathbf{\Pi}_{U_p}$ denotes the restriction of $\mathbf{\Pi}_U$ to the subdomain Ω_p .

5.2 Convergence error estimates

In this subsection, we show the convergence error estimates for the fully-discrete scheme. Importantly, the proposed convergence error estimate still holds for $c_0 = 0$. To begin, we recall the following discrete Gronwall inequality (cf. [43]), which will play an important role for later analysis.

Lemma 5.4 (Discrete Gronwall lemma). *Let $\tau > 0$, $B \geq 0$, and let a_n, b_n, c_n, d_n , be non-negative sequences such that $a_0 \leq B$ and*

$$a_n + \tau \sum_{l=0}^n b_l \leq \tau \sum_{l=0}^{n-1} d_l a_l + \tau \sum_{l=0}^n c_l + B, \quad n \geq 1.$$

Then,

$$a_n + \tau \sum_{l=0}^n b_l \leq \exp\left(\tau \sum_{l=0}^{n-1} d_l\right) \left(\tau \sum_{l=0}^n c_l + B\right).$$

To facilitate the subsequent analysis, we introduce the following notation for $n = 0, \dots, N$:

$$\begin{aligned} \underline{e}_{\sigma_i}^n &:= \sigma_i(t_n) - \underline{\sigma}_{ih}^n = (\sigma_i(t_n) - \mathbf{\Pi}_\Sigma \sigma_i(t_n)) + (\mathbf{\Pi}_\Sigma \sigma_i(t_n) - \underline{\sigma}_{ih}^n) := \underline{e}_{\sigma_i}^{I,n} + \underline{e}_{\sigma_i}^{h,n}, \quad i = p, f, \\ \mathbf{e}_{u_i}^n &:= \mathbf{u}_i(t_n) - \mathbf{u}_{ih}^n = (\mathbf{u}_i(t_n) - \mathbf{\Pi}_U \mathbf{u}_i(t_n)) + (\mathbf{\Pi}_U \mathbf{u}_i(t_n) - \mathbf{u}_{ih}^n) := \mathbf{e}_{u_i}^{I,n} + \mathbf{e}_{u_i}^{h,n}, \quad i = s, f, \\ \mathbf{e}_{u_p}^n &:= \mathbf{u}_p(t_n) - \mathbf{u}_{ph}^n = (\mathbf{u}_p(t_n) - \mathbf{\Pi}_{U_p} \mathbf{u}_p(t_n)) + (\mathbf{\Pi}_{U_p} \mathbf{u}_p(t_n) - \mathbf{u}_{ph}^n) := \mathbf{e}_{u_p}^{I,n} + \mathbf{e}_{u_p}^{h,n}, \\ e_p^n &:= p_p(t_n) - p_{ph}^n = (p_p(t_n) - \Pi_{Q_p} p_p(t_n)) + (\Pi_{Q_p} p_p(t_n) - p_{ph}^n) := e_p^{I,n} + e_p^{h,n}, \\ \widehat{\mathbf{e}}_u^n &:= \widehat{\mathbf{u}}(t_n) - \widehat{\mathbf{u}}_h^n = (\widehat{\mathbf{u}}(t_n) - \mathbf{\Pi}_\Lambda \widehat{\mathbf{u}}(t_n)) + (\mathbf{\Pi}_\Lambda \widehat{\mathbf{u}}(t_n) - \widehat{\mathbf{u}}_h^n) := \widehat{\mathbf{e}}_u^{I,n} + \widehat{\mathbf{e}}_u^{h,n}, \\ e_\lambda^n &:= \lambda(t_n) - \lambda_h^n = (\lambda(t_n) - \mathbf{\Pi}_\Gamma \lambda(t_n)) + (\mathbf{\Pi}_\Gamma \lambda(t_n) - \lambda_h^n) := e_\lambda^{I,n} + e_\lambda^{h,n}, \\ \mathbf{e}_\eta^n &:= \boldsymbol{\eta}(t_n) - \boldsymbol{\eta}_h^n = \boldsymbol{\eta}(t_n) - \mathbf{\Pi}_U \boldsymbol{\eta}(t_n) + \mathbf{\Pi}_U \boldsymbol{\eta}(t_n) - \boldsymbol{\eta}_h^n := \mathbf{e}_\eta^{I,n} + \mathbf{e}_\eta^{h,n}, \\ \widehat{\mathbf{e}}_\eta^n &:= \widehat{\boldsymbol{\eta}}(t_n) - \widehat{\boldsymbol{\eta}}_h^n = \widehat{\boldsymbol{\eta}}(t_n) - \mathbf{\Pi}_\Lambda \widehat{\boldsymbol{\eta}}(t_n) + \mathbf{\Pi}_\Lambda \widehat{\boldsymbol{\eta}}(t_n) - \widehat{\boldsymbol{\eta}}_h^n := \widehat{\mathbf{e}}_\eta^{I,n} + \widehat{\mathbf{e}}_\eta^{h,n}, \end{aligned}$$

where $\widehat{\boldsymbol{\eta}} := \boldsymbol{\eta}$ on $\mathcal{F}_{pr}^p$ and $\widehat{\boldsymbol{\eta}} := \boldsymbol{\eta}^t$ on $\mathcal{F}_{dl}^p$.

Recall that the discrete initial data is the elliptic projection of the true solution at the initial time satisfying (4.19), which yields the error equations

$$(\mathcal{A}_p(e_{\sigma_p}^0 + \alpha e_{pI}^0), \tau_p)_{\Omega_p} + (\mathcal{A}_f \underline{e}_{\sigma_f}^0, \tau_f)_{\Omega_f} + a_f(e_{u_f}^0, \tau_f) + a_p(e_\eta^0, \tau_p) + \sum_{e \in \mathcal{F}_h^\Gamma} (e_\lambda^0, \tau_p \mathbf{n}_p \cdot \mathbf{n}_p)_e$$

$$+ \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} ((\underline{e}_{\sigma_p}^0 \mathbf{n})^t, (\underline{\tau}_p \mathbf{n})^t)_e - \sum_{e \in \mathcal{F}_h^0 \cup \mathcal{F}_h^\Gamma} (\hat{\mathbf{e}}_u^0, [\underline{\tau}_p \mathbf{n}])_e = 0, \forall \underline{\tau} \in \underline{\Sigma}_h, \quad (5.26a)$$

$$- a_f(\mathbf{v}_f, \underline{e}_{\sigma_f}^0) - a_p(\mathbf{v}_s, \underline{e}_{\sigma_p}^0) = 0, \forall \mathbf{v} \in \mathbf{U}_h, \quad (5.26b)$$

$$(K^{-1} \nu_p \mathbf{e}_{u_p}^0, \mathbf{v}_p)_{\Omega_p} - b_p(\mathbf{e}_p^0, \mathbf{v}_p) = 0, \forall \mathbf{v}_p \in \mathbf{U}_{ph}, \quad (5.26c)$$

$$(\alpha \mathcal{A}_p(\underline{e}_{\sigma_p}^0 + \alpha \mathbf{e}_p^0 \underline{I}), q_p \underline{I})_{\Omega_p} + b_p(q_p, \mathbf{e}_{u_p}^0) + \sum_{e \in \mathcal{F}_h^\Gamma} (e_\lambda^0, q_p)_e = 0, \quad \forall q_p \in Q_{ph}, \quad (5.26d)$$

$$\sum_{e \in \mathcal{F}_h^{0,f}} ([\underline{e}_{\sigma_f}^0 \mathbf{n}], \hat{\mathbf{v}})_e + \sum_{e \in \mathcal{F}_h^{0,p}} ([\underline{e}_{\sigma_p}^0 \mathbf{n}], \hat{\mathbf{v}})_e + \sum_{e \in \mathcal{F}_h^\Gamma} (\underline{e}_{\sigma_p}^0 \mathbf{n}_p + \underline{e}_{\sigma_f}^0 \mathbf{n}_f, \hat{\mathbf{v}})_e = 0, \quad \forall \hat{\mathbf{v}} \in \mathbf{\Lambda}_h, \quad (5.26e)$$

$$\sum_{e \in \mathcal{F}_h^\Gamma} (\underline{e}_{\sigma_p}^0 \mathbf{n}_p \cdot \mathbf{n}_p + e_p^0, \mu)_e = 0 \quad \forall \mu \in \Gamma_h. \quad (5.26f)$$

The error estimate at the initial time is given in the next lemma.

Lemma 5.5 (Error estimate at the initial time $t = 0$). *Let $(\underline{\sigma}_h^0, \mathbf{u}_h^0, \mathbf{u}_{ph}^0, p_{ph}^0, \hat{\mathbf{u}}_h^0, \lambda_h^0) \in \underline{\Sigma}_h \times \mathbf{U}_h \times \mathbf{U}_h^p \times Q_h^p \times \mathbf{\Lambda}_h \times \Gamma_h$ be the solution of (4.19). Assuming sufficient smoothness of the true solution, there exists a positive constant C independent of h , c_0 , and λ_p such that*

$$\begin{aligned} & \|\underline{e}_{\sigma_f}^0\|_{L^2(\Omega_f)} + \|\mathbf{e}_{u_p}^0\|_{L^2(\Omega_p)} + \|\underline{e}_{\sigma_p}^0\|_{L^2(\Omega_p)} + \|e_p^0\|_{L^2(\Omega_p)} + \|(\mathbf{e}_\eta^{h,0}, \hat{\mathbf{e}}_\eta^{h,0})\|_{1,h,p} + \|(\mathbf{e}_{u_p}^{h,0}, e_\lambda^{h,0})\|_{\text{div}_p} \\ & + \delta^{-\frac{1}{2}} \|(\underline{e}_{\sigma_p}^{h,0} \mathbf{n}_p)^t\|_{L^2(\Gamma)} \leq Ch^{k+1}. \end{aligned} \quad (5.27)$$

Proof. The proof of the error estimate (5.27) follows the proof of Lemma 4.8, using the error equations (5.26). The details are omitted for the sake of brevity. $\square$

Subtracting (3.15a)–(3.15f) from (3.14a)–(3.14f) yields the following error equations.

