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Velocity-vorticity-pressure mixed formulation for the Kelvin–Voigt–Brinkman–Forchheimer equations

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In this paper, we propose and analyze a mixed formulation for the Kelvin–Voigt–Brinkman–Forchheimer equations for unsteady viscoelastic flows in porous media. Besides the velocity and pressure, our approach introduces the vorticity as a further unknown. Consequently, we obtain a three-field mixed variational formulation, where the aforementioned variables are the main unknowns of the system. We establish the existence and uniqueness of a solution for the weak formulation, and derive the corresponding stability bounds, employing a fixed-point strategy, along with monotone operators theory and Schauder theorem. Afterwards, we introduce a semidiscrete continuous-in-time approximation based on stable Stokes elements for the velocity and pressure, and continuous or discontinuous piecewise polynomial spaces for the vorticity. Additionally, employing backward Euler time discretization, we introduce a fully discrete finite element scheme. We prove well-posedness, derive stability bounds and establish the corresponding error estimates for both schemes. We provide several numerical results verifying the theoretical rates of convergence and illustrating the performance and flexibility of the method for a range of domain configurations and model parameters.

Keywords: Kelvin–Voigt–Brinkman–Forchheimer equations; mixed finite element methods; velocity-vorticity-pressure formulation.

1. Introduction

Fluid flows through porous media at high velocity occur in many industrial applications, such as environmental, chemical and petroleum engineering. For instance, in groundwater remediation and oil and gas extraction, the flow may be fast near injection or production wells or if the aquifer/reservoir is highly porous. Accurate modeling and simulation of such flows are imperative in these fields to optimize processes, ensure safety and minimize environmental impact. Mathematical models have been developed to address different aspects of these flows. The Forchheimer model (Forchheimer, 1901) addresses nonlinearities inherent in high velocity porous flow regimes. The Brinkman model (Brinkman, 1949) incorporates both viscous and permeability effects, enabling precise simulations of fluid movement in diverse environments, including highly porous media. On the other hand, many applications of interest involve flows of viscoelastic fluids through porous media, such as polymer injection and foam flooding in enhanced oil and gas recovery, blood perfusion through biological tissues, and industrial filters. The Kelvin–Voigt model (Kalantarov & Titi, 2009) provides a fundamental framework for describing the viscoelastic behavior of fluids, capturing both viscosity and elasticity. The Kelvin–Voigt–Brinkman–Forchheimer (KVBF) model (The Anh & Thi Trang, 2013), which generalizes and combines the advantages of the three models, is suitable for fast viscoelastic flows in highly porous media.

Concerning the literature, there are papers devoted to the mathematical analysis of the KVBF equations (see, e.g., The Anh & Thi Trang, 2013; Su & Qin, 2018; Mohan, 2020, and references therein). In The Anh & Thi Trang (2013), the existence of a weak solution to the KVBF problem in velocity-pressure formulation is proved by using the Faedo–Galerkin method. In addition, existence, uniqueness and stability of a stationary solution is studied when the external force is time-independent and small. Later on, the KVBF model with continuous delay is analyzed in Su & Qin (2018). In particular, the authors demonstrate that, following the establishment of pullback-$\mathcal{D}$ absorbing sets for the continuous solution process, the asymptotic compactness obtained through the decomposition method leads to the existence of pullback-$\mathcal{D}$ attractors. Meanwhile, the existence and uniqueness of a strong solution to the KVBF equations is obtained in Mohan (2020) by exploiting the m-accretive quantization of both the linear and nonlinear operators. Furthermore, the existence of an exponential attractor is established, along with a discussion concerning the inviscid limit of the 3D KVBF equations towards the 3D Navier–Stokes–Voigt system, and subsequently towards the simplified Bardina model. However, up to the authors' knowledge, there is no literature focused on the numerical analysis of the KVBF problem. On the other hand, several papers have been dedicated to the design and analysis of numerical schemes for simulating the Brinkman–Forchheimer equations. In Louaked *et al.* (2015), the authors introduce and analyze a perturbed compressible system that serves as an approximation to the Brinkman–Forchheimer equations. They also develop a numerical method for this perturbed system, which relies on a semi-implicit Euler scheme for time discretization and employs the lowest-order Raviart–Thomas elements for spatial discretization. A pressure stabilization finite element method is developed in Louaked *et al.* (2017). In Kou *et al.* (2019), a time-discrete scheme for a variable porosity Brinkman–Forchheimer model is applied for simulating wormhole propagation. In Caucao & Yotov (2021), a mixed formulation based on the pseudostress tensor and the velocity field is presented. By employing classical results on nonlinear monotone operators and a suitable regularization technique in Banach spaces, existence and uniqueness are proved. A fully discrete scheme is developed, which combines a finite element space discretization based on the Raviart–Thomas spaces for the pseudostress tensor and discontinuous piecewise polynomial elements for the velocity with a backward Euler time discretization. Sub-optimal error estimates are derived. These estimates are improved in Caucao *et al.* (2022), where a three-field formulation including the velocity gradient is developed and analyzed. A staggered DG method for a velocity–velocity gradient–pressure formulation of the unsteady Brinkman–Forchheimer problem is developed in Zhao *et al.* (2022). Well-posedness and error analysis are presented for the semi-discrete and fully discrete schemes. The method is robust with respect to the Brinkman parameter. More recently, a vorticity-based mixed variational formulation is analyzed in Anaya *et al.* (2023), where the velocity, vorticity and pressure are the main unknowns of the system. Existence and uniqueness of a weak solution, as well as stability bounds are derived by employing classical results on nonlinear monotone operators. A semidiscrete continuous-in-time mixed finite element approximation and a fully discrete scheme are introduced and optimal rates of convergence are established.

The purpose of the present work is to develop and analyze a new vorticity-based mixed formulation of the KVBF problem and to study a suitable conforming numerical discretization. To that end, unlike previous KVBF works and motivated by Anaya *et al.* (2015), Anaya *et al.* (2021) and Anaya *et al.* (2023), we introduce the vorticity as an additional unknown besides the fluid velocity and pressure. In addition to the advantage of providing a direct, accurate, and smooth approximation of the vorticity, our approach gives optimal theoretical convergence rates without requiring any small data or quasi-uniformity assumptions on the mesh. Furthermore, unlike Anaya *et al.* (2015), Anaya *et al.* (2021) or Anaya *et al.* (2023), our method does not require any augmentation process. It is also important to mention that another novelty and advantage of the present work is that it generalizes the model studied in Anaya *et al.* (2023) by including the nonlinear convective term and an additional time-derivative term, thus considering viscoelastic flows.

We establish the existence of a solution to the continuous weak formulation by employing techniques from Showalter (1997), Caucao *et al.* (2021) and Caucao *et al.* (2023), combined with a fixed-point argument, the Browder–Minty theorem and the Schauder theorem. The uniqueness is achieved by contradiction arguments in conjunction with Grönwall's inequality. Stability for the weak solution is established by means of an energy estimate. We further develop semidiscrete continuous-in-time and fully discrete finite element approximations. We emphasize that our formulation relies on the natural $\mathbf{H}^1$–L^2 spaces for the velocity-pressure pair, facilitating the use of classical stable Stokes elements such as the Taylor–Hood, Crouzeix–Raviart or MINI elements. Additionally, both continuous and discontinuous piecewise polynomial spaces can be utilized for discretizing the vorticity. We make use of the backward Euler method for the discretization in time. Adapting the tools employed for the analysis of the continuous problem, we prove well-posedness of the discrete schemes and derive the corresponding stability estimates. We further perform error analysis for the semidiscrete and fully discrete schemes, establishing optimal rates of convergence in space and time.

We have organized the contents of this paper as follows. In Section 2 we describe the model problem of interest and develop the velocity-vorticity-pressure variational formulation. In Section 3, we show that it is well posed using a fixed-point strategy, along with monotone operators theory and the classical Schauder theorem. Next, in Section 4 we present the semidiscrete continuous-in-time approximation, provide particular families of stable finite elements, and obtain error estimates for the proposed methods. Section 5 is devoted to the fully discrete approximation. The performance of the method is studied in Section 6 with several numerical examples in 2D and 3D, verifying the aforementioned rates of convergence, as well as illustrating its flexibility to handle spatially varying parameters in complex geometries. The paper ends with conclusions in Section 7.

In the remainder of this section, we introduce some standard notation and needed functional spaces. Let $\Omega \subset \mathrm{R}^d$, $d \in \{2,3\}$, denote a domain with Lipschitz boundary Γ. For $\mathrm{s} \geq 0$ and $\mathrm{p} \in [1,+\infty]$, we denote by $\mathrm{L}^{\mathrm{p}}(\Omega)$ and $\mathrm{W}^{\mathrm{s},\mathrm{p}}(\Omega)$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^{\mathrm{p}}(\Omega)}$ and $\|\cdot\|_{\mathrm{W}^{\mathrm{s},\mathrm{p}}(\Omega)}$, respectively. Note that $\mathrm{W}^{0,\mathrm{p}}(\Omega) = \mathrm{L}^{\mathrm{p}}(\Omega)$. If $\mathrm{p} = 2$, we write $\mathrm{H}^{\mathrm{s}}(\Omega)$ in place of $\mathrm{W}^{\mathrm{s},2}(\Omega)$ and denote the corresponding norm by $\|\cdot\|_{\mathrm{H}^{\mathrm{s}}(\Omega)}$. By $\mathbf{H}$ and $\mathbb{H}$ we will denote the corresponding vectorial and tensorial counterparts of a generic scalar functional space H. The $\mathrm{L}^2(\Omega)$ inner product for scalar, vector or tensor valued functions is denoted by $(\cdot,\cdot)_\Omega$. The $\mathrm{L}^2(\Gamma)$ inner product or duality pairing is denoted by $\langle\cdot,\cdot\rangle_\Gamma$. Moreover, given a separable Banach space V endowed with the norm $\|\cdot\|_{\mathrm{V}}$, we let $\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})$ be the space of classes of functions $f:(0,T) \to \mathrm{V}$ that are Bochner measurable and such that $\|f\|_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})} < \infty$, with

$$\|f\|^{\mathrm{p}}_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})} := \int_0^T \|f(t)\|^{\mathrm{p}}_{\mathrm{V}}\,\mathrm{d}t, \quad \|f\|_{\mathrm{L}^{\infty}(0,T;\mathrm{V})} := \operatorname*{ess\,sup}_{t\in[0,T]} \|f(t)\|_{\mathrm{V}}.$$

In turn, for any vector field $\mathbf{v} := (v_i)_{i=1,d}$, we set the gradient and divergence operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j} \right)_{i,j=1,d} \quad \text{and} \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{d} \frac{\partial v_j}{\partial x_j}.$$

In what follows, when no confusion arises, $|\cdot|$ denotes the Euclidean norm in R^n or $\mathrm{R}^{n\times n}$. In addition, in the sequel, we will make use of the well-known Hölder inequality given by

$$\int_\Omega |f\,g| \leq \|f\|_{\mathrm{L}^\mathrm{p}(\Omega)}\,\|g\|_{\mathrm{L}^\mathrm{q}(\Omega)} \quad \forall f \in \mathrm{L}^\mathrm{p}(\Omega),\ \forall g \in \mathrm{L}^\mathrm{q}(\Omega), \quad \text{with} \quad \frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1,$$

and Young's inequality, for $a, b \geq 0$ and $\delta > 0$,

$$a\,b \leq \frac{\delta^{\mathrm{p}/2}}{\mathrm{p}}\,a^\mathrm{p} + \frac{1}{\mathrm{q}\,\delta^{\mathrm{q}/2}}\,b^\mathrm{q}. \tag{1.1}$$

Finally, we recall the continuous injection i_p of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^\mathrm{p}(\Omega)$ for $\mathrm{p} \geq 1$ if $d = 2$ or $\mathrm{p} \in [1,6]$ if $d = 3$. More precisely, we have the following inequality:

$$\|w\|_{\mathrm{L}^\mathrm{p}(\Omega)} \leq \|i_\mathrm{p}\|\,\|w\|_{\mathrm{H}^1(\Omega)} \quad \forall w \in \mathrm{H}^1(\Omega), \tag{1.2}$$

with $\|i_\mathrm{p}\| > 0$ depending only on $|\Omega|$ and p (see Quarteroni & Valli, 1994, Theorem 1.3.4).

We will denote by $\mathbf{i}_\mathrm{p}$ the vectorial version of i_p.

2. The model problem and its velocity-vorticity-pressure formulation

Our model of interest is given by the Kelvin–Voigt–Brinkman–Forchheimer equations (see, e.g., The Anh & Thi Trang, 2013; Su & Qin, 2018; Mohan, 2020). More precisely, given the body force term $\mathbf{f}$ and a suitable initial data $\mathbf{u}_0$, the aforementioned system of equations is given by

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} - \kappa^2\,\frac{\partial\,\Delta\mathbf{u}}{\partial t} - \nu\,\Delta\mathbf{u} + (\nabla\mathbf{u})\mathbf{u} + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\mathbf{u} + \nabla p = \mathbf{f}, \quad \mathrm{div}(\mathbf{u}) = 0 \quad \text{in} \quad \Omega \times (0,T], \\ &\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma \times (0,T], \quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in} \quad \Omega, \quad (p,1)_\Omega = 0 \quad \text{in} \quad (0,T], \end{aligned} \tag{2.1}$$

where the unknowns are the velocity field $\mathbf{u}$ and the scalar pressure p. In addition, the constant $\kappa > 0$ is a length scale parameter characterizing the elasticity of the fluid, $\nu > 0$ is the Brinkman coefficient (or the effective viscosity), $\mathtt{D} > 0$ is the Darcy coefficient, $\mathtt{F} > 0$ is the Forchheimer coefficient and $\rho \in [3,4]$ is a given number.

We next introduce a new velocity-vorticity-pressure formulation for (2.1). To that end, we first define the trace operator $\boldsymbol{\gamma}_*$ and vorticity $\boldsymbol{\omega}$:

$$\boldsymbol{\gamma}_*(\mathbf{v}) := \begin{cases} \mathbf{v}\cdot\mathbf{t}, & \text{for } d = 2, \\ \mathbf{v}\times\mathbf{n}, & \text{for } d = 3, \end{cases} \quad \text{and} \quad \boldsymbol{\omega} := \mathbf{curl}(\mathbf{u}) = \begin{cases} \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2}, & \text{for } d = 2, \\ \nabla\times\mathbf{u}, & \text{for } d = 3. \end{cases}$$

Note that the **curl** of a two-dimensional vector field is a scalar, whereas the **curl** of a three-dimensional vector field is a vector. In order to avoid a multiplicity of notation, we nevertheless denote it like a vector, provided there is no confusion. In addition, in 2D, the **curl** of a scalar field q is a vector given by $\mathbf{curl}(q) = \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1}\right)^{\mathrm{t}}$. Then, employing the well-known identity (Girault & Raviart, 1986, Section I.2.3) in conjunction with the incompressibility condition $\mathrm{div}(\mathbf{u}) = 0$ in $\Omega \times (0,T]$, we deduce that

$$\mathbf{curl}(\boldsymbol{\omega}) = \mathbf{curl}(\mathbf{curl}(\mathbf{u})) = -\Delta\mathbf{u} + \nabla(\mathrm{div}(\mathbf{u})) = -\Delta\mathbf{u}, \tag{2.2}$$

from which we conclude that (2.1) can be equivalently rewritten as follows: Find $(\mathbf{u}, \boldsymbol{\omega}, p)$ in suitable spaces to be indicated below such that

$$\begin{aligned} \frac{\partial\,\mathbf{u}}{\partial\,t} - \kappa^2\,\frac{\partial\,\Delta\mathbf{u}}{\partial\,t} + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\mathbf{u} + (\nabla\mathbf{u})\mathbf{u} + \nu\,\mathbf{curl}(\boldsymbol{\omega}) + \nabla p \;&=\; \mathbf{f} \quad \text{in} \quad \Omega \times (0,T], \\ \boldsymbol{\omega} \;=\; \mathbf{curl}(\mathbf{u}), \quad \mathrm{div}(\mathbf{u}) \;=\; 0 \quad \text{in} \quad \Omega \times (0,T], \\ \mathbf{u} \;=\; \mathbf{0} \quad \text{on} \quad \Gamma \times (0,T], \quad \mathbf{u}(0) \;=\; \mathbf{u}_0 \quad \text{in} \quad \Omega, \quad (p,1)_\Omega \;=\; 0 \quad \text{in} \quad (0,T]. \end{aligned} \tag{2.3}$$

Next, multiplying the first equation of (2.3) by a suitable test function $\mathbf{v}$, we obtain

$$\begin{aligned} &(\partial_t\,\mathbf{u}, \mathbf{v})_\Omega - \kappa^2(\partial_t\,\Delta\mathbf{u}, \mathbf{v})_\Omega + \mathtt{D}\,(\mathbf{u}, \mathbf{v})_\Omega + \mathtt{F}\,(|\mathbf{u}|^{\rho-2}\mathbf{u}, \mathbf{v})_\Omega \\ &\qquad + ((\nabla\mathbf{u})\mathbf{u}, \mathbf{v})_\Omega + \nu\,(\mathbf{curl}(\boldsymbol{\omega}), \mathbf{v})_\Omega + (\nabla p, \mathbf{v})_\Omega \;=\; (\mathbf{f}, \mathbf{v})_\Omega, \end{aligned} \tag{2.4}$$

where we use the notation $\partial_t := \dfrac{\partial}{\partial\,t}$. Notice that the fourth and fifth terms in the left-hand side of (2.4) require $\mathbf{u}$ to live in a smaller space than $\mathbf{L}^2(\Omega)$. In particular, by applying Cauchy–Schwarz and Hölder's inequalities and then the continuous injection $\mathbf{i}_\rho$ (resp. $\mathbf{i}_4$) of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^\rho(\Omega)$ (resp. $\mathbf{L}^4(\Omega)$), with $\rho \in [3,4]$, we find that

$$\left|(|\mathbf{u}|^{\rho-2}\mathbf{u}, \mathbf{v})_\Omega\right| \;\leq\; \|\mathbf{u}\|^{\rho-1}_{\mathbf{L}^\rho(\Omega)}\,\|\mathbf{v}\|_{\mathbf{L}^\rho(\Omega)} \;\leq\; \|\mathbf{i}_\rho\|^\rho\,\|\mathbf{u}\|^{\rho-1}_{\mathbf{H}^1(\Omega)}\,\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \tag{2.5}$$

and

$$\left|((\nabla\mathbf{u})\mathbf{z}, \mathbf{v})_\Omega\right| \;\leq\; \|\nabla\mathbf{u}\|_{\mathbb{L}^2(\Omega)}\,\|\mathbf{z}\|_{\mathbf{L}^4(\Omega)}\,\|\mathbf{v}\|_{\mathbf{L}^4(\Omega)} \;\leq\; \|\mathbf{i}_4\|^2\,\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}\,\|\mathbf{z}\|_{\mathbf{H}^1(\Omega)}\,\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}, \tag{2.6}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{z} \in \mathbf{H}^1(\Omega)$, which, together with the Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on Γ (cf. (2.3)) suggest to look for the unknown $\mathbf{u}$ in $\mathbf{H}^1_0(\Omega)$ and to restrict the set of corresponding test functions $\mathbf{v}$ to the same space. If a non-homogeneous condition of the form $\mathbf{u} = \mathbf{u}_\mathrm{D}$ on $\Gamma \times (0,T]$ is prescribed, with boundary data $\mathbf{u}_\mathrm{D}$ satisfying $\int_\Gamma \mathbf{u}_\mathrm{D} \cdot \mathbf{n} = 0$ in $(0,T]$, a suitable lifting approach must be employed to ensure that both the velocity and its test functions belong to $\mathbf{H}^1_0(\Omega)$. Employing Green's formula (Girault & Raviart, 1986, Theorem I.2.11), the sixth term in the left-hand side in (2.4) can be rewritten as

$$(\mathbf{curl}(\boldsymbol{\omega}), \mathbf{v})_\Omega \;=\; (\boldsymbol{\omega}, \mathbf{curl}(\mathbf{v}))_\Omega - \langle\boldsymbol{\gamma}_*(\mathbf{v}), \boldsymbol{\omega}\rangle_\Gamma \;=\; (\boldsymbol{\omega}, \mathbf{curl}(\mathbf{v}))_\Omega \quad \forall\,\mathbf{v} \in \mathbf{H}^1_0(\Omega). \tag{2.7}$$

Thus, replacing back (2.7) into (2.4), integrating by parts the terms $(\partial_t \Delta\mathbf{u}, \mathbf{v})_\Omega$ and $(\nabla p, \mathbf{v})_\Omega$, and incorporating the second and third equations of (2.3) in a weak sense, we obtain the system

$$\begin{aligned}(\partial_t \mathbf{u}, \mathbf{v})_\Omega + \kappa^2(\partial_t \nabla\mathbf{u}, \nabla\mathbf{v})_\Omega + \mathtt{D}\,(\mathbf{u}, \mathbf{v})_\Omega + \mathtt{F}\,(|\mathbf{u}|^{\rho-2}\mathbf{u}, \mathbf{v})_\Omega \\ + ((\nabla\mathbf{u})\mathbf{u}, \mathbf{v})_\Omega + \nu\,(\boldsymbol{\omega}, \mathbf{curl}(\mathbf{v}))_\Omega - (p, \mathrm{div}(\mathbf{v}))_\Omega &= (\mathbf{f}, \mathbf{v})_\Omega, \end{aligned} \tag{2.8}$$

$$\nu\,(\boldsymbol{\omega}, \boldsymbol{\psi})_\Omega - \nu\,(\boldsymbol{\psi}, \mathbf{curl}(\mathbf{u}))_\Omega = 0, \tag{2.9}$$

$$(q, \mathrm{div}(\mathbf{u}))_\Omega = 0, \tag{2.10}$$

for all $(\mathbf{v}, \boldsymbol{\psi}, q) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega) \times \mathrm{L}_0^2(\Omega)$, where $\mathrm{L}_0^2(\Omega) := \big\{q \in \mathrm{L}^2(\Omega) : \quad (q, 1)_\Omega = 0\big\}$.

Next, in order to write the above formulation in a more suitable way for the analysis to be developed below, we set

$$\underline{\mathbf{u}} := (\mathbf{u}, \boldsymbol{\omega}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega),$$

with corresponding norm given by

$$\|\underline{\mathbf{v}}\| = \|(\mathbf{v}, \boldsymbol{\psi})\| := \Big(\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}\Big)^{1/2} \quad \forall\, \underline{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\psi}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega).$$

Hence, the weak form associated with the Kelvin–Voigt–Brinkman–Forchheimer equations (2.8)–(2.10) reads: Given $\mathbf{f}: [0, T] \to \mathbf{H}^{-1}(\Omega)$ and $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)$, find $(\underline{\mathbf{u}}, p): [0, T] \to \big(\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)\big) \times \mathrm{L}_0^2(\Omega)$ such that $\mathbf{u}(0) = \mathbf{u}_0$ and, for a.e. $t \in (0, T)$,

$$\begin{aligned}\frac{\partial}{\partial t}\,[\mathcal{E}(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] + [\mathcal{A}(\mathbf{u}(t))(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] + [\mathcal{B}'(p(t)), \underline{\mathbf{v}}] &= [\mathbf{F}(t), \underline{\mathbf{v}}] \quad && \forall\, \underline{\mathbf{v}} \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega), \\ -\,[\mathcal{B}(\underline{\mathbf{u}}(t)), q] &= 0 && \forall\, q \in \mathrm{L}_0^2(\Omega),\end{aligned} \tag{2.11}$$

where, given $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$, the operators $\mathcal{E}, \mathcal{A}(\mathbf{z}): \big(\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)\big) \to \big(\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)\big)'$ and $\mathcal{B}: \big(\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)\big) \to \mathrm{L}_0^2(\Omega)'$ are defined, respectively, as

$$\big[\mathcal{E}(\underline{\mathbf{u}}), \underline{\mathbf{v}}\big] := (\mathbf{u}, \mathbf{v})_\Omega + \kappa^2\,(\nabla\mathbf{u}, \nabla\mathbf{v})_\Omega, \tag{2.12}$$

$$\big[\mathcal{A}(\mathbf{z})(\underline{\mathbf{u}}), \underline{\mathbf{v}}\big] := \big[\mathbf{a}(\underline{\mathbf{u}}), \underline{\mathbf{v}}\big] + \big[\mathbf{c}(\mathbf{z})(\underline{\mathbf{u}}), \underline{\mathbf{v}}\big], \tag{2.13}$$

$$\begin{aligned}\big[\mathbf{a}(\underline{\mathbf{u}}), \underline{\mathbf{v}}\big] :=\; & \mathtt{D}\,(\mathbf{u}, \mathbf{v})_\Omega + \mathtt{F}\,(|\mathbf{u}|^{\rho-2}\mathbf{u}, \mathbf{v})_\Omega + \nu\,(\boldsymbol{\omega}, \boldsymbol{\psi})_\Omega \\ & + \nu\,(\boldsymbol{\omega}, \mathbf{curl}(\mathbf{v}))_\Omega - \nu\,(\boldsymbol{\psi}, \mathbf{curl}(\mathbf{u}))_\Omega,\end{aligned} \tag{2.14}$$

$$\big[\mathbf{c}(\mathbf{z})(\underline{\mathbf{u}}), \underline{\mathbf{v}}\big] := ((\nabla\mathbf{u})\mathbf{z}, \mathbf{v})_\Omega, \tag{2.15}$$

$$\big[\mathcal{B}(\underline{\mathbf{v}}), q\big] := -\,(q, \mathrm{div}(\mathbf{v}))_\Omega, \tag{2.16}$$

and $\mathbf{F} \in (\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega))'$ is the bounded linear functional given by

$$[\mathbf{F}, \underline{\mathbf{v}}] := (\mathbf{f}, \mathbf{v})_\Omega. \tag{2.17}$$

In all the terms above, $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators. In addition, we let $\mathcal{B}' : \mathrm{L}_0^2(\Omega) \to \left(\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)\right)'$ be the adjoint of $\mathcal{B}$, which satisfies $[\mathcal{B}'(q), \underline{\mathbf{v}}] = [\mathcal{B}(\underline{\mathbf{v}}), q]$ for all $\underline{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\psi}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$ and $q \in \mathrm{L}_0^2(\Omega)$.

Now we define the kernel space of the operator $\mathcal{B}$,

$$\mathbf{V} := \left\{ \underline{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\psi}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega) : \quad [\mathcal{B}(\underline{\mathbf{v}}), q] = 0 \quad \forall\, q \in \mathrm{L}_0^2(\Omega) \right\},$$

which from the definition of the operator $\mathcal{B}$ (cf. (2.16)) can be rewritten as

$$\mathbf{V} = \mathbf{K} \times \mathbf{L}^2(\Omega), \quad \text{where} \quad \mathbf{K} := \left\{ \mathbf{v} \in \mathbf{H}_0^1(\Omega) : \ \operatorname{div}(\mathbf{v}) = 0 \quad \text{in} \quad \Omega \right\}. \tag{2.18}$$

This leads us to the reduced problem: Given $\mathbf{f} : [0, T] \to \mathbf{H}^{-1}(\Omega)$ and $\mathbf{u}_0 \in \mathbf{K}$, find $\underline{\mathbf{u}} : [0, T] \to \mathbf{K} \times \mathbf{L}^2(\Omega)$ such that $\mathbf{u}(0) = \mathbf{u}_0$ and, for a.e. $t \in (0, T)$,

$$\frac{\partial}{\partial t} [\mathcal{E}(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] + [\mathcal{A}(\mathbf{u}(t))(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] = [\mathbf{F}(t), \underline{\mathbf{v}}] \quad \forall\, \underline{\mathbf{v}} \in \mathbf{K} \times \mathbf{L}^2(\Omega). \tag{2.19}$$

According to the definition of $\mathbf{K}$ (cf. (2.18)), owing to the inf-sup condition of $\mathcal{B}$ (cf. Ern & Guermond, 2004, Corollary B.71):

$$\sup_{\mathbf{0} \neq \underline{\mathbf{v}} \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)} \frac{[\mathcal{B}(\underline{\mathbf{v}}), q]}{\|\underline{\mathbf{v}}\|} \geq \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{\displaystyle\int_\Omega q \operatorname{div}(\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}} \geq \beta \, \|q\|_{\mathrm{L}^2(\Omega)} \quad \forall\, q \in \mathrm{L}_0^2(\Omega), \tag{2.20}$$

with $\beta > 0$, and using standard arguments, it is not difficult to show that the problem (2.19) is equivalent to (2.11). This result is stated next and the proof is omitted.

LEMMA 2.1. If $(\underline{\mathbf{u}}, p) : [0, T] \to \left(\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)\right) \times \mathrm{L}_0^2(\Omega)$ is a solution of (2.11), then $\mathbf{u} : [0, T] \to \mathbf{K}$ and $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\omega})$ is a solution of (2.19). Conversely, if $\underline{\mathbf{u}} : [0, T] \to \mathbf{K} \times \mathbf{L}^2(\Omega)$ is a solution of (2.19), then there exists a unique $p : [0, T] \to \mathrm{L}_0^2(\Omega)$ such that $(\underline{\mathbf{u}}, p)$ is a solution of (2.11).

3. Well-posedness of the model

In this section, we establish the solvability of (2.19) (equivalently of (2.11)). To that end, we first collect some previous results that will be used in the forthcoming analysis.

3.1 *Preliminary results*

We begin by recalling a key result, which will be used to establish the existence of a solution to (2.19). In what follows, an operator $\mathcal{A}$ from a real vector space E to its algebraic dual E' is symmetric and monotone if, respectively,

$$[\mathcal{A}(x), y] = [\mathcal{A}(y), x] \quad \text{and} \quad [\mathcal{A}(x) - \mathcal{A}(y), x - y] \geq 0 \quad \forall\, x, y \in E.$$

In addition, $Rg(\mathcal{A})$ denotes the range of $\mathcal{A}$ and a dual space with a seminorm is the space of linear functionals on a vector space that are continuous with respect to the seminorm. The following theorem is a slight simplification of Showalter (1997, Theorem IV.6.1(b)).

Theorem 3.1. Let the linear, symmetric and monotone operator $\mathcal{N}$ be given from the real vector space E to its algebraic dual E', and let E'_b be the Hilbert space which is the dual of E with the seminorm

$$|x|_b = [\mathcal{N}(x), x]^{1/2} \quad x \in E.$$

Let $\mathcal{M} : E \to E'_b$ be an operator with domain $\mathcal{D} = \left\{x \in E : \mathcal{M}(x) \in E'_b\right\}$. Assume that $\mathcal{M}$ is monotone and $Rg(\mathcal{N} + \mathcal{M}) = E'_b$. Then, for each $f \in \mathrm{W}^{1,1}(0, T; E'_b)$ and for each $u_0 \in \mathcal{D}$, there is a solution u of

$$\frac{\partial}{\partial t}\,(\mathcal{N}(u(t))) + \mathcal{M}\,(u(t)) \;=\; f(t) \quad \text{a.e.} \quad 0 < t < T, \tag{3.1}$$

with

$$\mathcal{N}(u) \in \mathrm{W}^{1,\infty}(0, T; E'_b), \quad u(t) \in \mathcal{D}, \quad \text{for all}\, 0 \leq t \leq T \quad \text{and} \quad \mathcal{N}(u(0)) = \mathcal{N}(u_0).$$

For the proof of the range condition in Theorem 3.1 we will utilize the Browder–Minty theorem (Ciarlet, 2013, Theorem 9.14-1) stated below.

Theorem 3.2. Let V be a real separable reflexive Banach space and let $\mathcal{A} : \mathrm{V} \to \mathrm{V}'$ be a coercive and hemicontinuous monotone operator. Then $\mathcal{A}$ is surjective, i.e., given any $f \in \mathrm{V}'$ there exists u such that

$$u \in \mathrm{V} \quad \text{and} \quad \mathcal{A}(u) = f.$$

If $\mathcal{A}$ is strictly monotone, then $\mathcal{A}$ is also injective.

We recall that an operator $\mathcal{A}$ is hemicontinuous if, for each $u, v, w \in V$, the real-valued function $t \mapsto [\mathcal{A}(u + t\,v), w]$ is continuous. In particular, if $\mathcal{A}$ is continuous, then it is hemicontinuous. Additionally, $\mathcal{A}$ is strictly monotone if $[\mathcal{A}(u) - \mathcal{A}(v), u - v] > 0$ for all $u \neq v$ in V and $\mathcal{A}$ is strongly monotone if there exists a constant $c > 0$ such that

$$[\mathcal{A}(u) - \mathcal{A}(v), u - v] \;\geq\; c\,\|u - v\|_V^2 \quad \forall\, u, v \in V.$$

It is clear that strong monotonicity implies strict monotonicity.

Next, we establish the stability properties of the operators involved in (2.11). We begin by observing that the operators $\mathcal{E}, \mathcal{B}$ and the functional $\mathbf{F}$ are linear. In turn, from (2.12), (2.16) and (2.17), and employing Hölder and Cauchy–Schwarz inequalities, there hold

$$\big|[\mathcal{B}(\underline{\mathbf{v}}), q]\big| \,\leq\, \sqrt{d}\,\|\underline{\mathbf{v}}\|\,\|q\|_{\mathrm{L}^2(\Omega)} \quad \forall\,(\underline{\mathbf{v}}, q) \in \Big(\mathbf{H}_0^1(\Omega)\times\mathbf{L}^2(\Omega)\Big)\times \mathrm{L}_0^2(\Omega), \tag{3.2}$$

$$\big|[\mathbf{F}, \underline{\mathbf{v}}]\big| \,\leq\, \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \,\leq\, \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|\underline{\mathbf{v}}\| \quad \forall\,\underline{\mathbf{v}} \in \mathbf{H}_0^1(\Omega)\times\mathbf{L}^2(\Omega), \tag{3.3}$$

and

$$\big|[\mathcal{E}(\underline{\mathbf{u}}), \underline{\mathbf{v}}]\big| \,\leq\, \max\{1,\kappa^2\}\,\|\underline{\mathbf{u}}\|\,\|\underline{\mathbf{v}}\|, \quad [\mathcal{E}(\underline{\mathbf{v}}), \underline{\mathbf{v}}] \,\geq\, \min\{1,\kappa^2\}\,\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)} \quad \forall\,\underline{\mathbf{u}}, \underline{\mathbf{v}} \in \mathbf{H}_0^1(\Omega)\times\mathbf{L}^2(\Omega), \tag{3.4}$$

which implies that $\mathcal{B}$ and $\mathbf{F}$ are bounded and continuous, and $\mathcal{E}$ is bounded, continuous, and monotone.

On the other hand, given $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$, it is readily seen the nonlinear operator $\mathcal{A}(\mathbf{z})$ (cf. (2.13)) is bounded. More precisely, employing the Cauchy–Schwarz inequality, (2.5), and (2.6), we obtain

$$\begin{aligned}
&\big|[\mathcal{A}(\mathbf{z})(\underline{\mathbf{u}}), \underline{\mathbf{v}}]\big| \\
&\quad = \big|\mathtt{D}\,(\mathbf{u},\mathbf{v})_\Omega + \mathtt{F}\,(|\mathbf{u}|^{\rho-2}\mathbf{u},\mathbf{v})_\Omega + \nu\,(\boldsymbol{\omega},\boldsymbol{\psi})_\Omega + \nu\,(\boldsymbol{\omega},\mathbf{curl}(\mathbf{v}))_\Omega - \nu\,(\boldsymbol{\psi},\mathbf{curl}(\mathbf{u}))_\Omega + ((\nabla\mathbf{u})\mathbf{z},\mathbf{v})_\Omega\big| \\
&\quad \leq\, C_{\mathcal{A}}\,\Big\{\Big(1+\|\mathbf{z}\|_{\mathbf{H}^1(\Omega)}\Big)\,\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|^{\rho-1}_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\omega}\|_{\mathbf{L}^2(\Omega)}\Big\}\,\|\underline{\mathbf{v}}\|,
\end{aligned} \tag{3.5}$$

with $C_{\mathcal{A}} > 0$ depending on $\mathtt{D}, \mathtt{F}, \nu, \|\mathbf{i}_4\|$ and $\|\mathbf{i}_\rho\|$. In addition, using similar arguments to (2.6), it is not difficult to see that the operator $\mathbf{c}(\mathbf{z})$ (cf. (2.15)) satisfies

$$\begin{aligned}
\big|[\mathbf{c}(\mathbf{z})(\underline{\mathbf{u}}_1-\underline{\mathbf{u}}_2), \underline{\mathbf{v}}]\big| \,&\leq\, \|\mathbf{z}\|_{\mathbf{L}^4(\Omega)}\,\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbf{H}^1(\Omega)}\,\|\mathbf{v}\|_{\mathbf{L}^4(\Omega)} \\
&\leq\, \|\mathbf{i}_4\|^2\,\|\mathbf{z}\|_{\mathbf{H}^1(\Omega)}\,\|\underline{\mathbf{u}}_1-\underline{\mathbf{u}}_2\|\,\|\underline{\mathbf{v}}\| \quad \forall\,\mathbf{z}\in\mathbf{H}_0^1(\Omega),\ \forall\,\underline{\mathbf{u}}_1,\underline{\mathbf{u}}_2,\underline{\mathbf{v}} \in \mathbf{H}_0^1(\Omega)\times\mathbf{L}^2(\Omega),
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\big|[\mathbf{c}(\mathbf{z}_1-\mathbf{z}_2)(\underline{\mathbf{u}}), \underline{\mathbf{v}}]\big| \,&\leq\, \|\mathbf{z}_1-\mathbf{z}_2\|_{\mathbf{L}^4(\Omega)}\,\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}\,\|\mathbf{v}\|_{\mathbf{L}^4(\Omega)} \\
&\leq\, \|\mathbf{i}_4\|^2\,\|\mathbf{z}_1-\mathbf{z}_2\|_{\mathbf{H}^1(\Omega)}\,\|\underline{\mathbf{u}}\|\,\|\underline{\mathbf{v}}\| \quad \forall\,\mathbf{z}_1,\mathbf{z}_2\in\mathbf{H}_0^1(\Omega),\ \forall\,\underline{\mathbf{u}},\underline{\mathbf{v}} \in \mathbf{H}_0^1(\Omega)\times\mathbf{L}^2(\Omega).
\end{aligned} \tag{3.7}$$

In turn, observe that for any $\mathbf{z} \in \mathbf{K}$ (cf. (2.18)), there holds

$$[\mathbf{c}(\mathbf{z})(\underline{\mathbf{v}}), \underline{\mathbf{v}}] = 0 \quad \forall\,\underline{\mathbf{v}} \in \mathbf{H}_0^1(\Omega)\times\mathbf{L}^2(\Omega). \tag{3.8}$$

Finally, given $\mathbf{u} \in \mathbf{K}$ (cf. (2.18)) and recalling the definition of the operators $\mathcal{E}$ and $\mathcal{A}(\mathbf{u})$ (cf. (2.12), (2.13)), we note that problem (2.19) can be written in the form of (3.1) with

$$E := \mathbf{K}\times\mathbf{L}^2(\Omega), \quad u := \underline{\mathbf{u}} = (\mathbf{u},\boldsymbol{\omega}), \quad \mathcal{N} := \mathcal{E}, \quad \mathcal{M} := \mathcal{A}(\mathbf{u}). \tag{3.9}$$

Let E_b' be the Hilbert space that is the dual of $\mathbf{K} \times \mathbf{L}^2(\Omega)$ with the seminorm induced by the operator $\mathcal{E}$ (cf. (2.12)), which thanks to the fact that $\kappa > 0$, is given by

$$\|\underline{\mathbf{v}}\|_{\mathcal{E}} := \Big\{ (\mathbf{v},\mathbf{v})_\Omega + \kappa^2\,(\nabla\mathbf{v},\nabla\mathbf{v})_\Omega \Big\}^{1/2} \equiv \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \quad \forall\,\underline{\mathbf{v}} = (\mathbf{v},\boldsymbol{\psi}) \in \mathbf{H}^1_0(\Omega)\times\mathbf{L}^2(\Omega).$$

Then we define the spaces

$$E_b' := \mathbf{H}^{-1}(\Omega)\times\{\mathbf{0}\}, \quad \mathcal{D} := \Big\{ \underline{\mathbf{u}} \in \mathbf{K}\times\mathbf{L}^2(\Omega) : \quad \mathcal{M}(\underline{\mathbf{u}}) \in E_b' \Big\}. \tag{3.10}$$

In the next section we prove the hypotheses of Theorem 3.1 to establish the well-posedness of (2.19).

3.2 *Range condition*

We begin with the verification of the range condition in Theorem 3.1. Let us consider the resolvent system associated with (2.19): Find $\underline{\mathbf{u}} = (\mathbf{u},\boldsymbol{\omega}) \in \mathbf{K}\times\mathbf{L}^2(\Omega)$ such that

$$[(\mathcal{E}+\mathcal{A}(\mathbf{u}))(\underline{\mathbf{u}}),\underline{\mathbf{v}}] = [\widehat{\mathbf{F}},\underline{\mathbf{v}}] \quad \forall\,\underline{\mathbf{v}} \in \mathbf{K}\times\mathbf{L}^2(\Omega), \tag{3.11}$$

where $\widehat{\mathbf{F}} \in \mathbf{H}^{-1}(\Omega)\times\{\mathbf{0}\}$ is a functional given by $\widehat{\mathbf{F}}(\underline{\mathbf{v}}) := (\widehat{\mathbf{f}},\mathbf{v})_\Omega$ for some $\widehat{\mathbf{f}} \in \mathbf{H}^{-1}(\Omega)$. In the following two sections we prove that (3.11) has a solution by employing a suitable fixed-point approach.

3.2.1 *A fixed-point strategy.*

Let us define the operator $\mathcal{J}:\mathbf{K}\to\mathbf{K}$ by

$$\mathcal{J}(\mathbf{z}) := \mathbf{u} \quad \forall\,\mathbf{z}\in\mathbf{K}, \tag{3.12}$$

where $\mathbf{u}$ is the first component of the solution of the partially linearized version of problem (3.11): Find $\underline{\mathbf{u}} = (\mathbf{u},\boldsymbol{\omega}) \in \mathbf{K}\times\mathbf{L}^2(\Omega)$ such that

$$[(\mathcal{E}+\mathcal{A}(\mathbf{z}))(\underline{\mathbf{u}}),\underline{\mathbf{v}}] = [\widehat{\mathbf{F}},\underline{\mathbf{v}}] \quad \forall\,\underline{\mathbf{v}} \in \mathbf{K}\times\mathbf{L}^2(\Omega). \tag{3.13}$$

It is clear that $\underline{\mathbf{u}} = (\mathbf{u},\boldsymbol{\omega}) \in \mathbf{K}\times\mathbf{L}^2(\Omega)$ is a solution of problem (3.11) if and only if $\mathcal{J}(\mathbf{u}) = \mathbf{u}$. In this way, to establish existence of solution of (3.11) it suffices to prove that $\mathcal{J}$ has a fixed-point in $\mathbf{K}$.

Before proceeding with the solvability analysis of (3.11), we first establish the well-definiteness of the fixed-point operator $\mathcal{J}$. To that end, in what follows we prove the hypothesis of the Browder–Minty theorem (cf. Theorem 3.2) applied to the problem (3.13). We begin by observing that, thanks to the reflexivity and separability of $\mathrm{L}^2(\Omega)$, it follows that $\mathbf{H}^1_0(\Omega), \mathbf{L}^2(\Omega)$ and $\mathrm{L}^2_0(\Omega)$ are reflexive and separable as well.

We continue by establishing a continuity bound of the nonlinear operator $\mathcal{E}+\mathcal{A}(\mathbf{z})$.

LEMMA 3.3. Let $\mathbf{z}\in\mathbf{K}$. Then, there exists $L_{\mathrm{KV}} > 0$, depending on $\mathtt{D}, \mathtt{F}, \nu, \kappa, \rho, \|\mathbf{i}_\rho\|, \|\mathbf{i}_4\|$ and $|\Omega|$, such that

$$\begin{aligned} &\|(\mathcal{E}+\mathcal{A}(\mathbf{z}))(\underline{\mathbf{u}}) - (\mathcal{E}+\mathcal{A}(\mathbf{z}))(\underline{\mathbf{v}})\| \\ &\quad \le L_{\mathrm{KV}} \left\{ \left(1 + \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|^{\rho-2}_{\mathbf{H}^1(\Omega)} + \|\mathbf{v}\|^{\rho-2}_{\mathbf{H}^1(\Omega)}\right) \|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\omega}-\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega)} \right\}, \end{aligned} \tag{3.14}$$

for all $\underline{\mathbf{u}} = (\mathbf{u},\boldsymbol{\omega}), \underline{\mathbf{v}} = (\mathbf{v},\boldsymbol{\psi}) \in \mathbf{K}\times\mathbf{L}^2(\Omega)$.