Lemma 5.6 (Error equations). *The following error equations hold:*

$$\begin{aligned} & (\mathcal{A}_p(d_t \underline{e}_{\sigma_p}^{n+1} + \alpha d_t \mathbf{e}_p^{n+1} \underline{I}), \underline{\tau}_p)_{\Omega_p} + a_p(\mathbf{e}_{u_s}^{n+1}, \underline{\tau}_p) + \sum_{e \in \mathcal{F}_h^\Gamma} (e_\lambda^{n+1}, \underline{\tau}_p \mathbf{n}_p \cdot \mathbf{n}_p)_e \\ & + \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} ((\underline{e}_{\sigma_p}^{n+1} \mathbf{n})^t, (\underline{\tau}_p \mathbf{n})^t)_e + (\mathcal{A}_f \underline{e}_{\sigma_f}^{n+1}, \underline{\tau}_f)_{\Omega_f} + a_f(\mathbf{e}_{u_f}^{n+1}, \underline{\tau}_f) \\ & - \sum_{e \in \mathcal{F}_h^0} (\hat{\mathbf{e}}_u^{n+1}, [\underline{\tau} \mathbf{n}])_e = (\underline{R}_1, \underline{\tau}_p)_{\Omega_p}, \end{aligned} \quad (5.28a)$$

$$- a_p(\mathbf{v}_s, \underline{e}_{\sigma_p}^{n+1}) - a_f(\mathbf{v}_f, \underline{e}_{\sigma_f}^{n+1}) = 0, \quad (5.28b)$$

$$(K^{-1} \nu_p \mathbf{e}_{u_p}^{n+1}, \mathbf{v}_p)_{\Omega_p} - b_p(\mathbf{e}_p^{n+1}, \mathbf{v}_p) = 0, \quad (5.28c)$$

$$\begin{aligned} & (c_0 d_t \mathbf{e}_p^{n+1}, q_p)_{\Omega_p} + (\alpha d_t \mathcal{A}(\underline{e}_{\sigma_p}^{n+1} + \alpha \mathbf{e}_p^{n+1} \underline{I}), q_p \underline{I})_{\Omega_p} + b_p(q_p, \mathbf{e}_{u_p}^{n+1}) \\ & + \sum_{e \in \mathcal{F}_h^\Gamma} (e_\lambda^{n+1}, q_p)_e = (c_0 R_2, q_p)_{\Omega_p} + (\mathcal{A}_p(R_1), \alpha q_p \underline{I})_{\Omega_p}, \end{aligned} \quad (5.28d)$$

$$\sum_{e \in \mathcal{F}_h^0} ([\underline{e}_{\sigma}^{n+1} \mathbf{n}], \hat{\mathbf{v}})_e = 0, \quad (5.28e)$$

$$\sum_{e \in \mathcal{F}_h^\Gamma} (e_p^{n+1} + \underline{e}_{\sigma_p}^{n+1} \mathbf{n}_p \cdot \mathbf{n}_p, \mu)_e = 0 \quad (5.28f)$$

for $(\underline{\tau}, \mathbf{v}, \mathbf{v}_p, q_p, \hat{\mathbf{v}}, \mu) \in \underline{\Sigma}_h \times \mathbf{U}_h \times \mathbf{U}_h^p \times Q_h^p \times \boldsymbol{\Lambda}_h \times \Gamma_h$. Here R_i , $i = 1, 2$ are defined by

$$\begin{aligned} \underline{R}_1 &:= d_t(\underline{\sigma}_p(t^{n+1}) + \alpha p_p(t^{n+1})\underline{I}) - \partial_t(\underline{\sigma}_p(t^{n+1}) + \alpha p_p(t^{n+1})\underline{I}), \\ \underline{R}_2 &:= d_t p_p(t^{n+1}) - \partial_t p_p(t^{n+1}). \end{aligned}$$

Using Taylor expansion, we have

$$|\underline{R}_1| \leq C\Delta t \quad \text{and} \quad |\underline{R}_2| \leq C\Delta t. \quad (5.29)$$

Theorem 5.1 (Convergence error estimates). *Assuming that the solution is smooth enough, there exists a positive constant C independent of h , Δt , c_0 and λ_p such that*

$$\begin{aligned} &\|\underline{e}_{\sigma_p}^N\|_{L^2(\Omega_p)}^2 + \delta^{-1} \sum_{n=0}^{N-1} \Delta t \|(\underline{e}_{\sigma_p}^{n+1} \mathbf{n})^t\|_{L^2(\Gamma)}^2 + \sum_{n=0}^{N-1} \Delta t \|\underline{e}_{\sigma_f}^{n+1}\|_{L^2(\Omega_f)}^2 + \|\underline{e}_{\sigma_f}^N\|_{L^2(\Omega_f)}^2 \\ &+ \sum_{n=0}^{N-1} \Delta t \|e_u^{n+1}\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \Delta t \|e_{u_p}^{n+1}\|_{L^2(\Omega_p)}^2 + \|e_{u_p}^N\|_{L^2(\Omega_p)}^2 \\ &+ \sum_{n=0}^{N-1} \|e_p^{n+1}\|_{L^2(\Omega_p)}^2 + \|e_p^N\|_{L^2(\Omega_p)}^2 \leq CT \left((\Delta t)^2 + h^{2(k+1)} \right). \end{aligned} \quad (5.30)$$

In addition, the following discrete error superconvergence results hold:

$$\|\underline{e}_{\sigma_f}^{h,n+1}\|_{\text{div},h,f} = 0 \quad \text{and} \quad \|\underline{e}_{\sigma_p}^{h,n+1}\|_{\text{div},h,p} = 0, \quad n = 0, \dots, N-1, \quad (5.31)$$

and

$$\sum_{n=0}^{N-1} \Delta t \|(\underline{e}_{u_p}^{h,n+1}, e_\lambda^{h,n+1})\|_{\text{div}_p}^2 + \sum_{n=0}^{N-1} \Delta t \|(\underline{e}_u^{h,n+1}, \hat{\underline{e}}_u^{h,n+1})\|_{1,h}^2 \leq CT \left((\Delta t)^2 + h^{2(k+1)} \right). \quad (5.32)$$

Proof. Setting $\underline{\tau}_p = \underline{e}_{\sigma_p}^{h,n+1}$, $\underline{\tau}_f = \underline{e}_{\sigma_f}^{h,n+1}$, $\mathbf{v}_s = \mathbf{e}_{u_s}^{h,n+1}$, $\mathbf{v}_f = \mathbf{e}_{u_f}^{h,n+1}$, $\hat{\mathbf{v}} = \hat{\mathbf{e}}_u^{h,n+1}$, $\mathbf{v}_p = \mathbf{e}_{u_p}^{h,n+1}$, $q_p = e_p^{h,n+1}$ and $\mu = e_\lambda^{h,n+1}$ in (5.28a)–(5.28f) and summing up the resulting equations yields:

$$\begin{aligned} &(d_t \mathcal{A}_p(\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I}), \underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I})_{\Omega_p} + \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} ((\underline{e}_{\sigma_p}^{h,n+1} \mathbf{n})^t, (\underline{e}_{\sigma_p}^{h,n+1} \mathbf{n})^t)_e \\ &+ (\mathcal{A}_f \underline{e}_{\sigma_f}^{h,n+1}, \underline{e}_{\sigma_f}^{h,n+1})_{\Omega_f} + (K^{-1} \nu_p \mathbf{e}_{u_p}^{h,n+1}, \mathbf{e}_{u_p}^{h,n+1})_{\Omega_p} + (c_0 d_t e_p^{h,n+1}, e_p^{h,n+1})_{\Omega_p} \\ &= (\underline{R}_1, \underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I})_{\Omega_p} - (d_t \mathcal{A}_p(\underline{e}_{\sigma_p}^{I,n+1} + \alpha e_p^{I,n+1} \underline{I}), \underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I})_{\Omega_p} \\ &- (\mathcal{A}_f \underline{e}_{\sigma_f}^{I,n+1}, \underline{e}_{\sigma_f}^{h,n+1})_{\Omega_f} - (K^{-1} \nu_p \mathbf{e}_{u_p}^{I,n+1}, \mathbf{e}_{u_p}^{h,n+1})_{\Omega_p} \\ &+ (c_0 \underline{R}_2, e_p^{h,n+1})_{\Omega_p} - (c_0 d_t e_p^{I,n+1}, e_p^{h,n+1})_{\Omega_p}, \end{aligned}$$

where we also used that

$$\begin{aligned} &a_p(\mathbf{e}_{u_s}^{I,n+1}, \underline{e}_{\sigma_p}^{h,n+1}) = 0, \quad a_f(\mathbf{e}_{u_f}^{I,n+1}, \underline{e}_{\sigma_f}^{h,n+1}) = 0, \quad a_p(\mathbf{e}_{u_s}^{h,n+1}, \underline{e}_{\sigma_p}^{I,n+1}) = 0, \quad a_f(\mathbf{e}_{u_f}^{h,n+1}, \underline{e}_{\sigma_f}^{I,n+1}) = 0, \\ &(e_\lambda^{I,n+1}, \underline{e}_{\sigma_p}^{h,n+1} \mathbf{n}_p \cdot \mathbf{n}_p)_e = 0, \quad ((\underline{e}_{\sigma_p}^{I,n+1} \mathbf{n})^t, (\underline{e}_{\sigma_p}^{h,n+1} \mathbf{n})^t)_e = 0, \quad (e_\lambda^{I,n+1}, e_p^{h,n+1})_e = 0 \quad \forall e \in \mathcal{F}_h^\Gamma, \\ &(e_p^{I,n+1}, e_\lambda^{h,n+1})_e = 0, \quad (\underline{e}_{\sigma_p}^{I,n+1} \mathbf{n}_p \cdot \mathbf{n}_p, e_\lambda^{h,n+1})_e = 0 \quad \forall e \in \mathcal{F}_h^\Gamma, \\ &(\hat{\underline{e}}_u^{h,n+1}, \llbracket \underline{e}_{\sigma}^{h,n+1} \mathbf{n} \rrbracket)_e = 0, \quad (\llbracket \underline{e}_{\sigma}^{I,n+1} \mathbf{n} \rrbracket, \hat{\underline{e}}_u^{h,n+1})_e = 0 \quad \forall e \in \mathcal{F}_h^0, \end{aligned}$$

owing to (5.4), (5.5), (5.6), (5.19), (5.22), (5.24), and (5.25).