Proof. Let $\mathbf{z} \in \mathbf{K}$ and let $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\omega}), \underline{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\psi}), \underline{\mathbf{w}} = (\mathbf{w}, \boldsymbol{\phi}) \in \mathbf{K} \times \mathbf{L}^2(\Omega)$. From the definition of the operators $\mathcal{E}, \mathcal{A}(\mathbf{z})$ (cf. (2.12), (2.13)), using the continuity bounds (3.4) and (3.6), and the Hölder and Cauchy–Schwarz inequalities, we deduce that

$$\begin{aligned}
[(\mathcal{E} + \mathcal{A}(\mathbf{z}))(\underline{\mathbf{u}}) - (\mathcal{E} + \mathcal{A}(\mathbf{z}))(\underline{\mathbf{v}}), \underline{\mathbf{w}}] \;&\leq\; \mathtt{F} \left\| |\mathbf{u}|^{\rho-2}\mathbf{u} - |\mathbf{v}|^{\rho-2}\mathbf{v} \right\|_{\mathbf{L}^{\upsilon}(\Omega)} \|\mathbf{w}\|_{\mathbf{L}^{\rho}(\Omega)} \\
&+ 2 \max\left\{1 + \mathtt{D}, \kappa^2, \nu\right\} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}\| \, \|\underline{\mathbf{w}}\| + \|\mathbf{i}_4\|^2 \, \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1(\Omega)} \|\underline{\mathbf{w}}\|, \qquad (3.15)
\end{aligned}$$

with $\upsilon \in [4/3, 3/2]$ and $1/\rho + 1/\upsilon = 1$. In turn, using Barrett & Liu (1993, Lemma 2.1, eq. (2.1a)) and the continuous injection $\mathbf{i}_\rho$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^\rho(\Omega)$ (cf. (1.2)), we deduce that there exists a constant $c_\rho > 0$, depending only on $|\Omega|$ and ρ, such that

$$\begin{aligned}
\left\| |\mathbf{u}|^{\rho-2}\mathbf{u} - |\mathbf{v}|^{\rho-2}\mathbf{v} \right\|_{\mathbf{L}^{\upsilon}(\Omega)} \|\mathbf{w}\|_{\mathbf{L}^{\rho}(\Omega)} \;&\leq\; c_\rho \left(\|\mathbf{u}\|_{\mathbf{L}^{\rho}(\Omega)} + \|\mathbf{v}\|_{\mathbf{L}^{\rho}(\Omega)} \right)^{\rho-2} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{L}^{\rho}(\Omega)} \|\mathbf{w}\|_{\mathbf{L}^{\rho}(\Omega)} \\
&\leq\; 2^{\rho-3} c_\rho \, \|\mathbf{i}_\rho\|^\rho \left(\|\mathbf{u}\|^{\rho-2}_{\mathbf{H}^1(\Omega)} + \|\mathbf{v}\|^{\rho-2}_{\mathbf{H}^1(\Omega)} \right) \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{w}\|_{\mathbf{H}^1(\Omega)}. \qquad (3.16)
\end{aligned}$$

Then, replacing back (3.16) into (3.15), and after simple computations, we obtain (3.14) with

$$L_{\mathrm{KV}} = \max\left\{2 \, \max\{1 + \mathtt{D}, \kappa^2, \nu\}, \|\mathbf{i}_4\|^2, 2^{\rho-3} \, \mathtt{F} \, \|\mathbf{i}_\rho\|^\rho \, c_\rho \right\}.$$

□

We continue our analysis by proving the coercivity and strong monotonicity of the nonlinear operator $\mathcal{E} + \mathcal{A}(\mathbf{z})$ (cf. (2.12), (2.13)).

LEMMA 3.4. Let $\mathbf{z} \in \mathbf{K}$ (cf. (2.18)). Then, there exists $\gamma_{\mathrm{KV}} > 0$, depending only on $\mathtt{D}$ and κ, such that

$$[(\mathcal{E} + \mathcal{A}(\mathbf{z}))(\underline{\mathbf{v}}), \underline{\mathbf{v}}] \;\geq\; \gamma_{\mathrm{KV}} \, \|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)} + \nu \, \|\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}, \qquad (3.17)$$

and

$$[(\mathcal{E} + \mathcal{A}(\mathbf{z}))(\underline{\mathbf{u}}) - (\mathcal{E} + \mathcal{A}(\mathbf{z}))(\underline{\mathbf{v}}), \underline{\mathbf{u}} - \underline{\mathbf{v}}] \;\geq\; \gamma_{\mathrm{KV}} \, \|\mathbf{u} - \mathbf{v}\|^2_{\mathbf{H}^1(\Omega)} + \nu \, \|\boldsymbol{\omega} - \boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}, \qquad (3.18)$$

for all $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\omega}), \underline{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\psi}) \in \mathbf{K} \times \mathbf{L}^2(\Omega)$.

Proof. Let $\mathbf{z} \in \mathbf{K}$ and let $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\omega}), \underline{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\psi}) \in \mathbf{K} \times \mathbf{L}^2(\Omega)$. Then, from the definition of the operators $\mathcal{E}, \mathcal{A}(\mathbf{z})$ (cf. (2.12), (2.13)) and the identity (3.8), we deduce that

$$\begin{aligned}
[(\mathcal{E} + \mathcal{A}(\mathbf{z}))(\underline{\mathbf{v}}), \underline{\mathbf{v}}] \;&=\; [\mathcal{E}(\underline{\mathbf{v}}), \underline{\mathbf{v}}] + [\mathbf{a}(\underline{\mathbf{v}}), \underline{\mathbf{v}}] + [\mathbf{c}(\mathbf{z})(\underline{\mathbf{v}}), \underline{\mathbf{v}}] \\
&=\; (1 + \mathtt{D}) \, \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2 \, \|\nabla \mathbf{v}\|^2_{\mathbb{L}^2(\Omega)} + \mathtt{F} \, \|\mathbf{v}\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu \, \|\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}, \qquad (3.19)
\end{aligned}$$

which, together with the fact that the term $\mathtt{F} \, \|\mathbf{v}\|^\rho_{\mathbf{L}^\rho(\Omega)}$ on the right-hand side of (3.19), which is positive, can be neglected, yields (3.17) with $\gamma_{\mathrm{KV}} := \min\{1 + \mathtt{D}, \kappa^2\}$.

On the other hand, proceeding as in (3.19) and using the fact that $\mathcal{E}$ and $\mathbf{c}(\mathbf{z})$ are linear, we get

$$[(\mathcal{E}+\mathcal{A}(\mathbf{z}))(\underline{\mathbf{u}}) - (\mathcal{E}+\mathcal{A}(\mathbf{z}))(\underline{\mathbf{v}}), \underline{\mathbf{u}}-\underline{\mathbf{v}}] \;=\; (1+\mathrm{D})\,\|\mathbf{u}-\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\,\|\nabla(\mathbf{u}-\mathbf{v})\|^2_{\mathbb{L}^2(\Omega)}$$
$$+\,\mathrm{F}\,(|\mathbf{u}|^{\rho-2}\mathbf{u} - |\mathbf{v}|^{\rho-2}\mathbf{v}, \mathbf{u}-\mathbf{v})_\Omega + \nu\,\|\boldsymbol{\omega}-\boldsymbol{\psi}\|^2_{\mathbf{L}^2(\Omega)}. \tag{3.20}$$

Thanks to (Barrett & Liu, 1993, Lemma 2.1, eq. (2.1b)), we know that there exists a constant $C_\rho > 0$, depending only on $|\Omega|$ and ρ, such that

$$\Big(|\mathbf{u}|^{\rho-2}\mathbf{u} - |\mathbf{v}|^{\rho-2}\mathbf{v}, \mathbf{u}-\mathbf{v}\Big)_\Omega \;\geq\; C_\rho\,\|\mathbf{u}-\mathbf{v}\|^\rho_{\mathbf{L}^\rho(\Omega)} \;>\; 0 \quad \forall\,\mathbf{u},\mathbf{v}\in\mathbf{L}^\rho(\Omega). \tag{3.21}$$

Thus, (3.20) yields (3.18) with the same constant γ_{KV} as in (3.17). □

Lemma 3.5. The operator $\mathcal{J}:\mathbf{K}\to\mathbf{K}$ introduced in (3.12) is well defined. In particular, for each $\mathbf{z}\in\mathbf{K}$, there exists a unique solution $\underline{\mathbf{u}} = (\mathbf{u},\boldsymbol{\omega}) \in \mathbf{K}\times\mathbf{L}^2(\Omega)$ to (3.13) and $\mathcal{J}(\mathbf{z}) = \mathbf{u}$. Moreover,

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \;\leq\; \frac{1}{\gamma_{\mathrm{KV}}}\,\|\widehat{\mathbf{f}}\|_{\mathbf{H}^{-1}(\Omega)}. \tag{3.22}$$

Proof. Let $\mathbf{z}\in\mathbf{K}$. Owing to the continuity, coercivity and strong monotonicity of the operator $\mathcal{E}+\mathcal{A}(\mathbf{z})$ (cf. Lemmas 3.3 and 3.4), the well-posedness of (3.13) is a direct consequence of the Browder–Minty theorem (cf. Theorem 3.2). This is clearly equivalent to the existence of a unique $\mathbf{u}\in\mathbf{K}$, such that $\mathcal{J}(\mathbf{z}) = \mathbf{u}$. Moreover, (3.22) follows readily by testing (3.13) with $\underline{\mathbf{v}} = \underline{\mathbf{u}}$ and using the coercivity bound of $\mathcal{E}+\mathcal{A}(\mathbf{z})$ (cf. (3.17)) and the continuity bound of $\widehat{\mathbf{F}}$ (cf. (3.3)). □

We next derive a continuity bound for the operator $\mathcal{J}$.

Lemma 3.6. For all $\mathbf{z}_1,\mathbf{z}_2\in\mathbf{K}$, there holds

$$\|\mathcal{J}(\mathbf{z}_1)-\mathcal{J}(\mathbf{z}_2)\|_{\mathbf{H}^1(\Omega)} \;\leq\; \frac{\|\mathbf{i}_4\|}{\gamma^2_{\mathrm{KV}}}\,\|\widehat{\mathbf{f}}\|_{\mathbf{H}^{-1}(\Omega)}\,\|\mathbf{z}_1-\mathbf{z}_2\|_{\mathbf{L}^4(\Omega)}. \tag{3.23}$$

Proof. Given $\mathbf{z}_1,\mathbf{z}_2\in\mathbf{K}$, we let $\mathbf{u}_1 = \mathcal{J}(\mathbf{z}_1)$ and $\mathbf{u}_2 = \mathcal{J}(\mathbf{z}_2)$. According to the definition of $\mathcal{J}$ (cf. (3.12)–(3.13)), it follows that

$$[(\mathcal{E}+\mathcal{A}(\mathbf{z}_1))(\underline{\mathbf{u}}_1) - (\mathcal{E}+\mathcal{A}(\mathbf{z}_2))(\underline{\mathbf{u}}_2), \underline{\mathbf{v}}] \;=\; 0 \quad \forall\,\underline{\mathbf{v}}\in\mathbf{K}\times\mathbf{L}^2(\Omega).$$

Taking $\underline{\mathbf{v}} = \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2$ in the above system, and recalling the definition of $\mathcal{E},\mathcal{A}(\mathbf{z})$ (cf. (2.12), (2.13)), as well as subtracting and adding the term $[\mathbf{c}(\mathbf{z}_1)(\underline{\mathbf{u}}_2), \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2]$ in order to rewrite $[\mathcal{A}(\mathbf{z}_2)(\underline{\mathbf{u}}_2), \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2]$ as $[\mathcal{A}(\mathbf{z}_1)(\underline{\mathbf{u}}_2), \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2] - [\mathbf{c}(\mathbf{z}_1-\mathbf{z}_2)(\underline{\mathbf{u}}_2), \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2]$, we obtain the identity

$$[(\mathcal{E}+\mathcal{A}(\mathbf{z}_1))(\underline{\mathbf{u}}_1) - (\mathcal{E}+\mathcal{A}(\mathbf{z}_1))(\underline{\mathbf{u}}_2), \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2] \;=\; -[\mathbf{c}(\mathbf{z}_1-\mathbf{z}_2)(\underline{\mathbf{u}}_2), \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2]. \tag{3.24}$$

Hence, using the strong monotonicity of $\mathcal{E} + \mathcal{A}(\mathbf{z})$ (cf. (3.18)) and the continuity bound of $\mathbf{c}$ (cf. (3.7)), we deduce that

$$\gamma_{\mathrm{KV}} \, \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{H}^1(\Omega)} \;\le\; \|\mathbf{i}_4\| \, \|\mathbf{u}_2\|_{\mathbf{H}^1(\Omega)} \, \|\mathbf{z}_1 - \mathbf{z}_2\|_{\mathbf{L}^4(\Omega)} \, \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{H}^1(\Omega)},$$

which, together with (3.22), implies (3.23). □

3.2.2 *Solvability analysis of the fixed-point equation.* Having proved the well-posedness of the problem (3.13), which ensures that the operator $\mathcal{J}$ is well defined, we now aim to establish existence of a fixed point of the operator $\mathcal{J}$. For this purpose, in what follows we verify the hypothesis of the Schauder fixed-point theorem in a suitable closed set.

Let $\mathbf{W}$ be the bounded and convex set defined by

$$\mathbf{W} := \left\{ \mathbf{z} \in \mathbf{K} : \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} \;\le\; \frac{1}{\gamma_{\mathrm{KV}}} \|\widehat{\mathbf{f}}\|_{\mathbf{H}^{-1}(\Omega)} \right\}. \tag{3.25}$$

The following lemma establishes the existence of a fixed point of $\mathcal{J}$ by means of the Schauder fixed point theorem.

Lemma 3.7. Let $\mathbf{W}$ be defined as in (3.25). Then the operator $\mathcal{J}$ has at least one fixed-point in $\mathbf{W}$, that is, the resolvent system (3.11) has a solution $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\omega}) \in \mathbf{W} \times \mathbf{L}^2(\Omega)$.

Proof. Given $\mathbf{z} \in \mathbf{W}$, we first recall from Lemma 3.5 that $\mathcal{J}$ is well defined and there exists a unique $\mathbf{u} \in \mathbf{K}$ such that $\mathcal{J}(\mathbf{z}) = \mathbf{u}$, which together with (3.22) implies that $\mathbf{u} \in \mathbf{W}$ and proves that $\mathcal{J}(\mathbf{W}) \subseteq \mathbf{W}$. Next, we observe from estimate (3.23) that $\mathcal{J}$ is continuous. In addition, using again (3.23), the compactness of the injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$ (see, e.g., Quarteroni & Valli, 1994, Theorem 1.3.5), and the well-known fact that every bounded sequence in a Hilbert space has a weakly convergent subsequence, we deduce that $\overline{\mathcal{J}(\mathbf{W})}$ is compact. Then, using the Schauder fixed point theorem written in the form (Ciarlet, 2013, Theorem 9.12-1(b)), we conclude that the operator $\mathcal{J}$ has at least one fixed-point in $\mathbf{W}$, that is, there exists $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\omega}) \in \mathbf{W} \times \mathbf{L}^2(\Omega)$ a solution to (3.11). □

3.3 *Construction of compatible initial data*

Now, we establish a suitable initial condition result, which is necessary to apply Theorem 3.1 to the context of (2.19).

Lemma 3.8. Assume the initial condition $\mathbf{u}_0 \in \mathbf{K}$ (cf. (2.18)). Then, there exists $\boldsymbol{\omega}_0 \in \mathbf{L}^2(\Omega)$ such that $\underline{\mathbf{u}}_0 = (\mathbf{u}_0, \boldsymbol{\omega}_0)$ and

$$\mathcal{A}(\mathbf{u}_0)(\underline{\mathbf{u}}_0) \in \mathbf{H}^{-1}(\Omega) \times \{\mathbf{0}\}. \tag{3.26}$$

Proof. We proceed as in Anaya *et al.* (2023, Lemma 3.7). In fact, we define $\boldsymbol{\omega}_0 := \mathbf{curl}(\mathbf{u}_0)$, with $\mathbf{u}_0 \in \mathbf{K}$ (cf. (2.18)). It follows that $\boldsymbol{\omega}_0 \in \mathbf{L}^2(\Omega)$. In addition, using (2.2), we get

$$\nu \, \mathbf{curl}(\boldsymbol{\omega}_0) \;=\; -\nu \, \Delta \mathbf{u}_0 \quad \text{in} \quad \Omega. \tag{3.27}$$

Next, multiplying the identities (3.27) and $\nu\,(\boldsymbol{\omega}_0 - \mathbf{curl}(\mathbf{u}_0)) = \mathbf{0}$ by $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ and $\boldsymbol{\psi} \in \mathbf{L}^2(\Omega)$, respectively, integrating by parts as in (2.7), and after minor algebraic manipulation, we obtain

$$[\mathcal{A}(\mathbf{u}_0)(\underline{\mathbf{u}}_0), \underline{\mathbf{v}}] = [\mathbf{F}_0, \underline{\mathbf{v}}] \quad \forall\, \underline{\mathbf{v}} \in \mathbf{H}^1_0(\Omega) \times \mathbf{L}^2(\Omega), \tag{3.28}$$

with $\mathbf{F}_0 = (\mathbf{f}_0, \mathbf{0})$ and

$$(\mathbf{f}_0, \mathbf{v})_\Omega := \nu\,(\nabla \mathbf{u}_0, \nabla \mathbf{v})_\Omega + \Big(\mathtt{D}\,\mathbf{u}_0 + \mathtt{F}\,|\mathbf{u}_0|^{\rho-2}\mathbf{u}_0 + (\nabla \mathbf{u}_0)\mathbf{u}_0, \mathbf{v}\Big)_\Omega,$$

which together with the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$ and $\mathbf{L}^\rho(\Omega)$, with $\rho \in [3,4]$, cf. (1.2), implies that

$$\big|(\mathbf{f}_0, \mathbf{v})_\Omega\big| \le C_0 \left\{ \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^{\rho-1}_{\mathbf{H}^1(\Omega)} \right\} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}, \tag{3.29}$$

with $C_0 := \max\left\{\nu + \mathtt{D}, \mathtt{F}\,\|\mathbf{i}_\rho\|^\rho, \|\mathbf{i}_4\|^2\right\}$. Thus, $\mathbf{F}_0 \in \mathbf{H}^{-1}(\Omega) \times \{\mathbf{0}\}$ so then (3.26) holds, completing the proof. □

Remark 3.1. The assumption on the initial condition $\mathbf{u}_0 \in \mathbf{K}$ (cf. (2.18)) is less restrictive than the one employed in Anaya *et al.* (2023, Lemma 3.7) (see also Caucao & Yotov, 2021, Lemma 3.6, Caucao *et al.*, 2022, Lemma 3.7 and Djoko & Razafimandimby, 2014, eq. (2.2)) for the analysis of the unsteady Brinkman–Forchheimer problem since the datum $\mathbf{f}_0$ is now in $\mathbf{H}^{-1}(\Omega)$ instead of $\mathbf{L}^2(\Omega)$. Note also that $\underline{\mathbf{u}}_0$ satisfying (3.26) is not unique. In addition, $(\underline{\mathbf{u}}_0, p_0) = ((\mathbf{u}_0, \mathbf{curl}(\mathbf{u}_0)), 0)$ can be chosen as initial condition for (2.11), that is, $(\underline{\mathbf{u}}_0, p_0)$ satisfy

$$[\mathcal{A}(\mathbf{u}_0)(\underline{\mathbf{u}}_0), \underline{\mathbf{v}}] + [\mathcal{B}'(p_0), \underline{\mathbf{v}}] = [\mathbf{F}_0, \underline{\mathbf{v}}] \quad \forall\, \underline{\mathbf{v}} \in \mathbf{H}^1_0(\Omega) \times \mathbf{L}^2(\Omega), \tag{3.30a}$$

$$-\,[\mathcal{B}(\underline{\mathbf{u}}_0), q] = 0 \qquad \forall\, q \in \mathrm{L}^2_0(\Omega). \tag{3.30b}$$

3.4 *Main result*

We now establish the well-posedness and stability bounds for the solution of problem (2.19).

Theorem 3.9. For each compatible initial data $\underline{\mathbf{u}}_0 = (\mathbf{u}_0, \boldsymbol{\omega}_0)$ constructed as in Lemma 3.8 and each $\mathbf{f} \in \mathrm{W}^{1,1}(0,T;\mathbf{H}^{-1}(\Omega))$, there exists a unique solution of (2.19), $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\omega}) : [0,T] \to \mathbf{K} \times \mathbf{L}^2(\Omega)$ with $\mathbf{u} \in \mathrm{W}^{1,\infty}(0,T;\mathbf{H}^{-1}(\Omega))$ and $\mathbf{u}(0) = \mathbf{u}_0$. In addition, $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 = \mathbf{curl}(\mathbf{u}_0)$ and there exists a constant $C_{\mathtt{KVr}} > 0$ only, depending on $\nu, \mathtt{D}$ and κ, such that

$$\begin{aligned}\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))} &+ \|\mathbf{u}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\omega}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} \\ &\le C_{\mathtt{KVr}}\,\sqrt{\exp(T)}\,\Big(\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{H}^{-1}(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)}\Big).\end{aligned} \tag{3.31}$$

Proof. We recall that (2.19) fits the problem in Theorem 3.1 with the definitions (3.9) and (3.10). Note that $\mathcal{N}$ is linear, symmetric and monotone since $\mathcal{E}$ is (cf. (3.4)). In addition, since $\mathcal{A}(\mathbf{u})$ is strongly monotone for any $\mathbf{u} \in \mathbf{K}$, it follows that $\mathcal{M}$ is monotone. On the other hand, from Lemma 3.7 we know that, given $(\widehat{\mathbf{f}}, \mathbf{0}) \in E'_b$, there exists $\underline{\mathbf{u}} \in \mathbf{K} \times \mathbf{L}^2(\Omega)$, such that $(\widehat{\mathbf{f}}, \mathbf{0}) = (\mathcal{N} + \mathcal{M})(\underline{\mathbf{u}})$, which implies

$Rg(\mathcal{N}+\mathcal{M}) = E_b'$. Finally, considering $\mathbf{u}_0 \in \mathbf{K}$, from a straightforward application of Lemma 3.8, we are able to find $\boldsymbol{\omega}_0 \in \mathbf{L}^2(\Omega)$ such that $\underline{\mathbf{u}}_0 = (\mathbf{u}_0, \boldsymbol{\omega}_0) \in \mathcal{D}$ and $(\mathbf{f}_0, \mathbf{0}) \in E_b'$. Therefore, applying Theorem 3.1 to our context, we conclude the existence of a solution $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\omega})$ to (2.19), with $\mathbf{u} \in \mathrm{W}^{1,\infty}(0,T;\mathbf{H}^{-1}(\Omega))$ and $\mathbf{u}(0) = \mathbf{u}_0$.