Multiplying both sides by Δt , we infer that

$$\begin{aligned}
& \frac{1}{2} \left(\|\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 - \|\underline{e}_{\sigma_p}^{h,n} + \alpha e_p^{h,n} \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 \right) + \Delta t \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} \|(\underline{e}_{\sigma_p}^{h,n+1} \mathbf{n})^t\|_{L^2(e)}^2 \\
& + \Delta t \|\underline{e}_{\sigma_f}^{h,n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 + \Delta t \|(K^{-1} \nu_p)^{\frac{1}{2}} \underline{e}_{u_p}^{h,n+1}\|_{L^2(\Omega_p)}^2 + \frac{c_0}{2} \left(\|e_p^{h,n+1}\|_{L^2(\Omega_p)}^2 - \|e_p^{h,n}\|_{L^2(\Omega_p)}^2 \right) \\
& \leq \Delta t \|\underline{R}_1\|_{\mathcal{A}_p(\Omega_p)} \|\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I}\|_{\mathcal{A}_p(\Omega_p)} \\
& + \Delta t \|d_t(\underline{e}_{\sigma_p}^{I,n+1} + \alpha e_p^{I,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)} \|\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I}\|_{\mathcal{A}_p(\Omega_p)} \\
& + \Delta t \|\underline{e}_{\sigma_f}^{I,n+1}\|_{\mathcal{A}_f(\Omega_f)} \|\underline{e}_{\sigma_f}^{h,n+1}\|_{\mathcal{A}_f(\Omega_f)} + \Delta t \|e_{u_p}^{I,n+1}\|_{L^2(\Omega_p)} \|e_{u_p}^{h,n+1}\|_{L^2(\Omega_p)} \\
& + c_0 \Delta t \|R_2\|_{L^2(\Omega_p)} \|e_p^{h,n+1}\|_{L^2(\Omega_p)} + c_0 \Delta t \|d_t e_p^{I,n+1}\|_{L^2(\Omega_p)} \|e_p^{h,n+1}\|_{L^2(\Omega_p)}.
\end{aligned}$$

Summing over $n = 0, 1, \dots, N-1$ and using Young's inequality yields

$$\begin{aligned}
& \frac{1}{2} \|\underline{e}_{\sigma_p}^{h,N} + \alpha e_p^{h,N} \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 + \sum_{n=0}^{N-1} \Delta t \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} \|(\underline{e}_{\sigma_p}^{h,n+1} \mathbf{n})^t\|_{L^2(e)}^2 \\
& + \sum_{n=0}^{N-1} \Delta t \|\underline{e}_{\sigma_f}^{h,n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 + \sum_{n=0}^{N-1} \Delta t \|e_{u_p}^{h,n+1}\|_{L^2(\Omega_p)}^2 + \frac{c_0}{2} \|e_p^{h,N}\|_{L^2(\Omega_p)}^2 \\
& \leq \sum_{n=0}^{N-1} \left(\Delta t T \|\underline{R}_1\|_{\mathcal{A}_p(\Omega_p)}^2 + \Delta t T \|d_t(\underline{e}_{\sigma_p}^{I,n+1} + \alpha e_p^{I,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 \right. \\
& + \frac{\Delta t}{2T} \|\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 + \Delta t \|\underline{e}_{\sigma_f}^{I,n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 + \frac{\Delta t}{4} \|\underline{e}_{\sigma_f}^{h,n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 \\
& + \Delta t \|e_{u_p}^{I,n+1}\|_{L^2(\Omega_p)}^2 + \frac{\Delta t}{4} \|e_{u_p}^{h,n+1}\|_{L^2(\Omega_p)}^2 + c_0 \Delta t \|R_2\|_{L^2(\Omega_p)}^2 + c_0 \Delta t \|d_t e_p^{I,n+1}\|_{L^2(\Omega_p)}^2 \Big) \\
& + \frac{1}{2} \|\underline{e}_{\sigma_p}^{h,0} + \alpha e_p^{h,0} \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 + \frac{c_0}{2} \|e_p^{h,0}\|_{L^2(\Omega_p)}^2.
\end{aligned}$$

Then, an application of the discrete Gronwall's inequality (cf. Lemma 5.4) for the term $\frac{\Delta t}{2T} \|\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2$ yields:

$$\begin{aligned}
& \frac{1}{2} \|\underline{e}_{\sigma_p}^{h,N} + \alpha e_p^{h,N} \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 + \sum_{n=0}^{N-1} \Delta t \delta^{-1} \sum_{e \in \mathcal{F}_h^\Gamma} \|(\underline{e}_{\sigma_p}^{h,n+1} \mathbf{n})^t\|_{L^2(e)}^2 \\
& + \sum_{n=0}^{N-1} \Delta t \|\underline{e}_{\sigma_f}^{h,n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 + \sum_{n=0}^{N-1} \Delta t \|(K \nu_p^{-1})^{-\frac{1}{2}} e_{u_p}^{h,n+1}\|_{L^2(\Omega_p)}^2 + \frac{c_0}{2} \|e_p^{h,N}\|_{L^2(\Omega_p)}^2 \\
& \leq \exp(1) \sum_{n=0}^{N-1} \left(T \Delta t \|\underline{R}_1\|_{\mathcal{A}_p(\Omega_p)}^2 + T \Delta t \|d_t(\underline{e}_{\sigma_p}^{I,n+1} + \alpha e_p^{I,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 + \Delta t \|\underline{e}_{\sigma_f}^{I,n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 \right. \\
& + \Delta t \|(K^{-1} \nu_p)^{\frac{1}{2}} \underline{e}_{u_p}^{I,n+1}\|_{L^2(\Omega_p)}^2 + \Delta t \|R_2\|_{L^2(\Omega_p)}^2 + c_0 \Delta t \|d_t e_p^{I,n+1}\|_{L^2(\Omega_p)}^2 \Big) \\
& + \left(\frac{1}{2} \|\underline{e}_{\sigma_p}^{h,0} + \alpha e_p^{h,0} \underline{I}\|_{\mathcal{A}_p(\Omega_p)}^2 + \frac{c_0}{2} \|e_p^{h,0}\|_{L^2(\Omega_p)}^2 \right) \\
& \leq CT((\Delta t)^2 + h^{2(k+1)}),
\end{aligned} \tag{5.33}$$

where the last bound follows from (4.32), (4.33), the initial error bounds (5.27), the time discretization error bounds (5.29), and the interpolation error estimates (5.7), (5.20) and (5.23).

Next, an application of (4.11) and (5.28d) gives, for $n = 0, \dots, N-1$,

$$\begin{aligned} \|(\mathbf{e}_{u_p}^{h,n+1}, e_\lambda^{h,n+1})\|_{\text{div}_p} &\leq C \sup_{0 \neq q_p \in Q_h^p} \frac{b_p(q_p, \mathbf{e}_{u_p}^{h,n+1}) + \sum_{e \in \mathcal{F}_h^\Gamma} (e_\lambda^{h,n+1}, q_p)_e}{\|q_p\|_{L^2(\Omega_p)}} \\ &\leq C \left(c_0 \|d_t e_p^{n+1}\|_{L^2(\Omega_p)} + \alpha \|d_t (\mathcal{A}_p(\underline{e}_{\sigma_p}^{n+1} + \alpha e_p^{n+1} \underline{I}))\|_{L^2(\Omega_p)} \right. \\ &\quad \left. + c_0 \|R_2\|_{L^2(\Omega_p)} + \|R_1\|_{\mathcal{A}_p(\Omega_p)} \right). \end{aligned} \quad (5.34)$$

The inf-sup condition (4.7) and (5.28a) yield

$$\begin{aligned} \sum_{n=0}^{N-1} \Delta t \|(\mathbf{e}_u^{h,n+1}, \hat{\mathbf{e}}_u^{h,n+1})\|_{1,h}^2 &\leq C \sum_{n=0}^{N-1} \left(\Delta t \|d_t (\underline{e}_{\sigma_p}^{n+1} + \alpha e_p^{n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 \right. \\ &\quad \left. + \Delta t \|\underline{e}_{\sigma_f}^{n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 + \Delta t \|\underline{R}_1\|_{L^2(\Omega_p)}^2 \right). \end{aligned} \quad (5.35)$$

The discrete Poincaré inequality (4.9), the inf-sup condition (4.12) and (5.28c) imply

$$\|\mathbf{e}_p^{h,n+1}\|_{L^2(\Omega_p)} \leq C \|e_p^{h,n+1}\|_Z \leq C \left(\|K^{-1} \nu_p \mathbf{e}_{u_p}^{h,n+1}\|_{L^2(\Omega_p)} + \|K^{-1} \nu_p \mathbf{e}_{u_p}^{I,n+1}\|_{L^2(\Omega_p)} \right). \quad (5.36)$$

The triangle inequality yields

$$\|\underline{e}_{\sigma_p}^{h,N}\|_{\mathcal{A}_p(\Omega_p)} \leq \|\underline{e}_{\sigma_p}^{h,N}\| + \alpha e_p^{h,N} \|\underline{I}\|_{\mathcal{A}_p(\Omega_p)} + \alpha \|\mathbf{e}_p^{h,N}\|_{L^2(\Omega_p)}. \quad (5.37)$$

On the other hand, in view of (5.28d) and (5.4), we have

$$-a_p(\mathbf{v}_s, \underline{e}_{\sigma_p}^{h,n+1}) - a_f(\mathbf{v}_f, \underline{e}_{\sigma_f}^{h,n+1}) = 0.$$

Proceeding in a similar manner to the proof of Lemma 3.3, we obtain

$$\|\underline{e}_{\sigma_f}^{h,n+1}\|_{\text{div},h,f} = 0 \quad \text{and} \quad \|\underline{e}_{\sigma_p}^{h,n+1}\|_{\text{div},h,p} = 0, \quad (5.38)$$

which, combined with (4.15) and (4.16), implies

$$\|\underline{e}_{\sigma_f}^{h,n+1}\|_{L^2(\Omega_f)} \leq C \left(\nu^{\frac{1}{2}} \|\underline{e}_{\sigma_f}^{h,n+1}\|_{\mathcal{A}_f(\Omega_f)} + \|\underline{e}_{\sigma_f}^{h,n+1}\|_{\text{div},h,f} \right) = C \nu^{\frac{1}{2}} \|\underline{e}_{\sigma_f}^{h,n+1}\|_{\mathcal{A}_f(\Omega_f)} \quad (5.39)$$

and

$$\|\underline{e}_{\sigma_p}^{h,n+1}\|_{L^2(\Omega_p)} \leq C \left(\mu_p^{\frac{1}{2}} \|\underline{e}_{\sigma_p}^{h,n+1}\|_{\mathcal{A}_f(\Omega_p)} + \|\underline{e}_{\sigma_p}^{h,n+1}\|_{\text{div},h,p} \right) = C \mu_p^{\frac{1}{2}} \|\underline{e}_{\sigma_p}^{h,n+1}\|_{\mathcal{A}_p(\Omega_p)}. \quad (5.40)$$