We next show the stability bound (3.31), which will be used to prove that the solution of (2.19) is unique. Indeed, to derive (3.31), we proceed as in Caucao & Yotov (2021, Theorem 3.3) and choose $\underline{\mathbf{v}} = \underline{\mathbf{u}}$ in (2.19) to get

$$\frac{1}{2}\,\partial_t \left(\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\,\|\nabla\mathbf{u}\|^2_{\mathbb{L}^2(\Omega)}\right) + [\mathcal{A}(\mathbf{u})(\underline{\mathbf{u}}), \underline{\mathbf{u}}] = (\mathbf{f}, \mathbf{u})_\Omega. \tag{3.32}$$

Next, from the definition of the operators $\mathcal{A}(\mathbf{z})$, $\mathbf{a}$ and $\mathbf{c}(\mathbf{z})$ (cf. (2.13), (2.14), (2.15)), employing similar arguments as in (3.17) (cf. (3.19)), particularly using the identity (3.8) to ensure that $[\mathbf{c}(\mathbf{u})(\underline{\mathbf{u}}), \underline{\mathbf{u}}] = 0$, together with the well-known inequality for dual norms: $(\mathbf{f}, \mathbf{u})_\Omega \le \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}$ and Young's inequality, we obtain

$$\frac{\widehat{\gamma}_{\mathrm{KV}}}{2}\,\partial_t\|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} + \mathtt{D}\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \mathtt{F}\,\|\mathbf{u}\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu\,\|\boldsymbol{\omega}\|^2_{\mathbf{L}^2(\Omega)} \le \frac{1}{2}\left(\|\mathbf{f}\|^2_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)}\right), \tag{3.33}$$

where $\widehat{\gamma}_{\mathrm{KV}} := \min\left\{1, \kappa^2\right\}$. Then, integrating (3.33) from 0 to $t \in (0, T]$, we obtain

$$\begin{aligned}\widehat{\gamma}_{\mathrm{KV}}\,\|\mathbf{u}(t)\|^2_{\mathbf{H}^1(\Omega)} &+ \int_0^t \left(2\,\mathtt{D}\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + 2\,\nu\,\|\boldsymbol{\omega}\|^2_{\mathbf{L}^2(\Omega)}\right)\mathrm{d}s \\ &\le \int_0^t \|\mathbf{f}\|^2_{\mathbf{H}^{-1}(\Omega)}\,\mathrm{d}s + \widehat{\gamma}_{\mathrm{KV}}\,\|\mathbf{u}(0)\|^2_{\mathbf{H}^1(\Omega)} + \int_0^t \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)}\,\mathrm{d}s,\end{aligned} \tag{3.34}$$

which, together with the Grönwall inequality and the fact that $\mathbf{u}(0) = \mathbf{u}_0$, yields (3.31). Notice that, in order to simplify the stability bound, we have neglected the positive term $\int_0^t \|\mathbf{u}\|^\rho_{\mathbf{L}^\rho(\Omega)}\,\mathrm{d}s$ in the left-hand side of (3.34), which also explains why the constant $C_{\mathrm{KV}\mathtt{r}}$ in (3.31) does not depend on $\mathtt{F}$.

The aforementioned uniqueness of (2.19) is now provided. In fact, let $\underline{\mathbf{u}}_i = (\mathbf{u}_i, \boldsymbol{\omega}_i)$, with $i \in \{1, 2\}$, be two solutions corresponding to the same data. Then, taking (2.19) with $\underline{\mathbf{v}} = \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2 \in \mathbf{K} \times \mathbf{L}^2(\Omega)$, subtracting the problems, we deduce that

$$\begin{aligned}&\tfrac{1}{2}\,\partial_t \left(\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\,\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2_{\mathbb{L}^2(\Omega)}\right) \\ &\quad + [\mathcal{A}(\mathbf{u}_1)(\underline{\mathbf{u}}_1) - \mathcal{A}(\mathbf{u}_1)(\underline{\mathbf{u}}_2), \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2] = -[\mathbf{c}(\mathbf{u}_1 - \mathbf{u}_2)(\underline{\mathbf{u}}_2), \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2].\end{aligned}$$

Then, using similar arguments to (3.18), the definition of the operator $\mathcal{A}(\mathbf{z})$ (cf. (2.13)), the identity (3.8), (3.21), and the continuity bound of $\mathbf{c}(\mathbf{z})$ (cf. (3.7)), we get

$$\begin{aligned}\frac{\widehat{\gamma}_{\mathrm{KV}}}{2}\,\partial_t\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{H}^1(\Omega)} &+ \mathtt{D}\,\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \nu\,\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|^2_{\mathbf{L}^2(\Omega)} \\ &\le \|\mathbf{i}_4\|^2\,\|\mathbf{u}_2\|_{\mathbf{H}^1(\Omega)}\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{H}^1(\Omega)},\end{aligned} \tag{3.35}$$

with $\widehat{\gamma}_{\mathtt{KV}}$ as in (3.33). Integrating in time (3.35) from 0 to $t \in (0,T]$, using the fact that $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))}$ is bounded by data (cf. (3.31)) in conjunction with the Grönwall inequality and algebraic manipulations, we obtain

$$\begin{aligned}\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_{\mathbf{H}^1(\Omega)} + \int_0^t \Big(\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|^2_{\mathbf{L}^2(\Omega)} \Big)\, \mathrm{d}s \\ \leq\, C\,\exp(T)\,\|\mathbf{u}_1(0) - \mathbf{u}_2(0)\|^2_{\mathbf{H}^1(\Omega)},\end{aligned}$$

with $C > 0$ depending on $\nu, \mathtt{D}, \kappa, \|\mathbf{i}_4\|$ and data. Therefore, recalling that $\mathbf{u}_1(0) = \mathbf{u}_2(0)$, it follows that $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ and $\boldsymbol{\omega}_1(t) = \boldsymbol{\omega}_2(t)$ for all $t \in (0,T]$.

Finally, since Theorem 3.1 implies that $\mathcal{M}(u) \in \mathrm{L}^\infty(0,T;E_b')$, we can take $t \to 0$ in all equations without time derivatives in (2.19). Using that the initial data $\underline{\mathbf{u}}_0 = (\mathbf{u}_0, \boldsymbol{\omega}_0)$ satisfies the same equations at $t = 0$ (cf. (3.26)), and that $\mathbf{u}(0) = \mathbf{u}_0$, we obtain

$$\nu\,(\boldsymbol{\omega}(0) - \boldsymbol{\omega}_0, \boldsymbol{\psi})_\Omega \;=\; 0 \quad \forall\, \boldsymbol{\psi} \in \mathbf{L}^2(\Omega). \tag{3.36}$$

Thus, taking $\boldsymbol{\psi} = \boldsymbol{\omega}(0) - \boldsymbol{\omega}_0$ in (3.36) we deduce that $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 = \mathbf{curl}(\mathbf{u}_0)$, completing the proof. □

We conclude this section by establishing the well-posedness and stability bounds for the solution of problem (2.11).

THEOREM 3.10. For each $\mathbf{f} \in \mathrm{W}^{1,1}(0,T;\mathbf{H}^{-1}(\Omega))$ and $\mathbf{u}_0 \in \mathbf{K}$, there exists a unique solution of (2.11), $(\underline{\mathbf{u}}, p) = ((\mathbf{u}, \boldsymbol{\omega}), p) : [0,T] \to \big(\mathbf{H}^1_0(\Omega) \times \mathbf{L}^2(\Omega)\big) \times \mathrm{L}^2_0(\Omega)$ with $\mathbf{u} \in \mathrm{W}^{1,\infty}(0,T;\mathbf{H}^{-1}(\Omega))$ and $\mathbf{u}(0) = \mathbf{u}_0$. In addition, $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 = \mathbf{curl}(\mathbf{u}_0)$ and there holds the stability bound (3.31) with the same constant $C_{\mathtt{KVr}}$ only, depending on $\nu, \mathtt{D}$ and κ. Moreover, there exists a constant $C_{\mathtt{KVp}} > 0$ only, depending on $|\Omega|, \|\mathbf{i}_\rho\|, \|\mathbf{i}_4\|, \nu, \mathtt{D}, \mathtt{F}, \kappa$ and β, such that

$$\|p\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} \;\leq\; C_{\mathtt{KVp}} \left(\sum_{j\in\{2,3,\rho\}} \Big\{ \sqrt{\exp(T)}\, \Big(\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{H}^{-1}(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} \Big) \Big\}^{j-1} + \|\mathbf{u}_0\|^{\rho/2}_{\mathbf{L}^\rho(\Omega)} \right). \tag{3.37}$$

Proof. We begin by recalling from Lemma 2.1 that the problems (2.11) and (2.19) are equivalent. Thus, the well-posedness of (2.11) follows from Theorem 3.9.

On the other hand, to derive (3.31) and (3.37), we first choose $\underline{\mathbf{v}} = \underline{\mathbf{u}}$ and $q = p$ in (2.11) to deduce (3.32)–(3.34) and consequently (3.31) also holds for the problem (2.11). In turn, starting from the inf-sup condition of $\mathcal{B}$ (cf. (2.20)), and then employing the first equation of (2.11) related to $\mathbf{v}$, the stability bounds of $\mathbf{F}, \mathcal{E}$ (cf. (3.3), (3.4)), the definition of $\mathcal{A}(\mathbf{z})$ (cf. (2.13)), and the continuous injections of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$ and $\mathbf{L}^\rho(\Omega)$, with $\rho \in [3,4]$, we deduce that

$$\begin{aligned}\beta\,\|p\|_{\mathrm{L}^2(\Omega)} &\leq \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{H}^1_0(\Omega)} \frac{[\mathbf{F}, (\mathbf{v}, \mathbf{0})] - [\partial_t\,\mathcal{E}(\underline{\mathbf{u}}), (\mathbf{v}, \mathbf{0})] - [\mathcal{A}(\mathbf{u})(\underline{\mathbf{u}}), (\mathbf{v}, \mathbf{0})]}{\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}} \\ &\leq\; \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + \mathtt{D}\,\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \nu\,\|\boldsymbol{\omega}\|_{\mathbf{L}^2(\Omega)} \\ &\quad +\; \|\mathbf{i}_4\|^2\,\|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} + \mathtt{F}\,\|\mathbf{i}_\rho\|^\rho\,\|\mathbf{u}\|^{\rho-1}_{\mathbf{H}^1(\Omega)} + (1+\kappa^2)\,\|\partial_t\,\mathbf{u}\|_{\mathbf{H}^1(\Omega)}.\end{aligned} \tag{3.38}$$

Then, taking square in (3.38) and integrating from 0 to $t \in (0,T]$, we deduce that there exits $C_1 > 0$, depending on $|\Omega|, \|\mathbf{i}_\rho\|, \|\mathbf{i}_4\|, \nu, \mathtt{F}, \mathtt{D}, \kappa$ and β, such that

$$\int_0^t \|p\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \leq C_1 \left\{ \int_0^t \left(\|\mathbf{f}\|^2_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\omega}\|^2_{\mathbf{L}^2(\Omega)} \right) \mathrm{d}s \right.$$
$$\left. + \int_0^t \left(\|\mathbf{u}\|^4_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|^{2(\rho-1)}_{\mathbf{H}^1(\Omega)} + \|\partial_t \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} \right) \mathrm{d}s \right\}. \tag{3.39}$$

Next, in order to bound the last term in (3.39), we differentiate in time the equations of (2.11) related to $\boldsymbol{\psi}$ and q, choose $(\underline{\mathbf{v}}, q) = ((\partial_t \mathbf{u}, \boldsymbol{\omega}), p)$, use (2.6) in conjunction with Cauchy–Schwarz and Young's inequalities, to find that

$$\frac{1}{2}\partial_t \left(\mathtt{D}\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathtt{F}}{\rho}\,\|\mathbf{u}\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu\,\|\boldsymbol{\omega}\|^2_{\mathbf{L}^2(\Omega)} \right) + \widehat{\gamma}_{\mathrm{KV}}\,\|\partial_t \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)}$$
$$\leq \left(\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{i}_4\|^2\,\|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} \right) \|\partial_t \mathbf{u}\|_{\mathbf{H}^1(\Omega)}$$
$$\leq C_2 \left(\|\mathbf{f}\|^2_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{u}\|^4_{\mathbf{H}^1(\Omega)} \right) + \frac{\widehat{\gamma}_{\mathrm{KV}}}{2}\,\|\partial_t \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)}, \tag{3.40}$$

with $\widehat{\gamma}_{\mathrm{KV}}$ as in (3.33) and $C_2 > 0$ only, depending on $\|\mathbf{i}_4\|$ and κ. Thus, integrating (3.40) from 0 to $t \in (0,T]$, we get

$$\mathtt{D}\,\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathtt{F}}{\rho}\,\|\mathbf{u}(t)\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu\,\|\boldsymbol{\omega}(t)\|^2_{\mathbf{L}^2(\Omega)} + \widehat{\gamma}_{\mathrm{KV}} \int_0^t \|\partial_t \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)}\,\mathrm{d}s$$
$$\leq 2C_2 \int_0^t \left(\|\mathbf{f}\|^2_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{u}\|^4_{\mathbf{H}^1(\Omega)} \right) \mathrm{d}s + \mathtt{D}\,\|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathtt{F}}{\rho}\,\|\mathbf{u}(0)\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu\,\|\boldsymbol{\omega}(0)\|^2_{\mathbf{L}^2(\Omega)}. \tag{3.41}$$

Combining (3.39) with (3.34) and (3.41), and using the fact that $(\mathbf{u}(0), \boldsymbol{\omega}(0)) = (\mathbf{u}_0, \boldsymbol{\omega}_0)$ and $\boldsymbol{\omega}_0 = \mathbf{curl}(\mathbf{u}_0)$ in Ω (cf. Lemma 3.8), we deduce that

$$\int_0^t \|p\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \leq C_3 \left\{ \int_0^t \|\mathbf{f}\|^2_{\mathbf{H}^{-1}(\Omega)}\,\mathrm{d}s + \|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^\rho_{\mathbf{L}^\rho(\Omega)} \right.$$
$$\left. + \int_0^t \left(\|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|^4_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|^{2(\rho-1)}_{\mathbf{H}^1(\Omega)} \right) \mathrm{d}s \right\}, \tag{3.42}$$

with $C_3 > 0$ only depending on $|\Omega|, \|\mathbf{i}_\rho\|, \|\mathbf{i}_4\|, \nu, \mathtt{F}, \mathtt{D}, \kappa$ and β. Finally, using (3.31) to bound $\|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)}$, $\|\mathbf{u}\|^4_{\mathbf{H}^1(\Omega)}$ and $\|\mathbf{u}\|^{2(\rho-1)}_{\mathbf{H}^1(\Omega)}$ in the left-hand side of (3.42), we obtain (3.37), concluding the proof. □

REMARK 3.2. Observe that (3.37) can be expanded to include a bound on $\|\partial_t \mathbf{u}\|_{\mathrm{L}^2(0,T;\mathbf{H}^1(\Omega))}$ and $\|\boldsymbol{\omega}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))}$, using (3.41). We also note that (3.31) will be employed in the next section to deal with the nonlinear terms associated to the operator $\mathcal{A}$ (cf. (2.13)), which is necessary to obtain the corresponding error estimate.

4. Semidiscrete continuous-in-time approximation

In this section, we introduce and analyze the semidiscrete continuous-in-time approximation of (2.11). We analyze its solvability by employing the strategy developed in Section 3. Finally, we derive the error estimates and obtain the corresponding rates of convergence.

4.1 *Existence and uniqueness of a solution*

Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω consisting of triangles K (when $d = 2$) or tetrahedra K (when $d = 3$) of diameter h_K, and define the mesh-size $h := \max\left\{h_K : K \in \mathcal{T}_h\right\}$. Let $(\mathbf{H}^{\mathbf{u}}_h, \mathrm{H}^p_h)$ be a pair of stable Stokes elements satisfying the discrete inf-sup condition: there exists a constant $\beta_{\mathtt{d}} > 0$, independent of h, such that

$$\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{H}^{\mathbf{u}}_h} \frac{\displaystyle\int_\Omega q_h\,\mathrm{div}(\mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{H}^1(\Omega)}} \geq \beta_{\mathtt{d}}\,\|q_h\|_{\mathrm{L}^2(\Omega)} \quad \forall\, q_h \in \mathrm{H}^p_h. \tag{4.1}$$

We refer the reader to Boffi *et al.* (2013) and Brezzi & Fortin (1991) for examples of stable Stokes elements. To simplify the presentation, we focus on Taylor–Hood (Taylor & Hood, 1973) finite elements for velocity and pressure, and continuous piecewise polynomials spaces for vorticity. Given an integer $l \geq 0$ and a subset S of R^d, we denote by $\mathrm{P}_l(S)$ the space of polynomials of total degree at most l defined on S. For any $k \geq 1$, we consider:

$$\begin{aligned}
\mathbf{H}^{\mathbf{u}}_h &:= \left\{\mathbf{v}_h \in [C(\overline{\Omega})]^d:\ \mathbf{v}_h|_K \in [\mathrm{P}_{k+1}(K)]^d \quad \forall\, K \in \mathcal{T}_h\right\} \cap \mathbf{H}^1_0(\Omega),\\
\mathrm{H}^p_h &:= \left\{q_h \in C(\overline{\Omega}):\ q_h|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\right\} \cap \mathrm{L}^2_0(\Omega),\\
\mathbf{H}^{\boldsymbol{\omega}}_h &:= \left\{\boldsymbol{\omega}_h \in [C(\overline{\Omega})]^{d(d-1)/2}:\ \boldsymbol{\omega}_h|_K \in [\mathrm{P}_k(K)]^{d(d-1)/2} \quad \forall\, K \in \mathcal{T}_h\right\}.
\end{aligned} \tag{4.2}$$

It is well known that the pair $(\mathbf{H}^{\mathbf{u}}_h, \mathrm{H}^p_h)$ in (4.2) satisfies (4.1) (cf. Boffi, 1994). We observe that similarly to Anaya *et al.* (2021) and Anaya *et al.* (2023), we can also consider discontinuous piecewise polynomials spaces for the vorticity, that is,

$$\mathbf{H}^{\boldsymbol{\omega}}_h := \left\{\boldsymbol{\omega}_h \in [\mathrm{L}^2(\Omega)]^{d(d-1)/2} :\ \boldsymbol{\omega}_h|_K \in [\mathrm{P}_k(K)]^{d(d-1)/2} \quad \forall\, K \in \mathcal{T}_h\right\}.$$

In addition to the Taylor–Hood elements for the velocity and pressure, in the numerical experiments in Section 6 we also consider the classical MINI-element (Boffi *et al.*, 2013, Sections 8.4.2, 8.6 and 8.7) and Crouzeix–Raviart elements with tangential jump penalization (see Crouzeix & Raviart, 1973 for the discrete inf-sup condition regarding the lowest-order case and, for instance, Carstensen & Sauter, 2022 for cubic order).

Now, defining $\underline{\mathbf{u}}_h := (\mathbf{u}_h, \boldsymbol{\omega}_h), \underline{\mathbf{v}}_h := (\mathbf{v}_h, \boldsymbol{\psi}_h) \in \mathbf{H}^{\mathbf{u}}_h \times \mathbf{H}^{\boldsymbol{\omega}}_h$, the semidiscrete continuous-in-time problem associated with (2.11) reads: Find $(\underline{\mathbf{u}}_h, p_h) : [0, T] \to \left(\mathbf{H}^{\mathbf{u}}_h \times \mathbf{H}^{\boldsymbol{\omega}}_h\right) \times \mathrm{H}^p_h$ such that, for a.e. $t \in$

$(0, T)$,

$$\frac{\partial}{\partial t}[\mathcal{E}(\underline{\mathbf{u}}_h(t)), \underline{\mathbf{v}}_h] + [\mathcal{A}_h(\mathbf{u}_h(t))(\underline{\mathbf{u}}_h(t)), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), p_h(t)] = [\mathbf{F}(t), \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}},$$

$$-[\mathcal{B}(\underline{\mathbf{u}}_h(t)), q_h] = 0 \qquad \forall\, q_h \in \mathrm{H}_h^p. \tag{4.3}$$

Here, $\mathcal{A}_h(\mathbf{z}_h) : (\mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}}) \to (\mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}})'$ is the discrete version of $\mathcal{A}(\mathbf{z})$ (with $\mathbf{z} \in \mathbf{H}_h^{\mathbf{u}}$ in place of $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$), which is defined by

$$[\mathcal{A}_h(\mathbf{z}_h)(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}_h] := [\mathbf{a}(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}_h] + [\mathbf{c}_h(\mathbf{z}_h)(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}_h], \tag{4.4}$$

where $\mathbf{c}_h(\mathbf{z}_h)$ is the well-known skew-symmetric convection form (Temam, 1977):

$$[\mathbf{c}_h(\mathbf{z}_h)(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}_h] := ((\nabla \mathbf{u}_h)\mathbf{z}_h, \mathbf{v}_h)_\Omega + \frac{1}{2}(\mathrm{div}(\mathbf{z}_h)\mathbf{u}_h, \mathbf{v}_h)_\Omega,$$

for all $\mathbf{u}_h, \mathbf{v}_h, \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$. Observe that integrating by parts, similarly to (3.8), there holds

$$[\mathbf{c}_h(\mathbf{z}_h)(\underline{\mathbf{v}}_h), \underline{\mathbf{v}}_h] = 0 \quad \forall\, \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}} \quad \text{and} \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}}. \tag{4.5}$$

As initial condition we take $(\underline{\mathbf{u}}_{h,0}, p_{h,0}) = ((\mathbf{u}_{h,0}, \boldsymbol{\omega}_{h,0}), p_{h,0})$ to be a suitable approximations of $(\underline{\mathbf{u}}_0, p_0) = ((\mathbf{u}_0, \boldsymbol{\omega}_0), 0)$, the solution of a slight modification of (3.30), that is, we chose $(\underline{\mathbf{u}}_{h,0}, p_{h,0})$, solving

$$\begin{aligned} (\nabla \mathbf{u}_{h,0}, \nabla \mathbf{v}_h)_\Omega + [\mathcal{A}_h(\mathbf{u}_{h,0})(\underline{\mathbf{u}}_{h,0}), \underline{\mathbf{v}}_h] + [\mathcal{B}'(p_{h,0}), \underline{\mathbf{v}}_h] &= [\mathbf{F}_0, \underline{\mathbf{v}}_h] + (\nabla \mathbf{u}_0, \nabla \mathbf{v}_h)_\Omega, \\ -[\mathcal{B}(\underline{\mathbf{u}}_{h,0}), q_h] &= 0, \end{aligned} \tag{4.6}$$

for all $\underline{\mathbf{v}}_h \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}}$ and $q_h \in \mathrm{H}_h^p$. The well-posedness of (4.6) follows from the discrete inf-sup condition (4.15) and similar arguments to the proof of Lemma 3.7. Alternatively, we can proceed as in Anaya *et al.* (2023, eq. (4.4)) and apply a fixed-point strategy in conjunction with Caucao *et al.* (2021, Theorem 3.1) to ensure existence and uniqueness of (4.6).