In addition, using (4.3) and (4.4) we have

$$\|\mathbf{e}_u^{h,n+1}\|_{L^2(\Omega)} \leq C \|(\mathbf{e}_u^{h,n+1}, \hat{\mathbf{e}}_u^{h,n+1})\|_{1,h}. \quad (5.41)$$

It remains to estimate $\sum_{n=0}^{N-1} \Delta t (\|d_t (\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 + c_0 \|d_t e_p^{n+1}\|_{L^2(\Omega_p)}^2)$, which appear on the right hand sides of (5.34) and (5.35). To this end, we take the difference of two successive time steps for (5.28b), (5.28c), (5.28e), and (5.28f), where for $n = 0$ we utilize (5.26b), (5.26c), (5.26e), and (5.26f). Testing (5.28) with $\underline{\tau}_p = \underline{e}_{\sigma_p}^{h,n+1} - \underline{e}_{\sigma_p}^{h,n}$, $\underline{\tau}_f = \underline{e}_{\sigma_f}^{h,n+1} - \underline{e}_{\sigma_f}^{h,n}$, $\mathbf{v}_s = \mathbf{e}_{u_s}^{h,n+1}$, $\mathbf{v}_f = \mathbf{e}_{u_f}^{h,n+1}$, $\hat{\mathbf{v}} = \hat{\mathbf{e}}_u^{h,n+1}$, $q_p = e_p^{h,n+1} - e_p^{h,n}$, $\mathbf{v}_p = \mathbf{e}_{u_p}^{h,n+1}$ and $\mu = e_\lambda^{h,n+1} - e_\lambda^{h,n}$ and summing up the resulting equations yields:

$$\Delta t \|d_t (\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 + \frac{1}{2} \left(\|\underline{e}_{\sigma_f}^{h,n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 - \|\underline{e}_{\sigma_f}^{h,n}\|_{\mathcal{A}_f(\Omega_f)}^2 \right)$$

$$\begin{aligned}
& + \|\underline{e}_{\sigma_f}^{h,n+1} - \underline{e}_{\sigma_f}^{h,n}\|_{\mathcal{A}_f(\Omega_f)}^2 \Big) + \frac{1}{2} \Big(\|e_{u_p}^{h,n+1}\|_{L^2(\Omega_p)}^2 - \|e_{u_p}^{h,n}\|_{L^2(\Omega_p)}^2 \Big) + c_0 \Delta t \|d_t e_p^{h,n+1}\|_{L^2(\Omega_p)}^2 \\
& \leq \Delta t \|R_1\|_{\mathcal{A}_p(\Omega_p)} \|d_t(\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)} \\
& \quad + \Delta t \|d_t(\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)} \|d_t(\underline{e}_{\sigma_p}^{I,n+1} + \alpha e_p^{I,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)} \\
& \quad + (\mathcal{A}_f \underline{e}_{\sigma_f}^{I,n+1}, \underline{e}_{\sigma_f}^{h,n+1} - \underline{e}_{\sigma_f}^{h,n})_{\Omega_f} + \|e_{u_p}^{I,n+1} - e_{u_p}^{I,n}\|_{L^2(\Omega_p)} \|e_{u_p}^{h,n+1}\|_{L^2(\Omega_p)} \\
& \quad + c_0 \Delta t \|R_2\|_{L^2(\Omega_p)} \|d_t e_p^{h,n+1}\|_{L^2(\Omega_p)} + c_0 \Delta t \|d_t e_p^{I,n+1}\|_{L^2(\Omega_p)} \|d_t e_p^{h,n+1}\|_{L^2(\Omega_p)}.
\end{aligned}$$

Summing up over all n ($n = 0, \dots, N-1$) gives

$$\begin{aligned}
& \Delta t \sum_{n=0}^{N-1} \|d_t(\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 + \frac{1}{2} \Big(\|\underline{e}_{\sigma_f}^{h,N}\|_{\mathcal{A}_f(\Omega_f)}^2 - \|\underline{e}_{\sigma_f}^{h,0}\|_{\mathcal{A}_f(\Omega_f)}^2 \\
& \quad + \sum_{n=0}^{N-1} \|\underline{e}_{\sigma_f}^{h,n+1} - \underline{e}_{\sigma_f}^{h,n}\|_{\mathcal{A}_f(\Omega_f)}^2 \Big) + \frac{1}{2} \Big(\|e_{u_p}^{h,N}\|_{L^2(\Omega_p)}^2 - \|e_{u_p}^{h,0}\|_{L^2(\Omega_p)}^2 \Big) + \sum_{n=0}^N c_0 \Delta t \|d_t e_p^{h,n+1}\|_{L^2(\Omega_p)}^2 \\
& \leq \sum_{n=0}^{N-1} 2\Delta t \|R_1\|_{\mathcal{A}_p(\Omega_p)}^2 + \frac{\Delta t}{4} \sum_{n=0}^{N-1} \|d_t(\underline{e}_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 \\
& \quad + \sum_{n=0}^{N-1} 2\Delta t \|d_t(\underline{e}_{\sigma_p}^{I,n+1} + \alpha e_p^{I,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 + \sum_{n=0}^{N-1} (\mathcal{A}_f \underline{e}_{\sigma_f}^{I,n+1}, \underline{e}_{\sigma_f}^{h,n+1} - \underline{e}_{\sigma_f}^{h,n})_{\Omega_f} \\
& \quad + \sum_{n=0}^{N-1} \frac{1}{\Delta t} \|e_{u_p}^{I,n+1} - e_{u_p}^{I,n}\|_{L^2(\Omega_p)}^2 + \sum_{n=0}^{N-1} \Delta t \|e_{u_p}^{h,n+1}\|_{L^2(\Omega_p)}^2 \\
& \quad + c_0 \Delta t \|R_2\|_{L^2(\Omega_p)}^2 + \sum_{n=0}^{N-1} \frac{c_0}{2} \Delta t \|d_t e_p^{h,n+1}\|_{L^2(\Omega_p)}^2 + \sum_{n=0}^{N-1} c_0 \Delta t \|d_t e_p^{I,n+1}\|_{L^2(\Omega_p)}^2. \tag{5.42}
\end{aligned}$$

The fourth term on the right-hand side of (5.42) can be bounded as follows by using summation by parts and the Cauchy-Schwarz inequality

$$\begin{aligned}
& \sum_{n=0}^{N-1} (\mathcal{A}_f \underline{e}_{\sigma_f}^{I,n+1}, \underline{e}_{\sigma_f}^{h,n+1} - \underline{e}_{\sigma_f}^{h,n})_{\Omega_f} \\
& = -(\mathcal{A}_f \underline{e}_{\sigma_f}^{I,1}, \underline{e}_{\sigma_f}^{h,0})_{\Omega_f} + \sum_{n=1}^{N-1} (\mathcal{A}_f (\underline{e}_{\sigma_f}^{I,n} - \underline{e}_{\sigma_f}^{I,n+1}), \underline{e}_{\sigma_f}^{h,n})_{\Omega_f} + (\mathcal{A}_f \underline{e}_{\sigma_f}^{I,N}, \underline{e}_{\sigma_f}^{h,N})_{\Omega_f} \\
& \leq \|\underline{e}_{\sigma_f}^{I,1}\|_{\mathcal{A}_f(\Omega_f)} \|\underline{e}_{\sigma_f}^{h,0}\|_{\mathcal{A}_f(\Omega_f)} + \sum_{n=1}^{N-1} \|\underline{e}_{\sigma_f}^{I,n} - \underline{e}_{\sigma_f}^{I,n+1}\|_{\mathcal{A}_f(\Omega_f)} \|\underline{e}_{\sigma_f}^{h,n}\|_{\mathcal{A}_f(\Omega_f)} \\
& \quad + \|\underline{e}_{\sigma_f}^{I,N}\|_{\mathcal{A}_f(\Omega_f)} \|\underline{e}_{\sigma_f}^{h,N}\|_{\mathcal{A}_f(\Omega_f)} \\
& \leq \|\underline{e}_{\sigma_f}^{I,1}\|_{\mathcal{A}_f(\Omega_f)}^2 + \|\underline{e}_{\sigma_f}^{h,0}\|_{\mathcal{A}_f(\Omega_f)}^2 + \sum_{n=1}^{N-1} \frac{1}{\Delta t} \|\underline{e}_{\sigma_f}^{I,n} - \underline{e}_{\sigma_f}^{I,n+1}\|_{\mathcal{A}_f(\Omega_f)}^2 \\
& \quad + \sum_{n=1}^{N-1} \Delta t \|\underline{e}_{\sigma_f}^{h,n}\|_{\mathcal{A}_f(\Omega_f)}^2 + \|\underline{e}_{\sigma_f}^{I,N}\|_{\mathcal{A}_f(\Omega_f)}^2 + \frac{1}{4} \|\underline{e}_{\sigma_f}^{h,N}\|_{\mathcal{A}_f(\Omega_f)}^2.
\end{aligned} \tag{5.43}$$

In view of (4.33), we have

$$\frac{1}{\Delta t} \|e_p^{I,n+1} - e_p^{I,n}\|_{L^2(\Omega_p)}^2 \leq C \int_{t_n}^{t_{n+1}} \|\partial_t(p - \Pi_{Q_p} p)\|_{L^2(\Omega_p)}^2,$$

$$\begin{aligned}
\frac{1}{\Delta t} \|e_{u_p}^{I,n+1} - e_{u_p}^{I,n}\|_{L^2(\Omega_p)}^2 &\leq C \int_{t_n}^{t_{n+1}} \|\partial_t(\mathbf{u}_p - \Pi_{U_p} \mathbf{u}_p)\|_{L^2(\Omega_p)}^2, \\
\frac{1}{\Delta t} \|e_{\sigma_f}^{I,n+1} - e_{\sigma_f}^{I,n}\|_{L^2(\Omega_f)}^2 &\leq C \int_{t_n}^{t_{n+1}} \|\partial_t(\underline{\sigma}_f - \Pi_{\Sigma} \underline{\sigma}_f)\|_{L^2(\Omega_f)}^2, \\
\Delta t \|d_t(e_{\sigma_p}^{I,n+1} + \alpha e_p^{I,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 &\leq C \int_{t_n}^{t_{n+1}} \|\partial_t(\sigma_p - \Pi_{\Sigma} \sigma_p + \alpha(p - \Pi_{Q_p} p) \underline{I})\|_{L^2(\Omega_p)}^2.
\end{aligned}$$