Next, we introduce the discrete kernel of $\mathcal{B}$, that is,

$$\mathbf{V}_h := \mathbf{K}_h \times \mathbf{H}_h^{\boldsymbol{\omega}}, \quad \text{where} \quad \mathbf{K}_h = \left\{\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}} : (q_h, \mathrm{div}(\mathbf{v}_h))_\Omega = 0 \quad \forall\, q_h \in \mathrm{H}_h^p\right\}. \tag{4.7}$$

Then, we can introduce the reduced problem: Given $\mathbf{f} : [0, T] \to \mathbf{H}^{-1}(\Omega)$, find $\underline{\mathbf{u}}_h : [0, T] \to \mathbf{K}_h \times \mathbf{H}_h^{\boldsymbol{\omega}}$ such that, for a.e. $t \in (0, T)$,

$$\frac{\partial}{\partial t}[\mathcal{E}(\underline{\mathbf{u}}_h(t)), \underline{\mathbf{v}}_h] + [\mathcal{A}_h(\mathbf{u}_h(t))(\underline{\mathbf{u}}_h(t)), \underline{\mathbf{v}}_h] = [\mathbf{F}(t), \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{K}_h \times \mathbf{H}_h^{\boldsymbol{\omega}}, \tag{4.8}$$

which, using (4.1) and similarly to Lemma 2.1, is equivalent to (4.3). As a preliminary initial condition for (4.8) we take $\underline{\mathbf{u}}_{h,0} := (\mathbf{u}_{h,0}, \boldsymbol{\omega}_{h,0})$ to be solution of the reduced problem associated to (4.6), that is, we chose $\underline{\mathbf{u}}_{h,0}$, solving

$$(\nabla \mathbf{u}_{h,0}, \nabla \mathbf{v}_h)_\Omega + [\mathcal{A}_h(\mathbf{u}_{h,0})(\underline{\mathbf{u}}_{h,0}), \underline{\mathbf{v}}_h] = [\mathbf{F}_0, \underline{\mathbf{v}}_h] + (\nabla \mathbf{u}_0, \nabla \mathbf{v}_h)_\Omega \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{K}_h \times \mathbf{H}_h^{\boldsymbol{\omega}}, \tag{4.9}$$

with $\mathbf{F}_0 \in \mathbf{H}^{-1}(\Omega) \times \{\mathbf{0}\}$ being the right-hand side of (3.28). Notice that the well-posedness of problem (4.9) follows from similar arguments to those employed in the proof of Lemma 3.7. In addition, taking $\underline{\mathbf{v}}_h = \underline{\mathbf{u}}_{h,0}$ in (4.9), we deduce from the definition of the operator $\mathcal{A}_h(\mathbf{u}_{h,0})$ (cf. (4.4)), the identity (4.5), and the continuity bound of $\mathbf{F}_0$ (cf. (3.29)), that there exists a constant $\widehat{C}_0 > 0$, depending only on $|\Omega|$, $\|\mathbf{i}_\rho\|$, $\|\mathbf{i}_4\|$, ν, $\mathtt{D}$ and $\mathtt{F}$, and hence independent of h, such that

$$\|\mathbf{u}_{h,0}\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_{h,0}\|^\rho_{\mathbf{L}^\rho(\Omega)} + \|\boldsymbol{\omega}_{h,0}\|^2_{\mathbf{L}^2(\Omega)} \le \widehat{C}_0 \left\{ \|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^4_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^{2(\rho-1)}_{\mathbf{H}^1(\Omega)} \right\}. \tag{4.10}$$

Thus, from (4.9), we deduce an initial condition for (4.8), that is, $\underline{\mathbf{u}}_{h,0} := (\mathbf{u}_{h,0}, \boldsymbol{\omega}_{h,0})$, the solution of

$$[\mathcal{A}_h(\mathbf{u}_{h,0})(\underline{\mathbf{u}}_{h,0}), \underline{\mathbf{v}}_h] = [\mathbf{F}_{h,0}, \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{K}_h \times \mathbf{H}_h^{\boldsymbol{\omega}}, \tag{4.11}$$

with $\mathbf{F}_{h,0} = (\mathbf{f}_{h,0}, \mathbf{0})$ and $(\mathbf{f}_{h,0}, \mathbf{v}_h)_\Omega := (\mathbf{f}_0, \underline{\mathbf{v}}_h)_\Omega + (\nabla \mathbf{u}_0, \nabla \mathbf{v}_h)_\Omega - (\nabla \mathbf{u}_{h,0}, \nabla \mathbf{v}_h)_\Omega$, which, thanks to (3.29) and (4.10), yields

$$\left|(\mathbf{f}_{h,0}, \mathbf{v}_h)_\Omega\right| \le C_{\mathrm{d},0} \left\{ \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^{\rho-1}_{\mathbf{H}^1(\Omega)} \right\} \|\mathbf{v}_h\|_{\mathbf{H}^1(\Omega)}, \tag{4.12}$$

with $C_{\mathrm{d},0} > 0$, depending only on $|\Omega|$, $\|\mathbf{i}_\rho\|$, $\|\mathbf{i}_4\|$, ν, $\mathtt{D}$ and $\mathtt{F}$. Thus, $\mathbf{F}_{h,0} \in \mathbf{H}^{-1}(\Omega) \times \{\mathbf{0}\}$. We observe that this choice is necessary to guarantee that the discrete initial data is compatible in the sense of Lemma 3.8, which is needed for the application of Theorem 3.1.

In this way, the well-posedness of (4.8) (equivalently of (4.3)), follows analogously to its continuous counterpart provided in Theorem 3.9. More precisely, we first address the discrete counterparts of Lemmas 3.3 and 3.4, whose proofs, being almost verbatim of the continuous ones, are omitted.

LEMMA 4.1. Let $\mathbf{z}_h \in \mathbf{K}_h$ (cf. (4.7)). Then, with the same constant γ_{KV} defined in (3.17), there holds

$$[(\mathcal{E} + \mathcal{A}_h(\mathbf{z}_h))(\underline{\mathbf{v}}_h), \underline{\mathbf{v}}_h] \ge \gamma_{\mathrm{KV}} \|\mathbf{v}_h\|^2_{\mathbf{H}^1(\Omega)} + \nu\, \|\boldsymbol{\psi}_h\|^2_{\mathbf{L}^2(\Omega)}, \tag{4.13}$$

and

$$[(\mathcal{E} + \mathcal{A}_h(\mathbf{z}_h))(\underline{\mathbf{u}}_h) - (\mathcal{E} + \mathcal{A}_h(\mathbf{z}_h))(\underline{\mathbf{v}}_h), \underline{\mathbf{u}}_h - \underline{\mathbf{v}}_h] \ge \gamma_{\mathrm{KV}} \|\mathbf{u}_h - \mathbf{v}_h\|^2_{\mathbf{H}^1(\Omega)} + \nu\, \|\boldsymbol{\omega}_h - \boldsymbol{\psi}_h\|^2_{\mathbf{L}^2(\Omega)}, \tag{4.14}$$

for all $\underline{\mathbf{u}}_h = (\mathbf{u}_h, \boldsymbol{\omega}_h)$, $\underline{\mathbf{v}}_h = (\mathbf{v}_h, \boldsymbol{\psi}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}}$. In addition, the operator $\mathcal{E} + \mathcal{A}_h : (\mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}}) \to (\mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}})'$ is continuous in the sense of (3.14), but with the constant

$$L_{\mathrm{KV,d}} = \max\left\{ 2\, \max\{1 + \mathtt{D}, \kappa^2, \nu\}, \|\mathbf{i}_4\|^2 \left(1 + \frac{\sqrt{d}}{2}\right), 2^{\rho-3}\, \mathtt{F}\, \|\mathbf{i}_\rho\|^\rho\, c_\rho \right\}.$$

We continue with the discrete inf-sup condition of $\mathcal{B}$.

Lemma 4.2. It holds that

$$\sup_{\mathbf{0}\neq\underline{\mathbf{v}}_h\in\mathbf{H}_h^{\mathbf{u}}\times\mathbf{H}_h^{\boldsymbol{\omega}}} \frac{[\mathcal{B}(\underline{\mathbf{v}}_h),q_h]}{\|\underline{\mathbf{v}}_h\|} \geq \beta_{\mathtt{d}}\,\|q_h\|_{\mathrm{L}^2(\Omega)} \quad \forall\, q_h\in\mathrm{H}_h^p, \tag{4.15}$$

where $\beta_{\mathtt{d}}$ is the inf-sup constant from (4.1).

Proof. The statement follows directly from (4.1). □

We are now in a position to establish the semi-discrete continuous in time analogue of Theorems 3.9 and 3.10. To that end, we first introduce the ball of $\mathbf{K}_h$ given by

$$\mathbf{W}_{\mathtt{d}} := \left\{ \mathbf{z}_h\in\mathbf{K}_h : \quad \|\mathbf{z}_h\|_{\mathbf{H}^1(\Omega)} \leq \frac{1}{\gamma_{\mathrm{KV}}}\|\widehat{\mathbf{f}}\|_{\mathbf{H}^{-1}(\Omega)} \right\}. \tag{4.16}$$

The aforementioned result is stated now.

Theorem 4.3. For each compatible initial data $(\underline{\mathbf{u}}_{h,0},p_{h,0}) := ((\mathbf{u}_{h,0},\boldsymbol{\omega}_{h,0}),p_{h,0})$ satisfying (4.11) and $\mathbf{f}\in\mathrm{W}^{1,1}(0,T;\mathbf{H}^{-1}(\Omega))$, there exists a unique solution to (4.3), $(\underline{\mathbf{u}}_h,p_h) = ((\mathbf{u}_h,\boldsymbol{\omega}_h),p_h):[0,T]\to(\mathbf{H}_h^{\mathbf{u}}\times\mathbf{H}_h^{\boldsymbol{\omega}})\times\mathrm{H}_h^p$, with $\mathbf{u}_h\in\mathrm{W}^{1,\infty}(0,T;\mathbf{H}_h^{\mathbf{u}})$ and $(\mathbf{u}_h(0),\boldsymbol{\omega}_h(0)) = (\mathbf{u}_{h,0},\boldsymbol{\omega}_{h,0})$. Moreover, there exists a constant $\widehat{C}_{\mathtt{KVr}}>0$, depending only on $|\Omega|$, $\|\mathbf{i}_\rho\|$, $\|\mathbf{i}_4\|$, ν, $\mathtt{D}$, $\mathtt{F}$ and κ, such that

$$\begin{aligned}
&\|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))} + \|\mathbf{u}_h\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\omega}_h\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} \\
&\qquad \leq \widehat{C}_{\mathtt{KVr}}\sqrt{\exp(T)}\,\Big(\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{H}^{-1}(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^{\rho-1}_{\mathbf{H}^1(\Omega)}\Big),
\end{aligned} \tag{4.17}$$

and a constant $\widehat{C}_{\mathtt{KVp}}>0$, depending only on $|\Omega|$, $\|\mathbf{i}_\rho\|$, $\|\mathbf{i}_4\|$, ν, $\mathtt{D}$, $\mathtt{F}$, κ and $\beta_{\mathtt{d}}$, such that

$$\begin{aligned}
&\|p_h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} \\
&\qquad \leq \widehat{C}_{\mathtt{KVp}}\sum_{j\in\{2,3,\rho\}}\left\{\sqrt{\exp(T)}\,\Big(\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{H}^{-1}(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^{\rho-1}_{\mathbf{H}^1(\Omega)}\Big)\right\}^{j-1}.
\end{aligned} \tag{4.18}$$

Proof. According to Lemma 4.1, the discrete inf-sup condition for $\mathcal{B}$ provided by (4.15) in Lemma 4.2, a fixed-point approach as the one used in Lemma 3.7, but now with $\mathbf{W}_{\mathtt{d}}$ (cf. (4.16)), and considering that $(\underline{\mathbf{u}}_{h,0},p_{h,0})$ satisfies (4.11), the proof of existence and uniqueness of solution of (4.8) (equivalently of (4.3)) with $\mathbf{u}_h\in\mathrm{W}^{1,\infty}(0,T;\mathbf{H}_h^{\mathbf{u}})$ and $\mathbf{u}_h(0)=\mathbf{u}_{h,0}$, follows similarly to the proof of Theorem 3.9 by applying Theorem 3.1. Moreover, from the discrete version of (3.36), we deduce that $\boldsymbol{\omega}_h(0)=\boldsymbol{\omega}_{h,0}$.

On the other hand, mimicking the steps followed in the proof of Theorems 3.9 and 3.10, we obtain, respectively, the discrete versions of (3.32)–(3.34) and (3.38)–(3.41). Then, using the fact that $(\mathbf{u}_h(0),\boldsymbol{\omega}_h(0)) = (\mathbf{u}_{h,0},\boldsymbol{\omega}_{h,0})$ and (4.10), we derive (4.17) and (4.18), thus completing the proof. □

4.2 *Error analysis*

Now we derive suitable error estimates for the semidiscrete scheme (4.3). To that end, we first recall that the discrete inf-sup condition of $\mathcal{B}$ (cf. (4.15)), along with a classical result on mixed methods (see, e.g., Gatica, 2014, eq. (2.89) in Theorem 2.6) ensure the existence of a constant $C > 0$, independent of h, such that:

$$\inf_{\underline{\mathbf{v}}_h\in\mathbf{V}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\| \leq C \inf_{\underline{\mathbf{v}}_h\in\mathbf{H}_h^{\mathbf{u}}\times\mathbf{H}_h^{\boldsymbol{\omega}}} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|. \tag{4.19}$$

Next, in order to obtain the theoretical rates of convergence for the discrete scheme (4.3), we recall the approximation properties of the finite element subspaces $\mathbf{H}_h^{\mathbf{u}}, \mathbf{H}_h^{\boldsymbol{\omega}}$ and H_h^p (cf. (4.2)) that can be found in Brezzi & Fortin (1991), Ern & Guermond (2004) and Boffi *et al.* (2013). Assume that $\mathbf{u} \in \mathbf{H}^{1+s}(\Omega), \boldsymbol{\omega} \in [\mathrm{H}^s(\Omega)]^{d(d-1)/2}$ and $p \in \mathrm{H}^s(\Omega)$, for some $s \in (1/2, k+1]$. Then there exists $C > 0$, independent of h, such that

$$\inf_{\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{u}}} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}^1(\Omega)} \leq C\, h^s\, \|\mathbf{u}\|_{\mathbf{H}^{1+s}(\Omega)}, \tag{4.20}$$

$$\inf_{\boldsymbol{\psi}_h\in\mathbf{H}_h^{\boldsymbol{\omega}}} \|\boldsymbol{\omega} - \boldsymbol{\psi}_h\|_{\mathbf{L}^2(\Omega)} \leq C\, h^s\, \|\boldsymbol{\omega}\|_{\mathbf{H}^s(\Omega)}, \tag{4.21}$$

$$\inf_{q_h\in\mathrm{H}_h^p} \|p - q_h\|_{\mathrm{L}^2(\Omega)} \leq C\, h^s\, \|p\|_{\mathrm{H}^s(\Omega)}. \tag{4.22}$$

Owing to (4.19) and (4.20)–(4.22), it follows that, under an extra regularity assumption on the exact solution, there exist positive constants $C(\underline{\mathbf{u}})$, $C(\partial_t\,\underline{\mathbf{u}})$, $C(p)$ and $C(\partial_t\, p)$, depending on $\mathbf{u}, \boldsymbol{\omega}$ and p, respectively, such that

$$\begin{gathered}\inf_{\underline{\mathbf{v}}_h\in\mathbf{V}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\| \leq C(\underline{\mathbf{u}})\, h^s, \quad \inf_{\underline{\mathbf{v}}_h\in\mathbf{V}_h} \|\partial_t\,\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\| \leq C(\partial_t\,\underline{\mathbf{u}})\, h^s, \\ \inf_{q_h\in\mathrm{H}_h^p} \|p - q_h\|_{\mathrm{L}^2(\Omega)} \leq C(p)\, h^s \quad\text{and}\quad \inf_{q_h\in\mathrm{H}_h^p} \|\partial_t\, p - q_h\|_{\mathrm{L}^2(\Omega)} \leq C(\partial_t\, p)\, h^s.\end{gathered} \tag{4.23}$$

In turn, in order to simplify the subsequent analysis, we write $\mathbf{e}_{\underline{\mathbf{u}}} = (\mathbf{e}_{\mathbf{u}}, \mathbf{e}_{\boldsymbol{\omega}}) = (\mathbf{u} - \mathbf{u}_h, \boldsymbol{\omega} - \boldsymbol{\omega}_h)$ and $\mathbf{e}_p = p - p_h$. Next, given arbitrary $\widehat{\underline{\mathbf{v}}}_h := (\widehat{\mathbf{v}}_h, \widehat{\boldsymbol{\psi}}_h) \colon [0,T] \to \mathbf{V}_h$ (cf. (4.7)) and $\widehat{q}_h \colon [0,T] \to \mathrm{H}_h^p$, as usual, we shall decompose the errors into

$$\mathbf{e}_{\underline{\mathbf{u}}} = \boldsymbol{\delta}_{\underline{\mathbf{u}}} + \boldsymbol{\eta}_{\underline{\mathbf{u}}} = (\boldsymbol{\delta}_{\mathbf{u}}, \boldsymbol{\delta}_{\boldsymbol{\omega}}) + (\boldsymbol{\eta}_{\mathbf{u}}, \boldsymbol{\eta}_{\boldsymbol{\omega}}), \quad \mathbf{e}_p = \boldsymbol{\delta}_p + \boldsymbol{\eta}_p, \tag{4.24}$$

with

$$\begin{gathered}\boldsymbol{\delta}_{\mathbf{u}} = \mathbf{u} - \widehat{\mathbf{v}}_h, \quad \boldsymbol{\delta}_{\boldsymbol{\omega}} = \boldsymbol{\omega} - \widehat{\boldsymbol{\psi}}_h, \quad \boldsymbol{\delta}_p = p - \widehat{q}_h, \\ \boldsymbol{\eta}_{\mathbf{u}} = \widehat{\mathbf{v}}_h - \mathbf{u}_h, \quad \boldsymbol{\eta}_{\boldsymbol{\omega}} = \widehat{\boldsymbol{\psi}}_h - \boldsymbol{\omega}_h, \quad \boldsymbol{\eta}_p = \widehat{q}_h - p_h.\end{gathered} \tag{4.25}$$

In addition, we stress for later use that for each $\underline{\mathbf{v}}_h : [0,T] \to \mathbf{V}_h$ (cf. (4.7)) it holds that $\partial_t\,\underline{\mathbf{v}}_h(t) \in \mathbf{V}_h$. In fact, given $(\underline{\mathbf{v}}_h, q_h) : [0,T] \to \mathbf{V}_h \times \mathrm{H}_h^p$, after simple algebraic computations, we obtain

$$[\mathcal{B}(\partial_t\,\underline{\mathbf{v}}_h), q_h] = \partial_t\left([\mathcal{B}(\underline{\mathbf{v}}_h), q_h]\right) - [\mathcal{B}(\underline{\mathbf{v}}_h), \partial_t\, q_h] = 0, \tag{4.26}$$

where, the latter is obtained by observing that $\partial_t\, q_h(t) \in \mathrm{H}_h^p$.

Finally, since the exact solution $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ satisfies $\mathrm{div}(\mathbf{u}) = 0$ in Ω, we have

$$[\mathcal{A}_h(\mathbf{u})(\underline{\mathbf{u}}), \underline{\mathbf{v}}_h] = [\mathcal{A}(\mathbf{u})(\underline{\mathbf{u}}), \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}}.$$

In this way, by subtracting the discrete and continuous problems (2.11) and (4.3), respectively, we obtain the following error system:

$$\frac{\partial}{\partial t}\left[\mathcal{E}(\mathbf{e}_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h\right] + \left[\mathcal{A}_h(\mathbf{u})(\underline{\mathbf{u}}) - \mathcal{A}_h(\mathbf{u}_h)(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}_h\right] + \left[\mathcal{B}(\underline{\mathbf{v}}_h), \mathbf{e}_p\right] = 0 \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}},$$
$$\left[\mathcal{B}(\mathbf{e}_{\underline{\mathbf{u}}}), q_h\right] = 0 \qquad \forall\, q_h \in \mathrm{H}_h^p. \tag{4.27}$$

We now establish the main result of this section, namely, the theoretical rate of convergence of the semidiscrete scheme (4.3). Note that optimal rates of convergences are obtained for all the unknowns.