As such, combining (5.42) with (5.33), (5.43), Lemma 5.5, the time discretization error bounds (5.29), and the interpolation error estimates (5.7), (5.20) and (5.23) gives

$$\begin{aligned}
&\Delta t \sum_{n=0}^{N-1} \|d_t(e_{\sigma_p}^{h,n+1} + \alpha e_p^{h,n+1} \underline{I})\|_{\mathcal{A}_p(\Omega_p)}^2 + \frac{1}{2} \|e_{\sigma_f}^{h,N}\|_{\mathcal{A}_f(\Omega_f)}^2 + \frac{1}{2} \|e_{u_p}^{h,N}\|_{L^2(\Omega_p)}^2 \\
&+ \sum_{n=0}^N c_0 \Delta t \|d_t e_p^{h,n+1}\|_{L^2(\Omega_p)}^2 \leq CT((\Delta t)^2 + h^{2(k+1)}).
\end{aligned} \tag{5.44}$$

Finally, combining (5.33)–(5.41), (5.44), and the approximation error bounds (5.7) and (5.20), we obtain estimates (5.30)–(5.32). $\square$

Remark 5.1. The constant in the error bounds (5.30) and (5.32) are independent of c_0 and λ_p , which establishes robustness in the limits of vanishing storativity, $c_0 \rightarrow 0$, and nearly incompressible elasticity, $\lambda_p \rightarrow \infty$. The constants may depend on ν , μ_p , ν_p , the spectral bounds of K , and δ . In particular, the compliance operators introduce factors ν^{-1} and μ_p^{-1} in (5.33), whereas (5.39) and (5.40) introduce factors ν and μ_p in the corresponding squared stress estimates. The Darcy terms involve the weight $\nu_p K^{-1}$. The factor δ^{-1} also appears explicitly in the interface term and it is retained when estimating the interface interpolation error.

Remark 5.2. With the numerical solution $\mathbf{u}_{sh}^j$ and the initial data $\boldsymbol{\eta}_h^0$ and $\hat{\boldsymbol{\eta}}_h^0$, cf. (4.19), we can recover $\boldsymbol{\eta}_h^j$ and $\hat{\boldsymbol{\eta}}_h^j$, $j = 1, \dots, N$, by

$$\boldsymbol{\eta}_h^j = \boldsymbol{\eta}_h^0 + \sum_{n=1}^j \Delta t \mathbf{u}_{sh}^n, \quad \hat{\boldsymbol{\eta}}_h^j = \hat{\boldsymbol{\eta}}_h^0 + \sum_{n=1}^j \Delta t \hat{\mathbf{u}}_{sh}^n. \tag{5.45}$$

Theorem 5.2. Under the assumptions of Theorem 5.1, there exists a positive constant C independent of h , Δt , c_0 , and λ_p such that

$$\|\mathbf{e}_\eta^N\|_{L^2(\Omega_p)}^2 + \|(\mathbf{e}_\eta^{h,N}, \hat{\mathbf{e}}_\eta^{h,N})\|_{1,h,p}^2 \leq C(\Delta t^2 + h^{2(k+1)}). \tag{5.46}$$

Proof. Using (5.45) and $\boldsymbol{\eta}(t_n) = \boldsymbol{\eta}(0) + \int_0^{t_n} \mathbf{u}_s$, we have

$$\begin{aligned}
\mathbf{e}_\eta^{h,N} &= \Pi_U \boldsymbol{\eta}(t_N) - \boldsymbol{\eta}_h^N = \Pi_U \boldsymbol{\eta}(0) + \Pi_U \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \mathbf{u}_s \, ds - \boldsymbol{\eta}_h^0 - \sum_{n=1}^N \Delta t \mathbf{u}_{sh}^n \\
&= \mathbf{e}_\eta^{h,0} + \sum_{n=1}^N \Delta t \mathbf{e}_{u_s}^{h,n} + \sum_{n=1}^N \Pi_U \left(\int_{t_{n-1}}^{t_n} \mathbf{u}_s \, ds - \Delta t \mathbf{u}_s(t_n) \right) \\
&=: \mathbf{e}_\eta^{h,0} + \sum_{n=1}^N \Delta t \mathbf{e}_{u_s}^{h,n} + \sum_{n=1}^N \Pi_U R_{u_s}^n,
\end{aligned}$$

$$\begin{aligned}
\widehat{\mathbf{e}}_\eta^{h,N} &= \mathbf{\Pi}_\Lambda \widehat{\boldsymbol{\eta}}(t_N) - \boldsymbol{\eta}_h^N = \mathbf{\Pi}_\Lambda \widehat{\boldsymbol{\eta}}(0) + \mathbf{\Pi}_\Lambda \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \widehat{\mathbf{u}}_s \, ds - \widehat{\boldsymbol{\eta}}_h^0 - \sum_{n=1}^N \Delta t \widehat{\mathbf{u}}_{sh}^n \\
&= \widehat{\mathbf{e}}_\eta^{h,0} + \sum_{n=1}^N \Delta t \widehat{\mathbf{e}}_{u_s}^{h,n} + \sum_{n=1}^N \mathbf{\Pi}_\Lambda \left(\int_{t_{n-1}}^{t_n} \widehat{\mathbf{u}}_s - \Delta t \widehat{\mathbf{u}}_s(t_n) \right) \\
&=: \widehat{\mathbf{e}}_\eta^{h,0} + \sum_{n=1}^N \Delta t \widehat{\mathbf{e}}_{u_s}^{h,n} + \sum_{n=1}^N \mathbf{\Pi}_\Lambda \widehat{R}_{u_s}^n.
\end{aligned}$$

Thus, using $\left(\sum_{i=1}^N a_i \right)^2 \leq N \sum_{i=1}^N a_i^2$, we have

$$\begin{aligned}
\|(\mathbf{e}_\eta^{h,N}, \widehat{\mathbf{e}}_\eta^{h,N})\|_{1,h,p}^2 &\leq C \left(\|(\mathbf{e}_\eta^{h,0}, \widehat{\mathbf{e}}_\eta^{h,0})\|_{1,h,p}^2 + \sum_{n=1}^N \Delta t \|(\mathbf{e}_{u_s}^{h,n}, \widehat{\mathbf{e}}_{u_s}^{h,n})\|_{1,h,p}^2 \right. \\
&\quad \left. + N \sum_{n=1}^N \|(\mathbf{\Pi}_U R_{u_s}^n, \mathbf{\Pi}_\Lambda \widehat{R}_{u_s}^n)\|_{1,h,p}^2 \right).
\end{aligned} \tag{5.47}$$

For the last term above we write

$$\begin{aligned}
\|(\mathbf{\Pi}_U R_{u_s}^n, \mathbf{\Pi}_\Lambda \widehat{R}_{u_s}^n)\|_{1,h,p}^2 &= \sum_{\tau \in \mathcal{T}_h^p} \|\nabla \mathbf{\Pi}_U R_{u_s}^n\|_{L^2(\tau)}^2 + \sum_{\tau \in \mathcal{T}_h^p} h_\tau^{-1} \|\mathbf{\Pi}_U R_{u_s}^n - \mathbf{\Pi}_\Lambda \widehat{R}_{u_s}^n\|_{L^2(\partial\tau)}^2 \\
&\leq \sum_{\tau \in \mathcal{T}_h^p} \|\nabla \mathbf{\Pi}_U R_{u_s}^n\|_{L^2(\tau)}^2 + 2 \sum_{\tau \in \mathcal{T}_h^p} h_\tau^{-1} \left(\|\mathbf{\Pi}_U R_{u_s}^n - R_{u_s}^n\|_{L^2(\partial\tau)}^2 + \|R_{u_s}^n - \mathbf{\Pi}_\Lambda R_{u_s}^n\|_{L^2(\partial\tau)}^2 \right) \\
&\leq C \sum_{\tau \in \mathcal{T}_h^p} \|\nabla R_{u_s}^n\|_{L^2(\tau)}^2,
\end{aligned} \tag{5.48}$$

where we used that $R_{u_s}^n = \widehat{R}_{u_s}^n$ on $\partial\tau$ in the first inequality and the approximation properties of $\mathbf{\Pi}_U$ and $\mathbf{\Pi}_\Lambda$, cf. (5.23) and (5.24), in the last inequality. Let $\boldsymbol{\theta} = \nabla \mathbf{u}_s$. An application of the fundamental theorem of calculus gives

$$\begin{aligned}
|\nabla R_{u_s}^n| &= \left| \int_{t_{n-1}}^{t_n} \boldsymbol{\theta}(t) dt - \Delta t \boldsymbol{\theta}(t_n) \right| = \left| \int_{t_{n-1}}^{t_n} \left(\boldsymbol{\theta}(t_n) - \int_t^{t_n} \partial_t \boldsymbol{\theta}(\tilde{t}) d\tilde{t} \right) dt - \Delta t \boldsymbol{\theta}(t_n) \right| \\
&= \left| \int_{t_{n-1}}^{t_n} \int_t^{t_n} \partial_t \boldsymbol{\theta}(\tilde{t}) d\tilde{t} dt \right| \leq \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} |\partial_t \boldsymbol{\theta}(\tilde{t})| d\tilde{t} dt = \Delta t \int_{t_{n-1}}^{t_n} |\partial_t \boldsymbol{\theta}(\tilde{t})| d\tilde{t},
\end{aligned}$$

implying that

$$\|\nabla R_{u_s}^n\|_{L^2(\tau)}^2 \leq \Delta t^2 \int_\tau \left(\int_{t_{n-1}}^{t_n} |\partial_t \boldsymbol{\theta}| \right)^2 \leq \Delta t^3 \int_\tau \int_{t_{n-1}}^{t_n} |\partial_t \boldsymbol{\theta}|^2 = \Delta t^3 \int_{t_{n-1}}^{t_n} \|\partial_t \boldsymbol{\theta}\|_{L^2(\tau)}^2,$$

which, combined with (5.48), gives

$$N \sum_{n=1}^N \|(\mathbf{\Pi}_U R_{u_s}^n, \mathbf{\Pi}_\Lambda \widehat{R}_{u_s}^n)\|_{1,h,p}^2 \leq C \Delta t^2 \|\partial_t \mathbf{u}_s\|_{L^2(0,T;H^1(\Omega_p))}^2. \tag{5.49}$$

Combining (5.47) and (5.49) and using the error estimate for $\|(\mathbf{e}_\eta^{h,0}, \widehat{\mathbf{e}}_\eta^{h,0})\|_{1,h,p}$ given in (5.27) and the error estimate for $\sum_{n=1}^N \Delta t \|(\mathbf{e}_u^n, \widehat{\mathbf{e}}_u^n)\|_{1,h}^2$ (cf. Theorem 5.1), we obtain

$$\|(\mathbf{e}_\eta^{h,N}, \widehat{\mathbf{e}}_\eta^{h,N})\|_{1,h,p}^2 \leq C(\Delta t^2 + h^{2(k+1)}).$$

Combining this with (4.3) and Lemma 4.1 restricted to $\overline{\Omega}_p$ and using (5.23) completes the proof. $\square$

Table 6.1: Convergence history for Example 6.1 with $k = 1$.

h	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)}$	Order	$\ \mathbf{u} - \mathbf{u}_h\ _{1,h}$	Order	$\ \underline{\sigma} - \underline{\sigma}_h\ _{L^2(\Omega)}$	Order
1/4	8.780e-03	–	1.943e-01	–	1.701e-02	–
1/8	2.306e-03	1.93	9.443e-02	1.04	4.843e-03	1.81
1/16	5.923e-04	1.96	4.643e-02	1.02	1.292e-03	1.91
1/32	1.483e-04	2.00	2.307e-02	1.01	3.335e-04	1.95
h	$\ \mathbf{u}_p - \mathbf{u}_{ph}\ _{L^2(\Omega_p)}$	Order	$\ p_p - p_{ph}\ _{L^2(\Omega_p)}$	Order	$\ (\mathbf{e}_{u_p}^h, \mathbf{e}_\lambda^h)\ _{\text{div}_p}$	Order
1/4	9.112e-04	–	1.232e-04	–	9.241e-03	–
1/8	2.700e-04	1.75	3.425e-05	1.85	3.300e-03	1.49
1/16	7.354e-05	1.88	9.091e-06	1.91	9.381e-04	1.81
1/32	1.862e-05	1.98	2.293e-06	1.99	2.390e-04	1.97

Table 6.2: Recovered displacement, fluid pressure, and divergence errors for Example 6.1 with $k = 1$.

h	$\ \boldsymbol{\eta} - \boldsymbol{\eta}_h\ _{L^2(\Omega_p)}$	Order	$\ \boldsymbol{\eta} - \boldsymbol{\eta}_h\ _{1,h,p}$	Order	$\ p_f - p_{fh}\ _{L^2(\Omega_f)}$	Order	$\ \nabla \cdot \mathbf{u}_{fh}\ _{L^2(\Omega_f)}$
1/4	7.583e-04	–	1.695e-03	–	8.970e-03	–	9.681e-15
1/8	2.163e-04	1.81	5.200e-04	1.70	2.575e-03	1.80	1.953e-14
1/16	5.822e-05	1.89	1.729e-04	1.59	6.918e-04	1.90	4.023e-14
1/32	1.470e-05	1.99	6.803e-05	1.35	1.793e-04	1.95	7.866e-14

6 Numerical experiments

In this section we present a series of numerical experiments in two dimensions designed to validate the performance and capabilities of the proposed scheme in terms of accuracy, robustness with respect to c_0 and λ_p , and structure preserving properties.

6.1 Convergence test

We consider $\Omega = (0, 1) \times (-1, 1)$, with $\Omega_f = (0, 1) \times (-1, 0)$, $\Omega_p = (0, 1) \times (0, 1)$. All physical parameters are set to one. We define the exact solution by

$$\boldsymbol{\eta} = \begin{cases} (-3xy^2 + \cos y) \sin^2(\pi t) \\ (y^3 + 1) \sin^2(\pi t) \end{cases}, \quad \mathbf{u}_f = \begin{cases} (-3xy^2 + \cos y) \pi \sin(2\pi t) \\ (y^3 + 1) \pi \sin(2\pi t) \end{cases}$$

and $p_f = p_p = (e^t - 1) \sin(\pi x) \cos(\pi y/2)$. In addition, $\mathbf{u}_s = \partial_t \boldsymbol{\eta}$.

The exact solution satisfies the interface conditions (2.16)–(2.19) at $y = 0$. The body forces and the poroelastic mass source are obtained by direct substitution into the governing equations. We take $T = 0.01$, use triangular grids with $k = 1$, and choose $\Delta t = Ch^2$. We report the convergence history against the meshsize in Tables 6.1 and 6.2. The observed convergence and superconvergence rates agree with the theoretical predictions. The computed fluid velocity is exactly divergence free.

We also present the test with $k = 2$. To balance the temporal and spatial errors, we now choose $\Delta t = Ch^3$. The convergence history against the meshsize for $k = 2$ is reported in Tables 6.3 and 6.4. Again, optimal convergence rates consistent with the theoretical results are obtained.

6.2 Parameter robustness test

In this example we test the convergence of the proposed scheme for several choices of the physical parameters. We particularly focus on robustness with respect to c_0 and λ_p . We consider $\bar{\Omega} = \bar{\Omega}_f \cup \bar{\Omega}_p$,

Table 6.3: Convergence history for Example 6.1 with $k = 2$.

h	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)}$	Order	$\ \mathbf{u} - \mathbf{u}_h\ _{1,h}$	Order	$\ \underline{\sigma} - \underline{\sigma}_h\ _{L^2(\Omega)}$	Order
1/4	5.219e-03	–	2.768e-02	–	2.247e-03	–
1/8	9.573e-04	2.45	5.711e-03	2.28	4.092e-04	2.46
1/16	1.288e-04	2.89	1.106e-03	2.37	5.492e-05	2.90
1/32	1.609e-05	3.00	2.468e-04	2.16	6.866e-06	3.00
h	$\ \mathbf{u}_p - \mathbf{u}_{ph}\ _{L^2(\Omega_p)}$	Order	$\ p_p - p_{ph}\ _{L^2(\Omega_p)}$	Order	$\ (\mathbf{e}_{u_p}^h, e_\lambda^h)\ _{\text{div}_p}$	Order
1/4	8.366e-04	–	9.243e-05	–	1.174e-02	–
1/8	1.628e-04	2.36	1.763e-05	2.39	2.371e-03	2.31
1/16	2.219e-05	2.88	2.391e-06	2.88	3.283e-04	2.85
1/32	2.776e-06	3.00	2.991e-07	3.00	4.123e-05	2.99

Table 6.4: Recovered displacement, fluid pressure, and divergence errors for Example 6.1 with $k = 2$.

h	$\ \boldsymbol{\eta} - \boldsymbol{\eta}_h\ _{L^2(\Omega_p)}$	Order	$\ \boldsymbol{\eta} - \boldsymbol{\eta}_h\ _{1,h,p}$	Order	$\ p_f - p_{fh}\ _{L^2(\Omega_f)}$	Order	$\ \nabla \cdot \mathbf{u}_{fh}\ _{L^2(\Omega_f)}$
1/4	7.571e-04	–	1.502e-03	–	2.361e-04	–	1.045e-14
1/8	1.376e-04	2.46	2.732e-04	2.46	4.472e-05	2.40	2.046e-14
1/16	1.846e-05	2.90	3.710e-05	2.88	6.081e-06	2.88	4.202e-14
1/32	2.307e-06	3.00	4.792e-06	2.95	7.612e-07	3.00	7.976e-14

where $\Omega_f = (0, 1) \times (-1, 0)$ and $\Omega_p = (0, 1)^2$. The exact solution is defined by

$$\boldsymbol{\eta} = \begin{cases} \sin(2\pi t)^2(x^2(1-x)^2y^2(1-y)(1+y)) \\ -\sin(2\pi t)^2 2x(1-x)(1-2x)(y^3/3 - y^5/5) \end{cases}$$

and $p_f = \sin(2\pi t) \sin(2\pi x) \sin(2\pi y)$, $p_p = \sin(2\pi t)^2 x(1-x)y(1-y)$. We set $\mathbf{u}_f = \mathbf{u}_s = \partial_t \boldsymbol{\eta}$. The source terms $\mathbf{f}_p$, g_p , and $\mathbf{h}$ are obtained by plugging the solution into the model equations. The exact solution is divergence free in the fluid region but does not satisfy all the homogeneous interface conditions (2.16)–(2.19). Corresponding interface source terms are therefore included. We test the convergence in space by solving the problem on a sequence of mesh refinements. In order to avoid the effect of the time discretization error we choose Δt proportional to h^2 . In Tables 6.5 and 6.6 we report the convergence history against the mesh size for all variables with all parameters set to one. We observe that optimal convergence and superconvergence rates matching the theoretical results are obtained and that the computed fluid velocity is exactly divergence free. We then perform numerical simulations to assess the scheme’s robustness with respect to the physical parameters. Tables 6.7 and 6.8 show the convergence history with $\lambda_p = 10^5$, $\mu_p = \nu_p = 1$, $\nu = 0.01$, $K = 0.01$, $c_0 = 1$ and $\delta = 10$. We again observe convergence rates in accordance with the theory. Finally, to further verify the robustness of our method, especially with respect to c_0 , we also carry out numerical simulations with $\lambda_p = 10^6$, $\mu_p = \nu_p = 1$, $\nu = 10^{-4}$, $K = 1$, $c_0 = 10^{-4}$ and $\delta = 1$. Optimal convergence and superconvergence are again achieved as shown in Tables 6.9 and 6.10. We further investigate the effect of c_0 with λ_p on the magnitude of the numerical errors. In Figure 6.1 we present the errors with a fixed mesh size $h = 1/4$ obtained by varying c_0 with $\lambda_p = 1$, choosing $c_0 = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$ (right panel) and by varying λ_p with $c_0 = 10^{-4}$, choosing $\lambda_p = 1, 10, 10^2, 10^3, 10^4$ (left panel). In both experiments, the errors remain nearly unchanged over the tested parameter ranges, providing numerical evidence of robustness with respect to these two physical parameters.