THEOREM 4.4. Let $((\mathbf{u}, \boldsymbol{\omega}), p) : [0, T] \to \left(\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)\right) \times \mathrm{L}_0^2(\Omega)$ with $\mathbf{u} \in \mathrm{W}^{1,\infty}(0, T; \mathbf{H}^{-1}(\Omega))$ and $((\mathbf{u}_h, \boldsymbol{\omega}_h), p_h) : [0, T] \to \left(\mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}}\right) \times \mathrm{H}_h^p$ with $\mathbf{u}_h \in \mathrm{W}^{1,\infty}(0, T; \mathbf{H}_h^{\mathbf{u}})$ be the unique solutions of the continuous and semidiscrete problems (2.11) and (4.3), respectively. Assume further that there exists $s \in (1/2, k+1]$, such that $\mathbf{u} \in \mathbf{H}^{1+s}(\Omega)$, $\boldsymbol{\omega} \in [\mathrm{H}^s(\Omega)]^{d(d-1)/2}$ and $p \in \mathrm{H}^s(\Omega)$. Then, there exists $C(\underline{\mathbf{u}}, p) > 0$, depending only on $C(\underline{\mathbf{u}}), C(\partial_t \underline{\mathbf{u}}), C(p), C(\partial_t p), \|\mathbf{i}_\rho\|, \|\mathbf{i}_4\|, |\Omega|, \nu, \mathtt{D}, \mathtt{F}, \kappa, \beta_{\mathtt{d}}, T, \|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{H}^{-1}(\Omega))}$ and $\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)}$, such that

$$\|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\boldsymbol{\omega}}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))}$$
$$+ \|\mathbf{e}_p\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} \leq C(\underline{\mathbf{u}}, p)\left(h^s + h^{s(\rho-1)}\right). \tag{4.28}$$

Proof. First, adding and subtracting suitable terms in the first equation of (4.27), with $\underline{\mathbf{v}}_h = \boldsymbol{\eta}_{\underline{\mathbf{u}}} = (\boldsymbol{\eta}_{\mathbf{u}}, \boldsymbol{\eta}_{\boldsymbol{\omega}}) : [0, T] \to \mathbf{V}_h$ (cf. (4.7)), and using the decomposition

$$[\mathcal{A}_h(\mathbf{u})(\underline{\mathbf{u}}) - \mathcal{A}_h(\mathbf{u}_h)(\underline{\mathbf{u}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] = [\mathcal{A}_h(\mathbf{u}_h)(\widehat{\underline{\mathbf{v}}}_h) - \mathcal{A}_h(\mathbf{u}_h)(\underline{\mathbf{u}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] + [\mathcal{A}_h(\mathbf{u})(\underline{\mathbf{u}}) - \mathcal{A}_h(\mathbf{u})(\widehat{\underline{\mathbf{v}}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}]$$
$$+ [\mathbf{c}_h(\mathbf{u} - \mathbf{u}_h)(\underline{\mathbf{u}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] + [\mathbf{c}_h(\mathbf{u})(\boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] - [\mathbf{c}_h(\mathbf{u}_h)(\boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\eta}_{\underline{\mathbf{u}}}], \tag{4.29}$$

where the last two terms can be neglected thanks to the identity (4.5), proceeding as in (4.14) to bound the first term in the right-hand side of (4.29), and using the definitions of the operators $\mathcal{E}$ and $\mathcal{B}$ (cf. (2.12), (2.16)), together with the fact that $\boldsymbol{\eta}_{\underline{\mathbf{u}}}(t) \in \mathbf{V}_h$, thus $[\mathcal{B}(\boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\eta}_p] = 0$, we deduce that

$$\frac{1}{2}\,\partial_t\left(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\|\nabla\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbb{L}^2(\Omega)}\right) + \mathtt{D}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \mathtt{F}\,C_\rho\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu\,\|\boldsymbol{\eta}_{\boldsymbol{\omega}}\|^2_{\mathbf{L}^2(\Omega)}$$
$$\leq -(\partial_t\boldsymbol{\delta}_{\mathbf{u}}, \boldsymbol{\eta}_{\mathbf{u}})_\Omega - \kappa^2\,(\partial_t\nabla\boldsymbol{\delta}_{\mathbf{u}}, \nabla\boldsymbol{\eta}_{\mathbf{u}})_\Omega - [\mathcal{A}_h(\mathbf{u})(\underline{\mathbf{u}}) - \mathcal{A}_h(\mathbf{u})(\widehat{\underline{\mathbf{v}}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}]$$
$$- [\mathbf{c}_h(\mathbf{u} - \mathbf{u}_h)(\underline{\mathbf{u}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] + (\boldsymbol{\delta}_p, \mathrm{div}(\boldsymbol{\eta}_{\mathbf{u}}))_\Omega. \tag{4.30}$$

The terms on the right-hand side can be bounded using the Cauchy–Schwarz, Hölder and Young's inequalities (cf. (1.1)), (3.14), as follows:

$$- (\partial_t \boldsymbol{\delta}_{\mathbf{u}}, \boldsymbol{\eta}_{\mathbf{u}})_\Omega - \kappa^2 (\partial_t \nabla \boldsymbol{\delta}_{\mathbf{u}}, \nabla \boldsymbol{\eta}_{\mathbf{u}})_\Omega \le \frac{\max\{1,\kappa^2\}}{2} \Big(\|\partial_t \boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} \Big), \tag{4.31}$$

$$\begin{aligned} &- [\mathcal{A}_h(\mathbf{u})(\underline{\mathbf{u}}) - \mathcal{A}_h(\mathbf{u})(\underline{\widehat{\mathbf{v}}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] \\ &\le \widetilde{C}_1 \Big\{ \Big(1 + \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|^{\rho-2}_{\mathbf{H}^1(\Omega)} - \|\widehat{\mathbf{v}}_h\|^{\rho-2}_{\mathbf{H}^1(\Omega)} \Big) \|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\omega}}\|_{\mathrm{L}^2(\Omega)} \Big\} \Big(\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\omega}}\|_{\mathbf{L}^2(\Omega)} \Big) \\ &\le \widehat{C}_1 \Big\{ \Big(1 + \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|^{\rho-2}_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^{\rho-2}_{\mathbf{H}^1(\Omega)} \Big) \|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\omega}}\|_{\mathrm{L}^2(\Omega)} \Big\} \Big(\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\omega}}\|_{\mathbf{L}^2(\Omega)} \Big) \\ &\le C_1 \Big\{ \Big(1 + \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|^{2(\rho-2)}_{\mathbf{H}^1(\Omega)} \Big) \|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^{2(\rho-1)}_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\omega}}\|^2_{\mathrm{L}^2(\Omega)} \Big\} \\ &\quad + \Big(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} + \frac{\nu}{2} \|\boldsymbol{\eta}_{\boldsymbol{\omega}}\|^2_{\mathbf{L}^2(\Omega)} \Big), \end{aligned} \tag{4.32}$$

$$\begin{aligned} - [\mathbf{c}_h(\mathbf{u} - \mathbf{u}_h)(\underline{\mathbf{u}}_h), \boldsymbol{\eta}_{\mathbf{u}}] &\le \left(1 + \frac{\sqrt{d}}{2} \right) \|\mathbf{i}_4\|^2 \, \|\mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} \Big(\|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} \Big) \|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}, \\ &\le C_2 \|\mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} \Big(\|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} \Big), \end{aligned} \tag{4.33}$$

$$(\delta_p, \mathrm{div}(\boldsymbol{\eta}_{\mathbf{u}}))_\Omega \le \frac{\sqrt{d}}{2} \Big(\|\delta_p\|^2_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} \Big), \tag{4.34}$$

where $C_1, C_2 > 0$ depend on $\|\mathbf{i}_4\|, \|\mathbf{i}_\rho\|, \kappa, \mathtt{D}, \mathtt{F}$ and ν. We note that in (4.32), we used the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^\rho(\Omega)$, with $\rho \in [3,4]$, cf. (1.2). Combining (4.30)–(4.34), and neglecting the term $\|\boldsymbol{\eta}_{\mathbf{u}}\|^\rho_{\mathbf{L}^\rho(\Omega)}$ in (4.30) to simplify the error estimate, we obtain

$$\begin{aligned} &\partial_t \Big(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2 \|\nabla \boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbb{L}^2(\Omega)} \Big) + \mathtt{D} \, \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \nu \, \|\boldsymbol{\eta}_{\boldsymbol{\omega}}\|^2_{\mathbf{L}^2(\Omega)} \\ &\le C_3 \Big(\|\partial_t \, \boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} + \Big(1 + \|\mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|^{2(\rho-2)}_{\mathbf{H}^1(\Omega)} \Big) \|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} \\ &\quad + \|\boldsymbol{\delta}_{\mathbf{u}}\|^{2(\rho-1)}_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\omega}}\|^2_{\mathbf{L}^2(\Omega)} + \|\delta_p\|^2_{\mathrm{L}^2(\Omega)} + \Big(1 + \|\mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} \Big) \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} \Big), \end{aligned} \tag{4.35}$$

with C_3 a positive constant, depending on $|\Omega|, \|\mathbf{i}_4\|, \|\mathbf{i}_\rho\|, \nu, \mathtt{D}, \mathtt{F}$ and κ. Integrating (4.35) from 0 to $t \in (0,T]$, recalling that $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))}$ and $\|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))}$ are bounded by data (cf. (3.31), (4.17)), we find that

$$\begin{aligned} &\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|^2_{\mathbf{H}^1(\Omega)} + \int_0^t \Big(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\omega}}\|^2_{\mathbf{L}^2(\Omega)} \Big) \, \mathrm{d}s \le C_4 \Big\{ \int_0^t \Big(\|\partial_t \, \boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 \Big) \, \mathrm{d}s \\ &\quad + \int_0^t \Big(\|\boldsymbol{\delta}_{\mathbf{u}}\|^{2(\rho-1)}_{\mathbf{H}^1(\Omega)} + \|\delta_p\|^2_{\mathrm{L}^2(\Omega)} \Big) \, \mathrm{d}s \Big\} + \widehat{C}_4 \Big\{ \int_0^t \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} \, \mathrm{d}s + \|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{H}^1(\Omega)} \Big\}, \end{aligned} \tag{4.36}$$

with $C_4, \widehat{C}_4 > 0$ depending on $|\Omega|, \|\mathbf{i}_4\|, \|\mathbf{i}_\rho\|, \nu, \mathtt{D}, \mathtt{F}, \kappa$ and data.

On the other hand, to estimate $\|\mathbf{e}_p\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))}$, we observe that from the discrete inf-sup condition of $\mathcal{B}$ (cf. (4.15)), the first equation of (4.27), and the continuity bounds of $\mathcal{B}, \mathcal{E}, \mathcal{A}_h$ (cf. (3.2), (3.4), (3.14)), there holds

$$\beta_{\mathtt{d}}\,\|\boldsymbol{\eta}_p\|_{\mathrm{L}^2(\Omega)} \le \sup_{\mathbf{0}\neq\underline{\mathbf{v}}_h\in\mathbf{H}_h^{\mathbf{u}}\times\mathbf{H}_h^{\boldsymbol{\omega}}} \frac{-\left([\partial_t\,\mathcal{E}(\mathbf{e}_{\underline{\mathbf{u}}}),\underline{\mathbf{v}}_h]+[\mathcal{A}_h(\mathbf{u})(\underline{\mathbf{u}})-\mathcal{A}_h(\mathbf{u}_h)(\underline{\mathbf{u}}_h),\underline{\mathbf{v}}_h]+[\mathcal{B}(\underline{\mathbf{v}}_h),\boldsymbol{\delta}_p]\right)}{\|\underline{\mathbf{v}}_h\|}$$
$$\le C\left(\|\partial_t\mathbf{e}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}+\left(1+\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}+\|\mathbf{u}\|^{\rho-2}_{\mathbf{H}^1(\Omega)}+\|\mathbf{u}_h\|^{\rho-2}_{\mathbf{H}^1(\Omega)}\right)\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}+\|\mathbf{e}_{\boldsymbol{\omega}}\|_{\mathbf{L}^2(\Omega)}+\|\boldsymbol{\delta}_p\|_{\mathrm{L}^2(\Omega)}\right),$$

with $C>0$ depending on $|\Omega|$, $\|\mathbf{i}_4\|$, $\|\mathbf{i}_\rho\|$, ν, $\mathtt{D}$, $\mathtt{F}$ and κ. Then, taking square in the above inequality, integrating from 0 to $t\in(0,T]$, using again the fact that $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))}$ and $\|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))}$ are bounded by data (cf. (3.31), (4.17)) and employing (4.36), we deduce that

$$\int_0^t \|\boldsymbol{\eta}_p\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \le C_5\int_0^t\left(\|\partial_t\,\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)}+\|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2+\|\boldsymbol{\delta}_{\mathbf{u}}\|^{2(\rho-1)}_{\mathbf{H}^1(\Omega)}+\|\boldsymbol{\delta}_p\|^2_{\mathrm{L}^2(\Omega)}\right)\,\mathrm{d}s$$
$$+\,\widehat{C}_5\left\{\int_0^t\left(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)}+\|\partial_t\,\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)}\right)\mathrm{d}s+\|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{H}^1(\Omega)}\right\}, \qquad (4.37)$$

with $C_5,\widehat{C}_5>0$ depending on $|\Omega|$, $\|\mathbf{i}_4\|$, $\|\mathbf{i}_\rho\|$, ν, $\mathtt{D}$, $\mathtt{F}$, $\beta_{\mathtt{d}}$, κ and data.

Bounds on time derivatives

In order to bound the term $\|\partial_t\,\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}$ in (4.37), we differentiate in time the equation of (4.27) related to $\boldsymbol{\psi}_h$ and choose $\underline{\mathbf{v}}_h=(\partial_t\,\boldsymbol{\eta}_{\mathbf{u}},\boldsymbol{\eta}_{\boldsymbol{\omega}})$ to find that

$$\min\{1,\kappa^2\}\|\partial_t\,\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)}+\frac{1}{2}\,\partial_t\left(\mathtt{D}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)}+\nu\,\|\boldsymbol{\eta}_{\boldsymbol{\omega}}\|^2_{\mathbf{L}^2(\Omega)}\right)$$
$$=-(\partial_t\,\boldsymbol{\delta}_{\mathbf{u}},\partial_t\,\boldsymbol{\eta}_{\mathbf{u}})_\Omega-\kappa^2\,(\partial_t\,\nabla\boldsymbol{\delta}_{\mathbf{u}},\partial_t\,\nabla\boldsymbol{\eta}_{\mathbf{u}})_\Omega-\mathtt{D}\,(\boldsymbol{\delta}_{\mathbf{u}},\partial_t\,\boldsymbol{\eta}_{\mathbf{u}})_\Omega-\nu(\partial_t\,\boldsymbol{\delta}_{\boldsymbol{\omega}},\boldsymbol{\eta}_{\boldsymbol{\omega}})_\Omega$$
$$-\,\nu(\boldsymbol{\delta}_{\boldsymbol{\omega}},\mathbf{curl}(\partial_t\,\boldsymbol{\eta}_{\mathbf{u}}))_\Omega+\nu(\boldsymbol{\eta}_{\boldsymbol{\omega}},\mathbf{curl}(\partial_t\boldsymbol{\delta}_{\mathbf{u}}))_\Omega+(\boldsymbol{\delta}_p,\mathrm{div}(\partial_t\boldsymbol{\eta}_{\mathbf{u}}))_\Omega$$
$$-\,\mathtt{F}\,(|\mathbf{u}|^{\rho-2}\mathbf{u}-|\mathbf{u}_h|^{\rho-2}\mathbf{u}_h,\partial_t\,\boldsymbol{\eta}_{\mathbf{u}})_\Omega-((\nabla\mathbf{u})\mathbf{u}-(\nabla\mathbf{u}_h)\mathbf{u}_h,\partial_t\boldsymbol{\eta}_{\mathbf{u}})_\Omega. \qquad (4.38)$$

Notice that $(\boldsymbol{\eta}_p,\mathrm{div}(\partial_t\boldsymbol{\eta}_{\mathbf{u}}))_\Omega=0$ since $(\boldsymbol{\eta}_{\mathbf{u}}(t),\mathbf{0})\in\mathbf{V}_h$ (cf. (4.7) and (4.26)). In turn, using the Hölder inequality, the estimate (3.16) and the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^\rho(\Omega)$, we deduce that there exists a constant $c_\rho>0$, depending on $|\Omega|$ and ρ such that

$$(|\mathbf{u}|^{\rho-2}\mathbf{u}-|\mathbf{u}_h|^{\rho-2}\mathbf{u}_h,\partial_t\,\boldsymbol{\eta}_{\mathbf{u}})_\Omega \le c_\rho\,\left(\|\mathbf{u}\|_{\mathbf{L}^\rho(\Omega)}+\|\mathbf{u}_h\|_{\mathbf{L}^\rho(\Omega)}\right)^{\rho-2}\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^{\rho-1}(\Omega)}\|\partial_t\,\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{L}^\rho(\Omega)}$$
$$\le c_\rho\,\|\mathbf{i}_\rho\|^\rho\,\left(\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}+\|\mathbf{u}_h\|_{\mathbf{H}^1(\Omega)}\right)^{\rho-2}\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}\|\partial_t\,\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}. \qquad (4.39)$$

Similarly, but now adding and subtracting the term $(\nabla\mathbf{u})\mathbf{u}_h$ (it also works with $(\nabla\mathbf{u}_h)\mathbf{u}$), using the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, we obtain

$$((\nabla\mathbf{u})\mathbf{u}-(\nabla\mathbf{u}_h)\mathbf{u}_h,\partial_t\boldsymbol{\eta}_{\mathbf{u}})_\Omega \le \left(\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^4(\Omega)}+\|\mathbf{u}_h\|_{\mathbf{L}^4(\Omega)}\|\nabla\mathbf{e}_{\mathbf{u}}\|_{\mathbb{L}^2(\Omega)}\right)\|\partial_t\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{L}^4(\Omega)}$$
$$\le \|\mathbf{i}_4\|^2\,\left(\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}+\|\mathbf{u}_h\|_{\mathbf{H}^1(\Omega)}\right)\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}\|\partial_t\,\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}. \qquad (4.40)$$

Thus, integrating (4.38) from 0 to $t \in (0,T]$, using the estimates (4.39) and (4.40), the Cauchy–Schwarz and Young's inequalities, and the fact that $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))}$ and $\|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))}$ are bounded by data (cf. (3.31), (4.17)), in a way similar to (4.31)–(4.34), we find that

$$\begin{aligned}
&\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\omega}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|\partial_t\,\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)}\,\mathrm{d}s \\
&\leq C_6\left(\int_0^t \Big(\|\partial_t\,\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\omega}}\|^2_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{\delta}_p\|^2_{\mathrm{L}^2(\Omega)}\Big)\,\mathrm{d}s\right. \\
&\quad \left. + \int_0^t \Big(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\omega}}\|^2_{\mathrm{L}^2(\Omega)}\Big)\,\mathrm{d}s + \|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\omega}}(0)\|^2_{\mathbf{L}^2(\Omega)}\right),
\end{aligned} \tag{4.41}$$

where $C_6 > 0$ depends on $|\Omega|, \|\mathbf{i}_4\|, \|\mathbf{i}_\rho\|, \nu, \mathtt{D}, \mathtt{F}, \kappa$ and data. Then, combining estimates (4.36), (4.37) and (4.41), using the Grönwall inequality, and some algebraic manipulations, we deduce that

$$\begin{aligned}
&\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\omega}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \Big(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\omega}}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_p\|^2_{\mathrm{L}^2(\Omega)} + \|\partial_t\,\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)}\Big)\,\mathrm{d}s \\
&\leq C_7\,\exp(T)\left(\int_0^t \Big(\|\partial_t\,\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\mathbf{u}}\|^{2(\rho-1)}_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\delta}_p\|^2_{\mathrm{L}^2(\Omega)}\Big)\,\mathrm{d}s \,+\, \|\boldsymbol{\eta}_{\underline{\mathbf{u}}}(0)\|^2\right),
\end{aligned} \tag{4.42}$$

with $C_7 > 0$ depending on $|\Omega|, \|\mathbf{i}_4\|, \|\mathbf{i}_\rho\|, \nu, \mathtt{D}, \mathtt{F}, \beta_{\mathtt{d}}, \kappa$ and data.

Bounds on initial data.

Finally, in order to bound the last term in (4.42), we subtract the continuous and discrete initial condition problems (3.30) and (4.6) to obtain the error system:

$$\begin{aligned}
(\nabla\mathbf{u}_0 - \nabla\mathbf{u}_{h,0}, \nabla\mathbf{v}_h)_\Omega + [\mathcal{A}_h(\mathbf{u}_0)(\underline{\mathbf{u}}_0) - \mathcal{A}_h(\mathbf{u}_{h,0})(\underline{\mathbf{u}}_{h,0}), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), p_0 - p_{h,0}] &= 0, \\
-\,[\mathcal{B}(\underline{\mathbf{u}}_0 - \underline{\mathbf{u}}_{h,0}), q_h] &= 0,
\end{aligned}$$

for all $\underline{\mathbf{v}}_h \in \mathbf{H}^{\mathbf{u}}_h \times \mathbf{H}^{\boldsymbol{\omega}}_h$ and $q_h \in \mathrm{H}^p_h$. Then, proceeding as in (4.35), recalling from Theorems 3.9 and 4.3 that $(\mathbf{u}(0), \boldsymbol{\omega}(0)) = (\mathbf{u}_0, \boldsymbol{\omega}_0)$ and $(\mathbf{u}_h(0), \boldsymbol{\omega}_h(0)) = (\mathbf{u}_{h,0}, \boldsymbol{\omega}_{h,0})$, respectively, we get

$$\|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\omega}}(0)\|^2_{\mathbf{L}^2(\Omega)} \;\leq\; \widetilde{C}_0\,\left(\|\boldsymbol{\delta}_{\underline{\mathbf{u}}_0}\|^2 + \|\boldsymbol{\delta}_{\mathbf{u}_0}\|^{2\,(\rho-1)}_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\delta}_{p_0}\|^2_{\mathrm{L}^2(\Omega)}\right), \tag{4.43}$$

where, similarly to (4.25), we denote $\boldsymbol{\delta}_{\underline{\mathbf{u}}_0} = (\boldsymbol{\delta}_{\mathbf{u}_0}, \boldsymbol{\delta}_{\boldsymbol{\omega}_0}) = (\mathbf{u}_0 - \widehat{\mathbf{v}}_h(0), \boldsymbol{\omega}_0 - \widehat{\boldsymbol{\psi}}_h(0))$ and $\boldsymbol{\delta}_{p_0} = p_0 - \widehat{q}_h(0)$, with arbitrary $(\widehat{\mathbf{v}}_h(0), \widehat{\boldsymbol{\psi}}_h(0)) \in \mathbf{V}_h$ and $\widehat{q}_h(0) \in \mathrm{H}^p_h$, and $\widetilde{C}_0$ is a positive constant, depending on $|\Omega|, \|\mathbf{i}_4\|, \|\mathbf{i}_\rho\|, \nu, \mathtt{D}, \mathtt{F}$ and κ.