Table 6.5: Convergence history for Example 6.2 with all parameters equal to one.

h	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)}$	Order	$\ (\mathbf{e}_u^h, \widehat{\mathbf{e}}_u^h)\ _{1,h}$	Order	$\ \sigma - \sigma_h\ _{L^2(\Omega)}$	Order
1/4	1.90e-02	–	3.36e-01	–	1.31e-01	–
1/8	4.70e-03	2.02	1.23e-01	1.45	3.97e-02	1.71
1/16	1.10e-03	2.11	3.78e-02	1.71	1.11e-02	1.83
1/32	2.55e-04	2.07	1.05e-02	1.85	3.00e-03	1.91
h	$\ \mathbf{u}_p - \mathbf{u}_{ph}\ _{L^2(\Omega_p)}$	Order	$\ p_p - p_{ph}\ _{L^2(\Omega_p)}$	Order	$\ (\mathbf{e}_{u_p}^h, e_\lambda^h)\ _{\text{div}_p}$	Order
1/4	2.30e-03	–	4.31e-04	–	1.27e-02	–
1/8	5.20e-04	2.13	1.07e-04	2.00	5.20e-03	1.30
1/16	1.25e-04	2.05	2.69e-05	1.99	1.70e-03	1.62
1/32	3.10e-05	2.01	6.70e-06	1.99	4.81e-04	1.81

Table 6.6: Convergence history of the recovered variable $\boldsymbol{\eta}_h$ and verification of Lemma 3.1 for Example 6.2 with all parameters equal to one.

h	$\ \boldsymbol{\eta} - \boldsymbol{\eta}_h\ _{L^2(\Omega_p)}$	Order	$\ (\mathbf{e}_\eta^h, \widehat{\mathbf{e}}_\eta^h)\ _{1,h,p}$	Order	$\nabla \cdot \mathbf{u}_{fh}$
1/4	2.30e-03	–	1.65e-02	–	2.53e-015
1/8	5.20e-04	2.13	5.90e-03	1.49	5.47e-015
1/16	1.25e-04	2.05	1.80e-03	1.71	1.13e-014
1/32	3.10e-05	2.01	4.98e-04	1.85	2.07e-014

Table 6.7: Convergence history for Example 6.2 with $\lambda_p = 10^5$, $\mu_p = \nu_p = 1$, $\nu = 0.01$, $K = 0.01$, $c_0 = 1$ and $\delta = 10$.

h	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)}$	Order	$\ (\mathbf{e}_u^h, \widehat{\mathbf{e}}_u^h)\ _{1,h}$	Order	$\ \sigma - \sigma_h\ _{L^2(\Omega)}$	Order
1/4	1.71e-02	–	2.72e-01	–	4.35e-0	–
1/8	4.40e-03	1.97	9.98e-02	1.45	1.14e-02	1.93
1/16	1.00e-03	2.06	3.05e-02	1.71	2.90e-03	1.97
1/32	2.52e-04	2.04	8.40e-03	1.86	7.25e-04	1.99
h	$\ \mathbf{u}_p - \mathbf{u}_{ph}\ _{L^2(\Omega_p)}$	Order	$\ p_p - p_{ph}\ _{L^2(\Omega_p)}$	Order	$\ (\mathbf{e}_{u_p}^h, e_\lambda^h)\ _{\text{div}_p}$	Order
1/4	4.04e-05	–	6.91e-04	–	4.48e-04	–
1/8	9.27e-06	2.12	1.74e-04	2.00	1.68e-04	1.42
1/16	2.05e-06	2.17	4.36e-05	1.99	4.53e-05	1.89
1/32	4.97e-07	2.04	1.09e-05	1.99	1.13e-05	2.01

Table 6.8: Convergence history of the recovered variable $\boldsymbol{\eta}_h$ and verification of Lemma 3.1 for Example 6.2 with $\lambda_p = 10^5$, $\mu_p = \nu_p = 1$, $\nu = 0.01$, $K = 0.01$, $c_0 = 1$ and $\delta = 10$.

h	$\ \boldsymbol{\eta} - \boldsymbol{\eta}_h\ _{L^2(\Omega_p)}$	Order	$\ (\mathbf{e}_\eta^h, \widehat{\mathbf{e}}_\eta^h)\ _{1,h,p}$	Order	$\nabla \cdot \mathbf{u}_{fh}$
1/4	2.30e-03	–	1.19e-02	–	2.53e-15
1/8	5.20e-04	2.13	4.20e-03	1.51	5.49e-015
1/16	1.25e-04	2.05	1.30e-03	1.73	1.12e-014
1/32	3.10e-05	2.01	3.45e-04	1.86	2.07e-014

Table 6.9: Convergence history for Example 6.2 with $\lambda_p = 10^6$, $\mu_p = 1$, $\nu = 10^{-4}$, $K = 1$, $c_0 = 10^{-4}$ and $\delta = 1$.

h	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)}$	Order	$\ (\mathbf{e}_u^h, \widehat{\mathbf{e}}_u^h)\ _{1,h}$	Order	$\ \sigma - \sigma_h\ _{L^2(\Omega)}$	Order
1/4	6.68e-02	–	2.07e-00	–	4.35e-0	–
1/8	4.60e-03	1.97	1.19e-01	4.11	1.14e-02	1.93
1/16	1.00e-03	2.06	3.06e-02	1.97	2.90e-03	1.97
1/32	2.52e-04	2.04	8.40e-03	1.86	7.25e-04	1.99
h	$\ \mathbf{u}_p - \mathbf{u}_{ph}\ _{L^2(\Omega_p)}$	Order	$\ p_p - p_{ph}\ _{L^2(\Omega_p)}$	Order	$\ (\mathbf{e}_{u_p}^h, e_\lambda^h)\ _{\text{div}_p}$	Order
1/4	1.70e-03	–	3.26e-04	–	2.04e-07	–
1/8	4.09e-04	2.12	8.19e-05	2.00	5.01e-08	2.03
1/16	1.03e-04	2.17	2.05e-05	1.99	1.24e-08	2.01
1/32	2.60e-05	2.04	5.13e-06	1.99	3.10e-09	1.99

Table 6.10: Convergence history of the recovered variable $\boldsymbol{\eta}_h$ and verification of Lemma 3.1 for Example 6.2 with $\lambda_p = 10^6$, $\mu_p = 1$, $\nu = 10^{-4}$, $K = 1$, $c_0 = 10^{-4}$ and $\delta = 1$.

h	$\ \boldsymbol{\eta} - \boldsymbol{\eta}_h\ _{L^2(\Omega_p)}$	Order	$\ (\mathbf{e}_\eta^h, \widehat{\mathbf{e}}_\eta^h)\ _{1,h,p}$	Order	$\nabla \cdot \mathbf{u}_{fh}$
1/4	2.30e-03	–	1.20e-02	–	2.71e-015
1/8	5.20e-04	2.13	4.20e-03	1.53	5.51e-015
1/16	1.25e-04	2.05	1.30e-03	1.73	1.12e-014
1/32	3.10e-05	2.01	3.45e-04	1.86	2.07e-014

6.3 Coupled surface and subsurface flows

In this example we simulate coupling of surface and subsurface flows, which can be used to describe the interaction between a river and an aquifer. The example is constructed in [38]. In the domain $\Omega = (0, 2) \times (-1, 1)$, we associate the upper half with the river flow modeled by Stokes equations, while the lower half represents the flow in the aquifer governed by the Biot system. We set $\mathbf{f}_p = \mathbf{h} = \mathbf{0}$ and $g_p = 0$ and the initial condition is also set to zero. The flow is driven through a parabolic fluid velocity on the left boundary of the fluid region with boundary conditions specified as follows:

$$\begin{aligned}
\mathbf{u}_f &= (-40y(y-1), 0)^t \text{ on } \Gamma_{f,\text{left}} = \{(x, y) : x = 0, y \in (0, 1)\}, \\
\mathbf{u}_f &= \mathbf{0} \text{ on } \Gamma_{f,\text{top}} \cup \Gamma_{f,\text{right}} := \{(x, y) : y = 1, x \in (0, 1)\} \cup \{(x, y) : x = 2, y \in (0, 1)\}, \\
p_p &= 0 \quad \text{and} \quad \underline{\sigma}_p \mathbf{n}_p = \mathbf{0} \text{ on } \Gamma_{p,\text{bottom}} := \{(x, y) : y = -1, x \in (0, 2)\}, \\
\mathbf{u}_p \cdot \mathbf{n}_p &= 0 \quad \text{and} \quad \mathbf{u}_s = \mathbf{0} \quad \text{on } \Gamma_{p,\text{left}} \cup \Gamma_{p,\text{right}} \\
&:= \{(x, y) : x = 0, y \in (-1, 0)\} \cup \{(x, y) : x = 2, y \in (-1, 0)\}.
\end{aligned}$$

We consider three cases with different values of $\delta, \lambda_p, c_0, K$ and μ_p , letting $\nu = \nu_p = 1$ for all the three cases, see Table 6.11. The final simulation time is $T = 3$ and the discretization parameters are $\Delta t = 0.06$ and $h = 1/64$. The computed solutions for Case 1–Case 3 are displayed in Figure 6.2–Figure 6.4. Case 2 represents a problem that exhibits both locking regimes for poroelasticity: (1) small storativity and (2) almost incompressible material. We observe from Figure 6.3 that the computed solution does not exhibit locking effects or oscillations. The behavior is qualitatively similar to Case 1. In Case 3, the Lamé coefficient μ_p is increased to 10^6 , resulting in a much stiffer poroelastic medium. The solution is again free of locking effects or oscillations. The stress plots in all three cases show that the computed

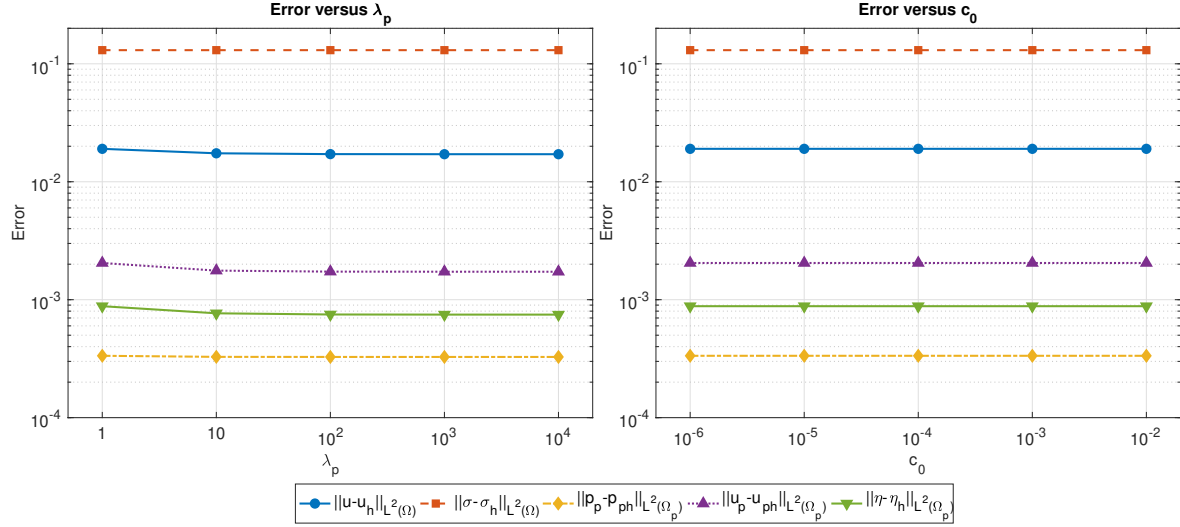


Figure 6.1: Errors with respect to different values of λ_p (left) and c_0 (right) of Example 6.2.