Thus, combining (4.42) with (4.43), and using the error decomposition (4.24), there holds

$$\|\mathbf{e}_{\mathbf{u}}(t)\|^2_{\mathbf{H}^1(\Omega)} + \int_0^t \Big(\|\mathbf{e}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}_{\boldsymbol{\omega}}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}_p\|^2_{\mathrm{L}^2(\Omega)}\Big)\,\mathrm{d}s \;\leq\; C_8\,\exp(T)\,\Psi(\underline{\mathbf{u}}, p), \tag{4.44}$$

where

$$\Psi(\underline{\mathbf{u}},p) := \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}(t)\|^2 + \int_0^t \left(\|\partial_t\,\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^{2\,(\rho-1)} + \|\boldsymbol{\delta}_p\|^2_{\mathrm{L}^2(\Omega)}\right)\,\mathrm{d}s$$
$$+\ \|\boldsymbol{\delta}_{\underline{\mathbf{u}}_0}\|^2 + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}_0}\|^{2\,(\rho-1)} + \|\boldsymbol{\delta}_{p_0}\|^2_{\mathrm{L}^2(\Omega)},$$

with $C_8 > 0$ depending on $|\Omega|$, $\|\mathbf{i}_4\|$, $\|\mathbf{i}_\rho\|$, ν, $\mathtt{D}$, $\mathtt{F}$, $\beta_{\mathtt{d}}$, κ and data. Finally, using the fact that $\widehat{\underline{\mathbf{v}}}_h : [0,T] \to \mathbf{V}_h$ and $\widehat{q}_h : [0,T] \to \mathrm{H}_h^p$ are arbitrary, taking infimum in (4.44) over the corresponding discrete subspaces $\mathbf{V}_h$ and H_h^p, and applying the approximation properties (4.23), we derive (4.28) and conclude the proof. □

REMARK 4.1. Observe that (4.28) can be expanded to include a bound on $\|\partial_t\,\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{H}^1(\Omega))}$ and $\|\mathbf{e}_{\boldsymbol{\omega}}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))}$, using (4.42).

5. Fully discrete approximation

In this section we introduce and analyze a fully discrete approximation of (2.11) (cf. (4.3)). To that end, for the time discretization we employ the backward Euler method. Let Δt be the time step, $T = N\Delta t$ and let $t_n = n\Delta t$, $n = 0, ..., N$. Let $d_t u^n = (\Delta t)^{-1}(u^n - u^{n-1})$ be the first order (backward) discrete time derivative, where $u^n := u(t_n)$. Then the fully discrete method reads: given $\mathbf{f}^n \in \mathbf{H}^{-1}(\Omega)$ and $(\underline{\mathbf{u}}_h^0, p_h^0) = ((\mathbf{u}_{h,0}, \boldsymbol{\omega}_{h,0}), p_{h,0})$ satisfying (4.11), find $(\underline{\mathbf{u}}_h^n, p_h^n) := ((\mathbf{u}_h^n, \boldsymbol{\omega}_h^n), p_h^n) \in (\mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}}) \times \mathrm{H}_h^p$, $n = 1, ..., N$, such that

$$d_t[\mathcal{E}(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] + [\mathcal{A}_h(\mathbf{u}_h^n)(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), p_h^n] = [\mathbf{F}^n, \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}},$$
$$-[\mathcal{B}(\underline{\mathbf{u}}_h^n), q_h] = 0 \qquad \forall\, q_h \in \mathrm{H}_h^p, \tag{5.1}$$

where $[\mathbf{F}^n, \underline{\mathbf{v}}_h] := (\mathbf{f}^n, \mathbf{v}_h)_\Omega$.

In what follows, given a separable Banach space V endowed with the norm $\|\cdot\|_{\mathrm{V}}$, we make use of the following discrete in time norms:

$$\|u\|^2_{\ell^2(0,T;\mathrm{V})} := \Delta t \sum_{n=1}^N \|u^n\|^2_{\mathrm{V}} \quad \text{and} \quad \|u\|_{\ell^\infty(0,T;\mathrm{V})} := \max_{0\le n\le N} \|u^n\|_{\mathrm{V}}. \tag{5.2}$$

We also recall the well-known identity:

$$(d_t u_h^n, u_h^n)_\Omega = \frac{1}{2}\, d_t\, \|u_h^n\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\,\Delta t\, \|d_t\, u_h^n\|^2_{\mathrm{L}^2(\Omega)}, \tag{5.3}$$

which follows from the definition of the discrete time derivative $d_t u_h^n = (\Delta t)^{-1}(u_h^n - u_h^{n-1})$ and the polarization identity $(a-b, a) = \frac{1}{2}\left(|a|^2 - |b|^2 + |a-b|^2\right)$, applied with $a = u_h^n$ and $b = u_h^{n-1}$. In addition, we state for later use the following discrete Grönwall inequality (Quarteroni & Valli, 1994, Lemma 1.4.2):

LEMMA 5.1. Let $\Delta t > 0, B \geq 0$ and let $a_n, b_n, c_n, d_n, n \geq 0$, be non-negative sequence such that $a_0 \leq B$ and

$$a_n + \Delta t \sum_{l=1}^{n} b_l \leq \Delta t \sum_{l=1}^{n-1} d_l\, a_l + \Delta t \sum_{l=1}^{n} c_l + B, \quad n \geq 1.$$

Then,

$$a_n + \Delta t \sum_{l=1}^{n} b_l \leq \exp\left(\Delta t \sum_{l=1}^{n-1} d_l \right)\left(\Delta t \sum_{l=1}^{n} c_l + B \right), \quad n \geq 1.$$

Next, we state the main results for method (5.1).

THEOREM 5.2. For each $(\underline{\mathbf{u}}_h^0, p_h^0) := ((\mathbf{u}_{h,0}, \boldsymbol{\omega}_{h,0}), p_{h,0})$ satisfying (4.6) and $\mathbf{f}^n \in \mathbf{H}^{-1}(\Omega)$, $n = 1, ..., N$, there exist a unique solution $(\underline{\mathbf{u}}_h^n, \underset{\sim}{p}_h^n) := ((\mathbf{u}_h^n, \boldsymbol{\omega}_h^n), p_h^n) \in (\mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}}) \times \mathrm{H}_h^p$ to (5.1), with $n = 1, ..., N$. Moreover, there exists a constant $\widetilde{C}_{\mathtt{KVr}} > 0$, depending only on $|\Omega|$, $\|\mathbf{i}_\rho\|$, $\|\mathbf{i}_4\|$, ν, $\mathtt{D}$, $\mathtt{F}$ and κ, such that

$$\begin{aligned} &\|\mathbf{u}_h\|_{\ell^\infty(0,T;\mathbf{H}^1(\Omega))} + \Delta t \|d_t \mathbf{u}_h\|_{\ell^2(0,T;\mathbf{H}^1(\Omega))} + \|\mathbf{u}_h\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\omega}_h\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} \\ &\quad \leq \widetilde{C}_{\mathtt{KVr}} \sqrt{\exp(T)} \left(\|\mathbf{f}\|_{\ell^2(0,T;\mathbf{H}^{-1}(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^{\rho-1}_{\mathbf{H}^1(\Omega)} \right), \end{aligned} \tag{5.4}$$

and a constant $\widetilde{C}_{\mathtt{KVp}} > 0$ depending only on $|\Omega|$, $\|\mathbf{i}_\rho\|$, $\|\mathbf{i}_4\|$, ν, $\mathtt{D}$, $\mathtt{F}$, κ and $\beta_{\mathtt{d}}$, such that

$$\begin{aligned} &\|p_h\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))} \\ &\quad \leq \widetilde{C}_{\mathtt{KVp}} \sum_{j \in \{2,3,\rho\}} \left\{ \sqrt{\exp(T)} \left(\|\mathbf{f}\|_{\ell^2(0,T;\mathbf{H}^{-1}(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^{\rho-1}_{\mathbf{H}^1(\Omega)} \right) \right\}^{j-1}. \end{aligned} \tag{5.5}$$

Proof. Existence of a solution of the fully discrete problem (5.1) at each time step t_n, $n = 1, ..., N$, can be established by induction. In particular, assuming that a solution exists at t_{n-1}, existence of a solution at t_n follows from similar arguments to those employed in the proof of Lemma 3.7, using the discrete inf-sup condition (4.15). We postpone the proof of uniqueness until after the stability bound.

The derivation of (5.4) and (5.5) can be obtained similarly as in the proof of Theorems 3.9 and 3.10, respectively. In fact, we choose $(\underline{\mathbf{v}}_h, q_h) = (\underline{\mathbf{u}}_h^n, p_h^n)$ in (5.1), use the identity (5.3), the definition of the operator $\mathcal{A}_h$ (cf. (4.4)), the well-known inequality for dual norms: $(\mathbf{f}^n, \mathbf{u}_h^n)_\Omega \leq \|\mathbf{f}^n\|_{\mathbf{H}^{-1}(\Omega)} \|\mathbf{u}_h^n\|_{\mathbf{H}^1(\Omega)}$ and Young's inequality (cf. (1.1)), to obtain

$$\begin{aligned} &\frac{1}{2} d_t \left(\|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2 \|\nabla \mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} \right) + \frac{1}{2} \Delta t \left(\|d_t \mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2 \|d_t \nabla \mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} \right) \\ &\quad + \mathtt{D} \|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \mathtt{F} \|\mathbf{u}_h^n\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu \|\boldsymbol{\omega}_h^n\|^2_{\mathbf{L}^2(\Omega)} \leq \frac{1}{2} \left(\|\mathbf{f}^n\|^2_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} \right). \end{aligned} \tag{5.6}$$

Then, summing up over the time index $n = 1, ..., m$, with $m = 1, \ldots, N$, in (5.6) and multiplying by Δt, we get

$$\widehat{\gamma}_{\mathrm{KV}} \, \|\mathbf{u}_h^m\|^2_{\mathbf{H}^1(\Omega)} + \widehat{\gamma}_{\mathrm{KV}} \, (\Delta t)^2 \sum_{n=1}^m \|d_t \mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} + 2\,\Delta t \sum_{n=1}^m \left(\mathtt{D}\, \|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \nu \, \|\boldsymbol{\omega}_h^n\|^2_{\mathbf{L}^2(\Omega)} \right)$$
$$\leq \Delta t \sum_{n=1}^m \|\mathbf{f}^n\|^2_{\mathbf{H}^{-1}(\Omega)} + \widehat{\gamma}_{\mathrm{KV}} \, \|\mathbf{u}_h^0\|^2_{\mathbf{H}^1(\Omega)} + \Delta t \sum_{n=1}^m \|\mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)}, \tag{5.7}$$

with $\widehat{\gamma}_{\mathrm{KV}}$ as in (3.33). Notice that, in order to simplify the stability bound, we have neglected the term $\|\mathbf{u}_h^n\|^\rho_{\mathbf{L}^\rho(\Omega)}$ in the left-hand side of (5.6). Thus, analogously to (3.34), applying the discrete Grönwall inequality (cf. Lemma 5.1) in (5.7) and recalling that $N\,\Delta t = T$, and using the estimate (4.10), we deduce the stability bound (5.4). Unlike its continuous counterpart (3.31), however, the constant $\widetilde{C}_{\mathtt{KVr}}$ in (5.4) depends on $\mathtt{F}$ due to the use of (4.10).

On the other hand, from the discrete inf-sup condition of $\mathcal{B}$ (cf. (4.1)) and the first equation of (5.1) related to $\mathbf{v}_h$, we deduce the discrete version of (3.38), that is,

$$\beta_{\mathtt{d}} \, \|p_h^n\|_{\mathrm{L}^2(\Omega)} \leq \|\mathbf{f}^n\|_{\mathbf{H}^{-1}(\Omega)} + \mathtt{D}\, \|\mathbf{u}_h^n\|_{\mathbf{L}^2(\Omega)} + \nu \, \|\boldsymbol{\omega}_h^n\|_{\mathbf{L}^2(\Omega)}$$
$$+ \|\mathbf{i}_4\|^2 \, \|\mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} + \mathtt{F}\, \|\mathbf{i}_\rho\|^\rho \, \|\mathbf{u}_h^n\|^{\rho-1}_{\mathbf{H}^1(\Omega)} + (1+\kappa^2) \|d_t \mathbf{u}_h^n\|_{\mathbf{H}^1(\Omega)}. \tag{5.8}$$

Then, squaring (5.8), summing up over the time index $n = 1, ..., m$, with $m = 1, \ldots, N$, and multiplying by Δt, we deduce analogously to (3.39), that there exists $C_1 > 0$, depending on $|\Omega|$, $\|\mathbf{i}_4\|$, $\|\mathbf{i}_\rho\|$, ν, $\mathtt{D}$, $\mathtt{F}$, κ and $\beta_{\mathtt{d}}$, such that

$$\Delta t \sum_{n=1}^m \|p_h^n\|^2_{\mathrm{L}^2(\Omega)} \leq C_1 \left\{ \Delta t \sum_{n=1}^m \left(\|\mathbf{f}^n\|^2_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\omega}_h^n\|^2_{\mathbf{L}^2(\Omega)} \right) \right.$$
$$\left. + \Delta t \sum_{n=1}^m \left(\|\mathbf{u}_h^n\|^4_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_h^n\|^{2(\rho-1)}_{\mathbf{H}^1(\Omega)} + \|d_t \mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} \right) \right\}. \tag{5.9}$$

Next, in order to bound the last term in (5.9), we choose $(\underline{\mathbf{v}}_h, q_h) = ((d_t\, \mathbf{u}_h^n, \boldsymbol{\omega}_h^n), p_h^n)$ in (5.1), apply some algebraic manipulation, use the identity (5.3) and the Cauchy–Schwarz and Young's inequalities, to obtain the discrete version of (3.40):

$$\widehat{\gamma}_{\mathrm{KV}} \, \|d_t \mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} + \frac{1}{2} d_t \left(\mathtt{D}\, \|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \nu \, \|\boldsymbol{\omega}_h^n\|^2_{\mathbf{L}^2(\Omega)} \right) + \frac{1}{2} \Delta t \left(\mathtt{D}\, \|d_t \mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \nu \, \|d_t \boldsymbol{\omega}_h^n\|^2_{\mathbf{L}^2(\Omega)} \right)$$
$$+ \mathtt{F}\, (|\mathbf{u}_h^n|^{\rho-2} \mathbf{u}_h^n, d_t \mathbf{u}_h^n)_\Omega \leq \left(\|\mathbf{f}^n\|_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{i}_4\|^2 \left(1 + \frac{\sqrt{d}}{2} \right) \|\mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} \right) \|d_t \mathbf{u}_h^n\|_{\mathbf{H}^1(\Omega)}$$
$$\leq C_2 \left(\|\mathbf{f}^n\|^2_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{u}_h^n\|^4_{\mathbf{H}^1(\Omega)} \right) + \frac{\widehat{\gamma}_{\mathrm{KV}}}{2} \, \|d_t \mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)}, \tag{5.10}$$

with $\widehat{\gamma}_{\mathrm{KV}}$ as in (3.33) and $C_2 > 0$, depending on $\|\mathbf{i}_4\|, d$ and κ. In turn, employing Hölder and Young's inequalities, we are able to deduce (cf. Caucao *et al.*, 2022, eq. (5.13)):

$$(|\mathbf{u}_h^n|^{\rho-2}\mathbf{u}_h^n, d_t\mathbf{u}_h^n)_\Omega \geq \frac{(\Delta t)^{-1}}{\rho}\left(\|\mathbf{u}_h^n\|^\rho_{\mathbf{L}^\rho(\Omega)} - \|\mathbf{u}_h^{n-1}\|^\rho_{\mathbf{L}^\rho(\Omega)}\right) = \frac{1}{\rho}\, d_t\, \|\mathbf{u}_h^n\|^\rho_{\mathbf{L}^\rho(\Omega)}. \tag{5.11}$$

Thus, combining (5.10) with (5.11), using Young's inequality, summing up over the time index $n = 1, ..., m$, with $m = 1, \ldots, N$ and multiplying by Δt, we get the discrete version of (3.41):

$$\begin{aligned} &\mathtt{D}\, \|\mathbf{u}_h^m\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathtt{F}}{\rho}\|\mathbf{u}_h^m\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu\, \|\boldsymbol{\omega}_h^m\|^2_{\mathbf{L}^2(\Omega)} + \widehat{\gamma}_{\mathrm{KV}}\, \Delta t \sum_{n=1}^m \|d_t\mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} \\ &\quad \leq 2C_2\, \Delta t \sum_{n=1}^m \left(\|\mathbf{f}^n\|^2_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{u}_h^n\|^4_{\mathbf{H}^1(\Omega)}\right) + \mathtt{D}\, \|\mathbf{u}_h^0\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathtt{F}}{\rho}\, \|\mathbf{u}_h^0\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu\, \|\boldsymbol{\omega}_h^0\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{5.12}$$

Combining (5.9) with (5.7) and (5.12), using the fact that $(\mathbf{u}_h^0, \boldsymbol{\omega}_h^0) = (\mathbf{u}_{h,0}, \boldsymbol{\omega}_{h,0})$ and (4.10), we deduce that

$$\begin{aligned} \Delta t \sum_{n=1}^m \|p_h^n\|^2_{\mathrm{L}^2(\Omega)} \leq C_3 \Bigg\{ &\Delta t \sum_{n=1}^m \|\mathbf{f}^n\|^2_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^4_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^{2(\rho-1)}_{\mathbf{H}^1(\Omega)} \\ &+ \Delta t \sum_{n=1}^m \left(\|\mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_h^n\|^4_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_h^n\|^{2\,(\rho-1)}_{\mathbf{H}^1(\Omega)}\right)\Bigg\}, \end{aligned} \tag{5.13}$$

with $m = 1, \ldots, N$ and $C_3 > 0$, depending on $|\Omega|, \|\mathbf{i}_4\|, \|\mathbf{i}_\rho\|, \nu, \mathtt{D}, \mathtt{F}, \kappa, d$ and $\beta_{\mathtt{d}}$. Then, using (5.4) to bound $\|\mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)}$, $\|\mathbf{u}_h^n\|^4_{\mathbf{H}^1(\Omega)}$ and $\|\mathbf{u}_h^n\|^{2\,(\rho-1)}_{\mathbf{H}^1(\Omega)}$ in the left-hand side of (5.13), we obtain (5.5).

Finally, uniqueness of the solution at each time step can be established using that $\|\mathbf{u}_h\|_{\ell^\infty(0,T;\mathbf{H}^1(\Omega))}$ is bounded by data (cf. (5.4)), following the argument showing uniqueness of the weak solution in Theorem 3.9. In particular, starting from the time-discrete version of (3.35), the uniqueness follows from summing in time and using the discrete Grönwall inequality (cf. Lemma 5.1). □

Now, we proceed with establishing rates of convergence for the fully discrete scheme (5.1). To that end, we subtract the fully discrete problem (5.1) from the continuous counterparts (2.11) at each time step $n = 1, \ldots, N$, to obtain the following error system:

$$\begin{aligned} d_t\, [\mathcal{E}(\mathbf{e}^n_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] + [\mathcal{A}_h(\mathbf{u}^n)(\underline{\mathbf{u}}^n) - \mathcal{A}_h(\mathbf{u}_h^n)(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), \mathbf{e}_p^n] &= [r_n(\mathbf{u}), \mathbf{v}_h], \\ [\mathcal{B}(\mathbf{e}^n_{\underline{\mathbf{u}}}), q_h] &= 0, \end{aligned} \tag{5.14}$$

for all $\underline{\mathbf{v}}_h \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\omega}}$ and $q_h \in \mathrm{H}_h^p$, where

$$[r_n(\mathbf{u}), \mathbf{v}_h] := (r_n(\mathbf{u}), \mathbf{v}_h)_\Omega + \kappa^2\, (r_n(\nabla\mathbf{u}), \nabla\mathbf{v}_h)_\Omega,$$

and r_n denotes the difference between the time derivative and its discrete analog, that is

$$r_n(\mathbf{u}) \;=\; d_t\,\mathbf{u}^n - \partial_t\,\mathbf{u}(t_n).$$

In addition, we recall from Bukač *et al.* (2015, Lemma 4) that for sufficiently smooth **u**, there holds

$$\Delta t \sum_{n=1}^{N} \|r_n(\mathbf{u})\|^2_{\mathbf{H}^1(\Omega)} \,\le\, C(\partial_{tt}\,\mathbf{u})\,(\Delta t)^2, \quad \text{with} \quad C(\partial_{tt}\,\mathbf{u}) \,:=\, C\,\|\partial_{tt}\,\mathbf{u}\|^2_{\mathrm{L}^2(0,T;\mathbf{H}^1(\Omega))}. \tag{5.15}$$

Then, the proof of the theoretical rate of convergence of the fully discrete scheme (5.1) follows the structure of the proof of Theorem 4.4, using discrete-in-time arguments as in the proof of Theorem 5.2, the discrete Grönwall inequality (cf. Lemma 5.1) and the estimate (5.15) (see Caucao *et al.*, 2022, Theorem 5.4 for a similar approach).

THEOREM 5.3. Let the assumptions of Theorem 4.4 hold. Then, for the solution of the fully discrete problem (5.1) there exists $\widehat{C}(\underline{\mathbf{u}},p) > 0$, depending only on $C(\underline{\mathbf{u}}), C(\partial_t\,\underline{\mathbf{u}}), C(\partial_{tt}\,\underline{\mathbf{u}}), C(p), C(\partial_t\,p), |\Omega|$, $\|\mathbf{i}_\rho\|, \|\mathbf{i}_4\|, \nu, \mathtt{D}, \mathtt{F}, \kappa, \beta_{\mathrm{d}}, T, \|\mathbf{f}\|_{\ell^2(0,T;\mathbf{H}^{-1}(\Omega))}$ and $\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)}$, such that

$$\begin{aligned}\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{H}^1(\Omega))} &+ \Delta t\,\|d_t\,\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{H}^1(\Omega))} + \|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} \\ &+ \|\mathbf{e}_{\boldsymbol{\omega}}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))} \,\le\, \widehat{C}(\underline{\mathbf{u}},p)\,\Big(h^s + h^{s\,(\rho-1)} + \Delta t\Big).\end{aligned} \tag{5.16}$$

REMARK 5.1. For the fully discrete scheme (5.1) we have considered the backward Euler method only for the sake of simplicity. The analysis developed in Section 5 can be adapted to other time discretizations, such as BDF schemes or the Crank–Nicholson method.

6. Numerical results

In this section, we present three numerical results that illustrate the performance of the fully discrete method (5.1). The implementation is based on a `FreeFEM` code (Hecht, 2012). We use quasi-uniform triangulations and the finite element subspaces detailed in Section 4.1 (cf. (4.2)). The nonlinearity is handled using a Newton–Raphson algorithm with a fixed tolerance of $\mathsf{tol} = 1\mathrm{E}{-}06$. The iterative process is stopped when the relative error between two consecutive iterations of the complete coefficient vector, namely $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{\mathtt{DoF}}}{\|\mathbf{coeff}^{m+1}\|_{\mathtt{DoF}}} \le \mathsf{tol},$$

where $\|\cdot\|_{\mathtt{DoF}}$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{DoF}}$, with $\mathtt{DoF}$ denoting the total number of degrees of freedom defined by the finite element subspaces $\mathbf{H}_h^{\mathbf{u}}, \mathbf{H}_h^{\boldsymbol{\omega}}$ and H_h^p (cf. (4.2)).