Table 6.11: Parameter configurations for Example 6.3.

Parameters	Case 1	Case 2	Case 3
δ	1	1	100
λ_p	1	10^6	10^6
c_0	1	10^{-4}	10^{-4}
K	1	1	10^{-4}
μ_p	1	1	10^6

stress has a continuous normal component across the interface.

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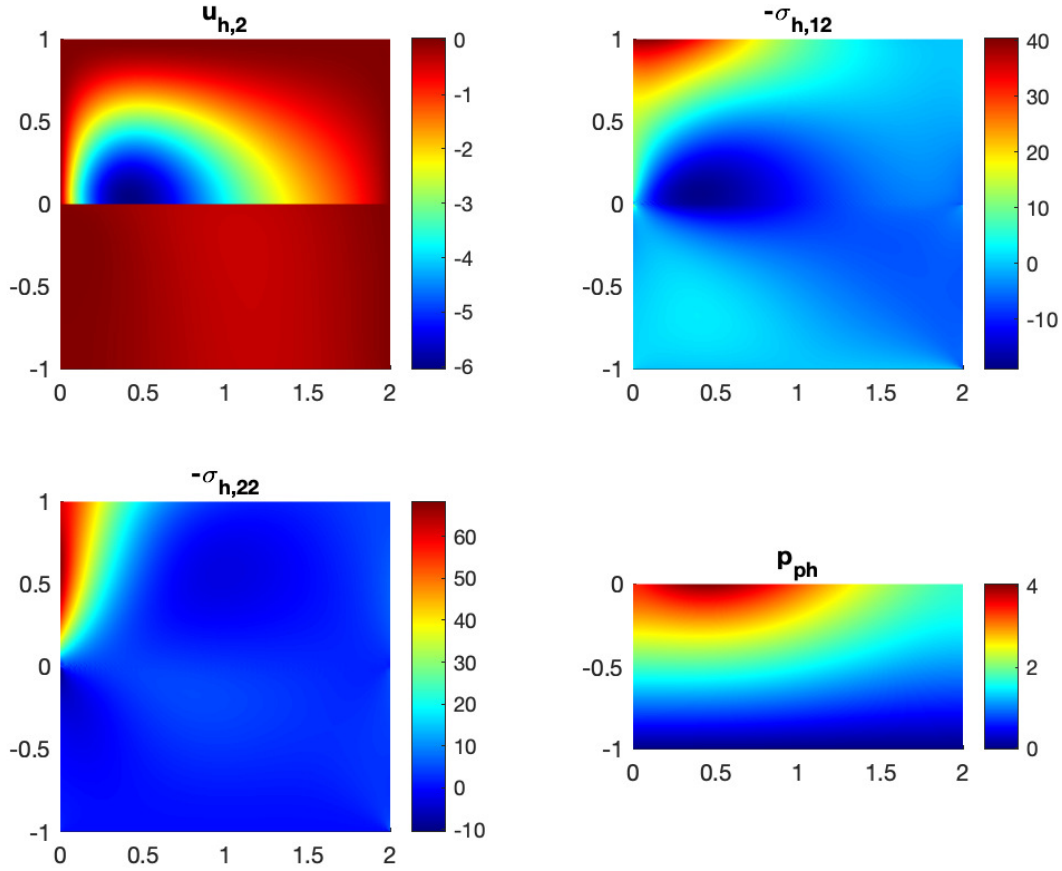


Figure 6.2: Computed solution for Case 1 of Example 6.3.

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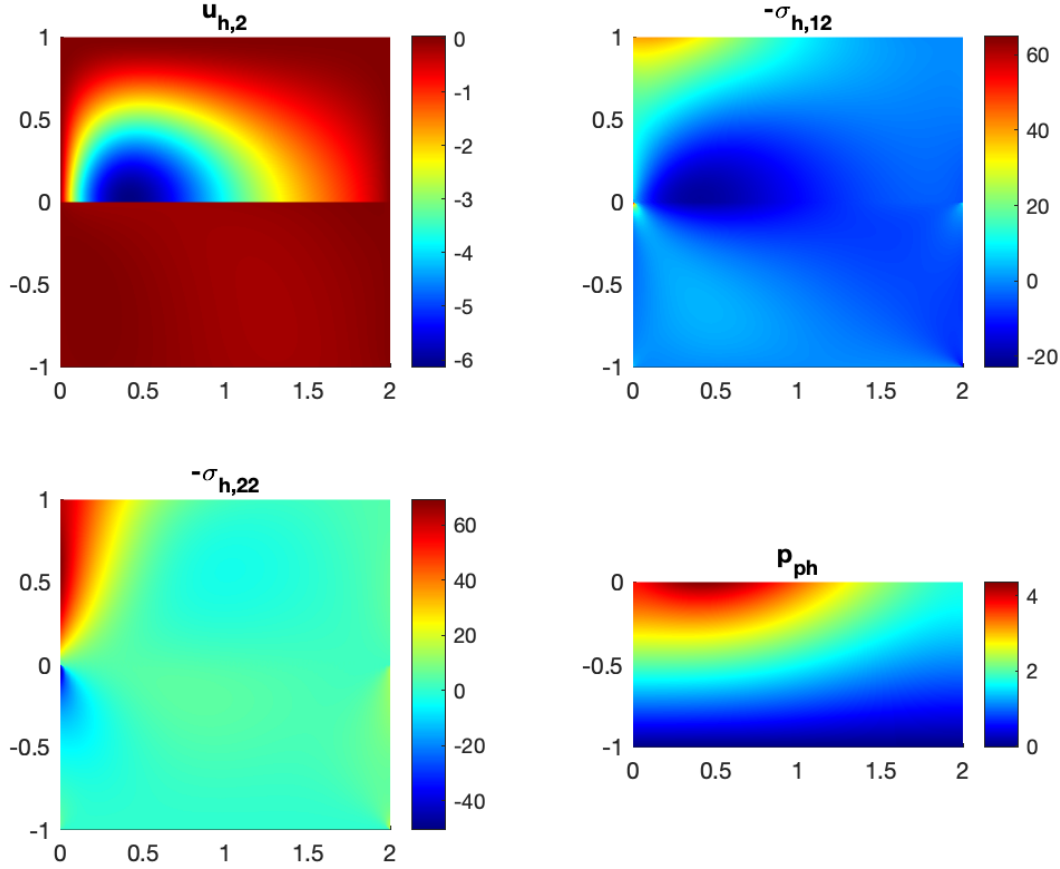


Figure 6.3: Computed solution for Case 2 of Example 6.3.

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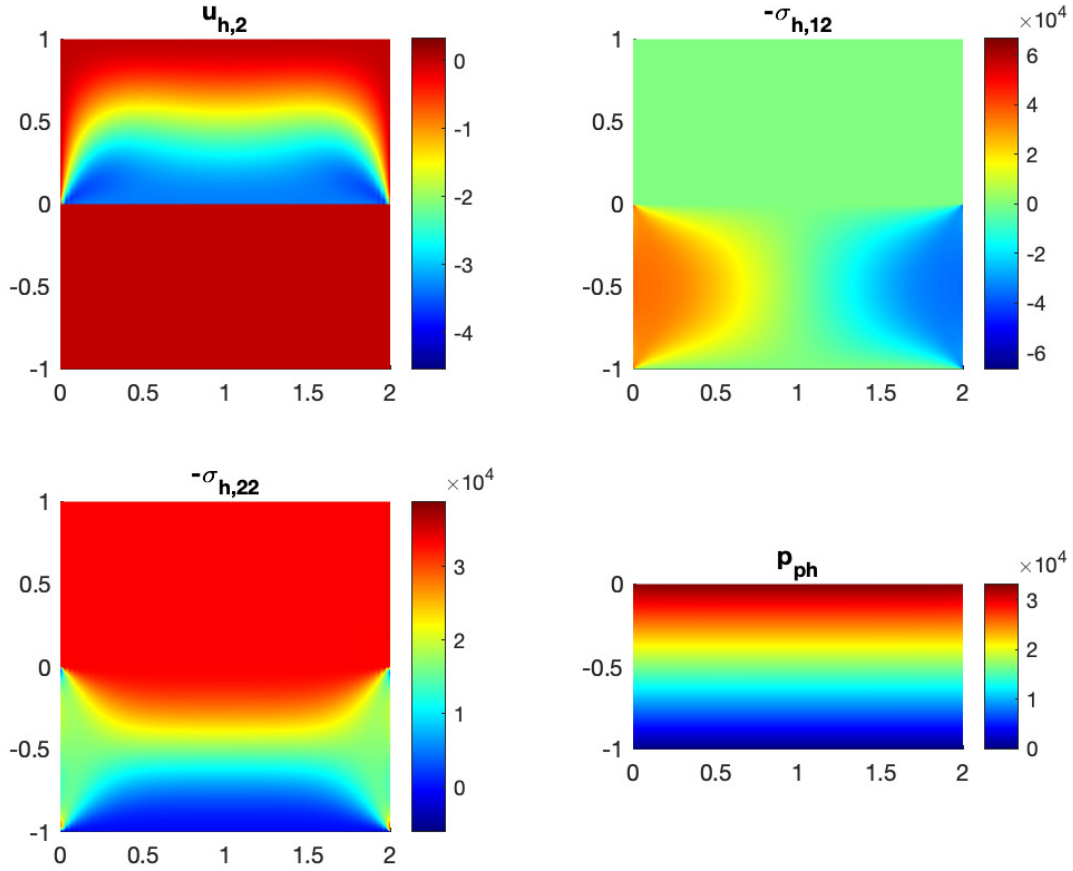


Figure 6.4: Computed solution for Case 3 of Example 6.3.

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