Examples 1 and 2 are used to corroborate the rate of convergence in two- and three-dimensional domains, respectively. The total simulation time for these examples is $T = 0.001$ and the time step is $\Delta t = 10^{-4}$. The time step is sufficiently small, so that the time discretization error does not affect the convergence rates. On the other hand, Example 3 is utilized to analyze the method's behavior under various scenarios, considering different Darcy and Forchheimer coefficients, as well as varying values

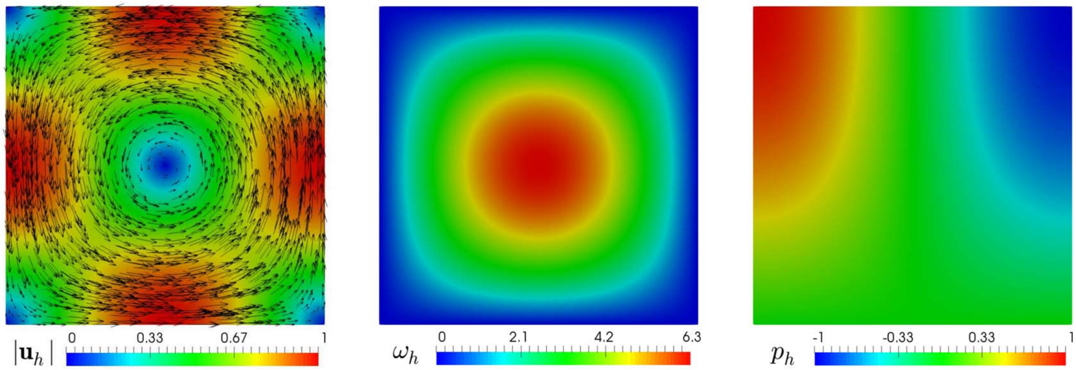

FIG. 1. [Example 1] Computed magnitude of the velocity, vorticity and pressure fields at time $T = 0.001$.

of the elasticity parameter κ. For these cases, the total simulation time and the time step are chosen as $T = 1$ and $\Delta t = 10^{-2}$, respectively.

6.1 *Example 1: Two-dimensional smooth exact solution*

In this test we study the convergence for the space discretization using an analytical solution. The domain is the square $\Omega = (0, 1)^2$. We consider $\rho = 3, \nu = 1, \mathtt{D} = 1, \mathtt{F} = 10, \kappa = 1$, and the datum $\mathbf{f}$ is adjusted so that the exact solution is given by the smooth functions:

$$\mathbf{u} = \exp(t) \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix} \quad \text{and} \quad p = \exp(t)\cos(\pi x)\sin\left(\frac{\pi y}{2}\right).$$

The model problem is then complemented with the appropriate Dirichlet boundary condition and initial data.

In Fig. 1, we display the solution obtained with the Crouzeix–Raviart-based approximation, 39, 146 triangle elements and 176 926 DoF at time $T = 0.001$. Table 1 shows the convergence history for a sequence of quasi-uniform mesh refinements, including the average number of Newton iterations. The results confirm that the optimal spatial rates of convergence $\mathcal{O}(h^{k+1})$ provided by Theorem 5.3 (see also Theorem 4.4) are attained for the Taylor–Hood based scheme, with $k = 1$. In addition, optimal order $\mathcal{O}(h)$ is also obtained for the MINI-element and Crouzeix–Raviart based discretizations. The Newton's method exhibits behavior independent of the mesh size, converging in 2.1 iterations in average in all cases.

6.2 *Example 2: Three-dimensional smooth exact solution*

In the second example, we consider the cube domain $\Omega = (0, 1)^3$ and the exact solution

$$\mathbf{u} = \exp(t) \begin{pmatrix} \sin(\pi x)\cos(\pi y)\cos(\pi z) \\ -2\cos(\pi x)\sin(\pi y)\cos(\pi z) \\ \cos(\pi x)\cos(\pi y)\sin(\pi z) \end{pmatrix} \quad \text{and} \quad p = \exp(t)\,(x - 0.5)^3 \sin(y + z).$$

TABLE 1 *[Example 1] Number of degrees of freedom, mesh sizes, average number of Newton iterations, errors, and rates of convergence with* $\rho = 3, \nu = 1, \mathtt{D} = 1, \mathtt{F} = 10$ *and* $\kappa = 1$

Taylor–Hood-based discretization

			$\lVert\mathbf{e_u}\rVert_{\ell^\infty(0,T;\mathbf{H}^1(\Omega))}$		$\lVert\mathbf{e_u}\rVert_{\ell^2(0,T;\mathbf{L}^2(\Omega))}$		$\lVert\mathbf{e}_\omega\rVert_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$		$\lVert\mathbf{e}_p\rVert_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$	
DoF	h	iter	error	rate	error	rate	error	rate	error	rate
232	0.373	2.1	1.95E-01	–	2.03E-04	–	6.37E-03	–	3.76E-00	–
860	0.196	2.1	3.57E-02	2.653	1.80E-05	3.782	1.18E-03	2.636	5.15E-01	3.105
3216	0.097	2.1	8.69E-03	2.001	2.17E-06	2.998	2.90E-04	1.986	9.16E-02	2.448
12468	0.048	2.1	2.00E-03	2.074	2.51E-07	3.049	6.76E-05	2.058	1.42E-02	2.631
49142	0.025	2.1	5.21E-04	2.013	3.29E-08	3.042	1.78E-05	2.000	4.03E-03	1.888
197270	0.013	2.1	1.27E-04	2.160	3.94E-09	3.252	4.30E-06	2.176	8.17E-04	2.447

MINI-element-based discretization

			$\lVert\mathbf{e_u}\rVert_{\ell^\infty(0,T;\mathbf{H}^1(\Omega))}$		$\lVert\mathbf{e_u}\rVert_{\ell^2(0,T;\mathbf{L}^2(\Omega))}$		$\lVert\mathbf{e}_\omega\rVert_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$		$\lVert\mathbf{e}_p\rVert_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$	
DoF	h	iter	error	rate	error	rate	error	rate	error	rate
180	0.373	2.1	1.22E-00	–	1.23E-03	–	9.50E-03	–	2.52E+01	–
676	0.196	2.1	6.62E-01	0.961	2.95E-04	2.224	2.43E-03	2.132	6.37E-00	2.149
2548	0.097	2.1	3.30E-01	0.985	7.39E-05	1.962	8.68E-04	1.456	3.31E-00	0.925
9924	0.048	2.1	1.68E-01	0.957	1.87E-05	1.943	3.62E-04	1.236	1.50E-00	1.125
39212	0.025	2.1	8.47E-02	1.024	4.69E-06	2.068	1.70E-04	1.130	7.25E-01	1.084
157612	0.013	2.1	4.14E-02	1.098	1.15E-06	2.162	7.52E-05	1.252	3.51E-01	1.110

Crouzeix–Raviart-based discretization

			$\lVert\mathbf{e_u}\rVert_{\ell^\infty(0,T;\mathbf{H}^1(\Omega))}$		$\lVert\mathbf{e_u}\rVert_{\ell^2(0,T;\mathbf{L}^2(\Omega))}$		$\lVert\mathbf{e}_\omega\rVert_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$		$\lVert\mathbf{e}_p\rVert_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$	
DoF	h	iter	error	rate	error	rate	error	rate	error	rate
187	0.373	2.1	7.59E-01	–	1.05E-03	–	1.45E-02	–	9.57E-00	–
733	0.196	2.1	3.87E-01	1.050	2.61E-04	2.170	4.82E-03	1.722	5.67E-00	0.819
2815	0.097	2.1	1.95E-01	0.974	6.59E-05	1.951	1.74E-03	1.447	3.04E-00	0.885
11065	0.048	2.1	9.86E-02	0.962	1.70E-05	1.918	7.61E-04	1.168	1.47E-00	1.023
43918	0.025	2.1	4.93E-02	1.037	4.22E-06	2.083	3.75E-04	1.057	7.91E-01	0.930
176926	0.013	2.1	2.44E-02	1.081	1.03E-06	2.167	1.70E-04	1.219	3.76E-01	1.138

Similarly to the first example, we consider the parameters $\rho = 4, \nu = 1, \mathtt{D} = 1, \mathtt{F} = 10$ and $\kappa = 1$, and the right-hand side function $\mathbf{f}$ is computed from (2.3) using the above solution.

The numerical solution obtained with the Taylor–Hood-based approximation, 63,888 tetrahedral elements and 322 043 DoF at time $T = 0.001$ is shown in Fig. 2. The convergence history for a set of quasi-uniform mesh refinements using Taylor–Hood and MINI-element-based approximations is shown in Table 2. Again, the mixed finite element method converges optimally with order $\mathcal{O}(h^2)$ and $\mathcal{O}(h)$, respectively, as it was proved by Theorem 5.3 (see also Theorem 4.4).

6.3 *Example 3: Flow through porous media with channel network*

Finally, inspired by Ambartsumyan *et al.* (2019, Section 5.2.4), we focus on a flow through a porous medium with a channel network. We consider the square domain $\Omega = (-1, 1)^2$ with an internal channel network denoted as Ω_c. The domain configuration and the prescribed mesh are described in the plots

TABLE 2 *[Example 2] Number of degrees of freedom, mesh sizes, average number of Newton iterations, errors, and rates of convergence with $\rho = 4, \nu = 1, \mathtt{D} = 1, \mathtt{F} = 10$ and $\kappa = 1$*

Taylor–Hood-based discretization										
			$\|\mathbf{e_u}\|_{\ell^\infty(0,T;\mathbf{H}^1(\Omega))}$		$\|\mathbf{e_u}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_{\boldsymbol{\omega}}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$	
DoF	h	iter	error	rate	error	rate	error	rate	error	rate
483	0.707	2.1	1.56E-00	–	2.76E-03	–	4.48E-02	–	6.44E+01	–
2687	0.354	2.1	4.36E-01	1.842	3.79E-04	2.869	1.33E-02	1.750	4.33E-00	3.895
17655	0.177	2.1	1.12E-01	1.956	4.89E-05	2.952	3.09E-03	2.106	2.75E-01	3.976
86667	0.101	2.1	3.69E-02	1.988	9.21E-06	2.984	9.83E-04	2.047	3.05E-02	3.930
322043	0.064	2.1	1.50E-02	1.996	2.38E-06	2.994	3.95E-04	2.019	5.21E-03	3.909

MINI-element-based discretization										
			$\|\mathbf{e_u}\|_{\ell^\infty(0,T;\mathbf{H}^1(\Omega))}$		$\|\mathbf{e_u}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_{\boldsymbol{\omega}}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$	
DoF	h	iter	error	rate	error	rate	error	rate	error	rate
333	0.707	2.1	7.55E-00	–	1.08E-02	–	6.97E-02	–	1.27E+03	–
2027	0.354	2.1	4.53E-00	0.738	3.52E-03	1.615	2.36E-02	1.563	6.43E+02	0.986
14319	0.177	2.1	2.27E-00	0.999	8.77E-04	2.004	6.58E-03	1.842	1.87E+02	1.777
73017	0.101	2.1	1.29E-00	1.010	2.79E-04	2.046	2.32E-03	1.862	6.79E+01	1.812
276833	0.064	2.1	8.17E-01	1.007	1.12E-04	2.023	1.01E-03	1.848	3.02E+01	1.791

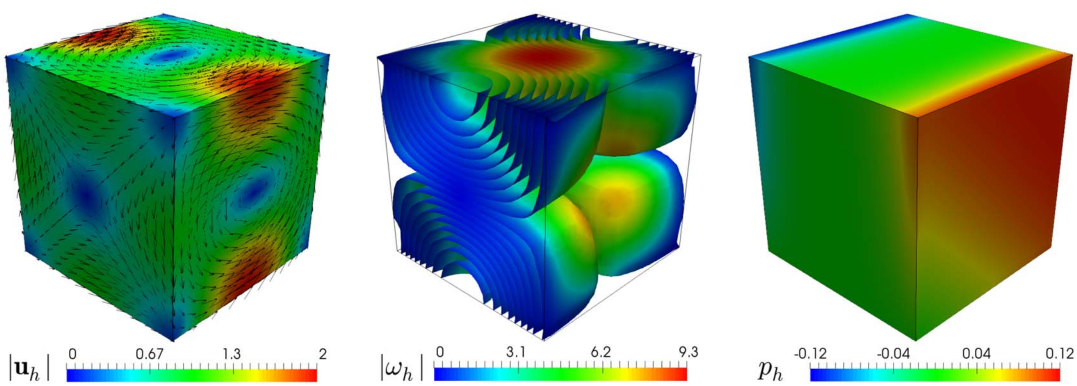

FIG. 2. [Example 2] Computed magnitude of the velocity and vorticity, and pressure field at time $T = 0.001$.

of the first column of Fig. 3. First, we consider the Kelvin–Voigt–Brinkman–Forchheimer model (2.3) in the whole domain Ω, with parameters $\rho = 3, \nu = 1$ and $\kappa = 1$, but with different values of the parameters D and F for the interior and the exterior of the channel, that is,

$$\mathtt{D} = \begin{cases} 1 & \text{in } \Omega_c \\ 1000 & \text{in } \Omega \setminus \Omega_c \end{cases} \quad \text{and} \quad \mathtt{F} = \begin{cases} 10 & \text{in } \Omega_c \\ 1 & \text{in } \Omega \setminus \Omega_c \end{cases}. \tag{6.1}$$

The parameter choice corresponds to high permeability ($\mathtt{D} = 1$) in the channel and increased inertial effect ($\mathtt{F} = 10$), compared to low permeability ($\mathtt{D} = 1000$) in the porous medium and reduced inertial effect ($\mathtt{F} = 1$). In addition, the body force term is $\mathbf{f} = \mathbf{0}$, the initial condition is zero, and the boundaries

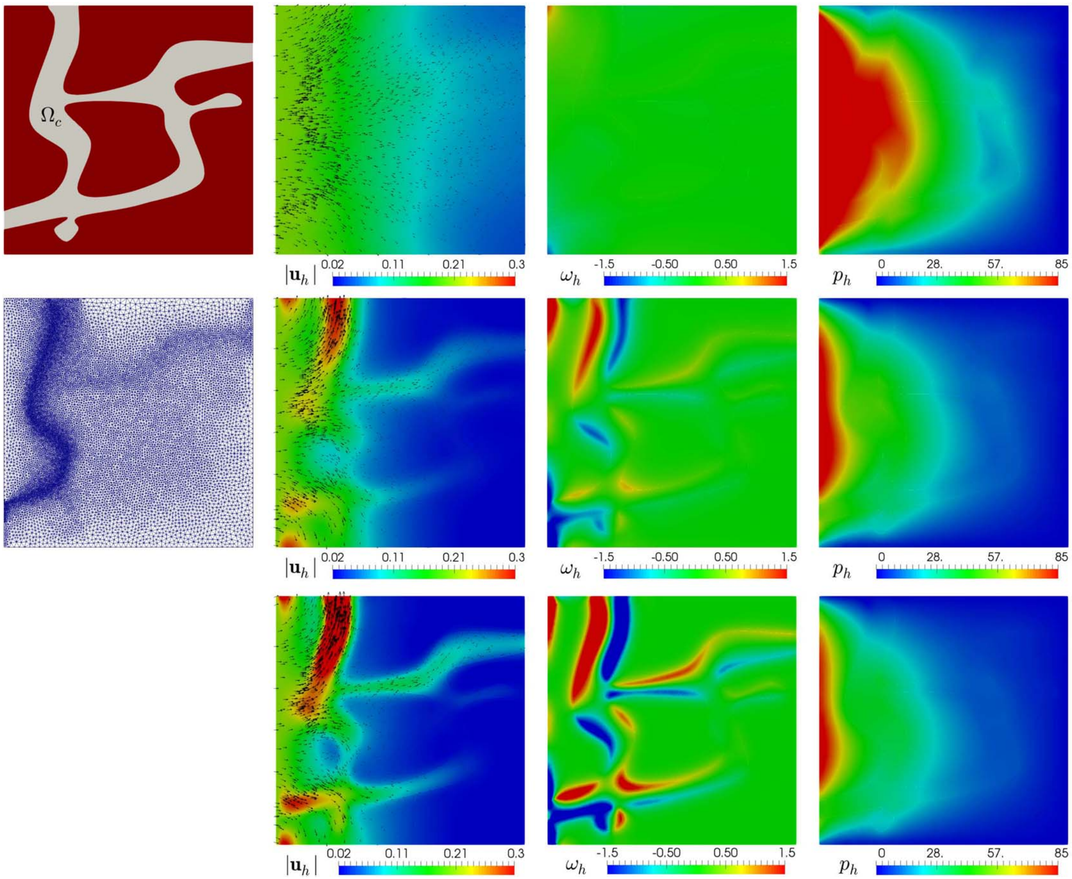

FIG. 3. [Example 3] Domain configuration and prescribed mesh (plots in first column), and computed magnitude of the velocity, vorticity and pressure field at time $T = 0.01$ (top plots), at time $T = 0.2$ (middle plots), and at time $T = 1$ (bottom plots).

conditions are

$$\mathbf{u}\cdot\mathbf{n} = 0.2, \quad \mathbf{u}\cdot\mathbf{t} = 0 \quad \text{on} \quad \Gamma_{\text{left}}, \quad \left(\kappa^2 \frac{\partial\, \nabla\mathbf{u}}{\partial\, t} - p\,\mathbf{I}\right)\mathbf{n} + \nu\,\boldsymbol{\omega}\mathbf{t} = \mathbf{0} \quad \text{on} \quad \Gamma \setminus \Gamma_{\text{left}},$$

which corresponds to inflow on the left boundary and zero viscoelastic stress outflow on the rest of the boundary.

In Fig. 3, we display the computed magnitude of the velocity, vorticity and pressure at times $T = 0.01, 0.2$ and 1, which were obtained using the MINI-element-based approximation on a mesh with 27 287 triangle elements and 109 682 DoF. As expected, we observe a faster flow through the channel network, accompanied by a significant change in vorticity across the interface between the channel and the porous medium. The pressure field decreases as time increases. This example illustrates the Kelvin–Voigt–Brinkman–Forchheimer model's capability to handle heterogeneous media with spatially varying parameters. It also demonstrates our three-field mixed finite element method's ability to resolve

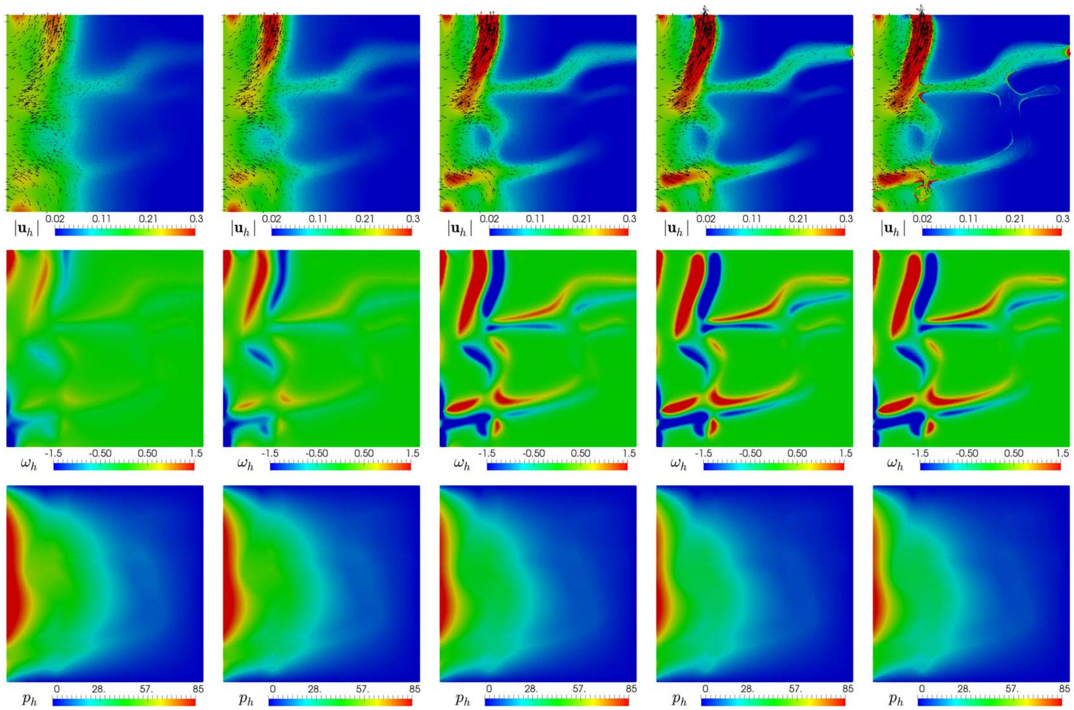

FIG. 4. [Example 3] Computed magnitude of the velocity, vorticity and pressure field at time $T = 1$, with $\rho = 3$, channel setting $\mathtt{F} = 10$ and $\mathtt{D} = 1$, and porous media setting $\mathtt{F} = 1$ and $\mathtt{D} = 1000$, for $\kappa \in \{3, 2, 1, 0.1, 0.01\}$ (from left to right).

sharp vorticities in the presence of strong jump discontinuities in the parameters. We further study the robustness of the method with respect to the elasticity parameter κ. In Fig. 4 we display the computed magnitude of the velocity, vorticity, and pressure for the settings given by (6.1), considering $\kappa \in \{3, 2, 1, 0.1, 0.01\}$. We observe that the elasticity parameter κ has a dissipative effect, reducing the velocity in the channel and slightly affecting the pressure in the entire domain, while the vorticity increases as κ decreases. This study illustrates that the method produces stable and physically reasonable results across a wide range of physical parameters, such as $\mathtt{D}$, $\mathtt{F}$ and κ.

7. Conclusions

In this paper, we presented a new velocity-vorticity-pressure formulation for the Kelvin–Voigt–Brinkman–Forchheimer equations and its mixed finite element approximation. The system models fast unsteady viscoelastic flows in highly porous media. The formulation has several advantages, including an accurate and smooth approximation of the vorticity, well posedness for large data, and optimal convergence rates without a mesh quasi-uniformity assumption. Well-posedness of the weak formulation, as well as stability and error analysis for the semidiscrete and fully discrete mixed finite element approximations are presented. The numerical results illustrate that the method is robust for a wide range of parameters, the ability of the system to model heterogeneous media exhibiting both Stokes and Darcy flow regimes, as well as the dissipative effect of the elasticity parameter. Future research directions include the study of a pseudostress and skew-symmetric-based mixed formulation for the problem, along with the use of Banach space techniques, as in Colmenares *et al.* (2020), Caucao *et al.* (2021) and Caucao *et al.* (2022), to naturally impose the non-homogeneous Dirichlet boundary condition and to obtain direct approximations of physical variables of interest, such as the velocity gradient and the viscoelastic stress tensor.

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