A multipoint stress–multipoint flux mixed finite element method for the Biot system of poroelasticity on cuboid grids

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Abstract. We present a mixed finite element method for a five-field formulation of the Biot system of poroelasticity on cuboid grids, which simplifies to a finite volume type system. The approach integrates a mixed stress–displacement–rotation formulation for elasticity with weak stress symmetry and a mixed velocity–pressure formulation for Darcy flow. A suitable choice of mixed finite element spaces with degrees of freedom for stress, rotation, and velocity associated with mesh vertices, and the use of the vertex quadrature rule allow for local elimination of stress, rotation, and velocity, resulting in a positive definite cell-centered displacement–pressure system. We provide stability and error analysis, demonstrating first-order convergence for all variables in their natural norms. Numerical experiments validate the theoretical convergence rates.

Keywords: Poroelasticity · Mixed finite element · Multipoint stress.

1 Introduction

The Biot system of poroelasticity [6] describes fluid flow through deformable porous media and has been extensively studied due to its diverse applications. These span geosciences, including groundwater remediation, hydraulic fracturing, and carbon sequestration, as well as biomedical fields, such as arterial flow modeling and organ tissue mechanics. The system comprises an equilibrium equation for the solid and a mass balance equation for the fluid, forming a fully coupled system: fluid pressure affects solid stress, while the divergence of solid displacement influences fluid content.

Extensive research has been conducted on the numerical solutions of the Biot system. Fully mixed formulations of the Biot system have been investigated [9, 12]. In [12], a stress and displacement mixed elasticity formulation is coupled with a mixed velocity–pressure formulation for Darcy flow. This approach was further developed in [9] by introducing a weakly symmetric stress–displacement–rotation elasticity formulation. They provide several benefits, including locking-free behavior, robustness with respect to physical parameters, local conservation of mass and momentum, and accurate approximations of stress and velocity with continuous normal components across element interfaces. Furthermore, these methods can accommodate discontinuous full-tensor permeabilities and Lamé coefficients, which frequently appear in subsurface flow modeling. However, a key challenge of fully mixed methods is the large algebraic system of saddle point type that must be solved at each time step.

In this paper we develop a mixed finite element method on cuboid grids for the five-field stress–displacement–rotation–velocity–pressure Biot formulation from [9], which reduces to a positive definite cell-centered displacement–pressure system. This method was previously introduced for simplicial and quadrilateral grids [4]. We refer to the method as a coupled multipoint stress mixed finite element (MSMFE) – multipoint flux mixed finite element (MFMFE) method. It builds upon the MSMFE method for elasticity [2, 3] and the MFMFE method for Darcy flow [10, 8]. We employ a suitable choice of mixed finite element spaces with degrees of freedom for stress, rotation, and velocity associated with mesh vertices, combined with a vertex quadrature rule for the stress, stress–rotation, and velocity bilinear forms. The vertex quadrature rule localizes the interaction of degrees of freedom at the vertices, leading to block-diagonal matrices. As a result, stress and velocity, followed by rotation, can be locally eliminated through small vertex-based linear systems. This elimination process reduces the five-field saddle point system to a positive definite cell-centered displacement–pressure system. We establish solvability, stability, and first-order error estimates. The analysis employs recent results on MSMFE methods on cuboid grids [11]. Numerical experiments are presented to illustrate the theory.

2 Model problem and a fully mixed weak formulation

In this section, we describe the poroelasticity system and its fully mixed formulation, incorporating weak stress symmetry. Let Ω be a simply connected, bounded domain in $\mathbb{R}^3$ occupied by a fluid-saturated poroelastic medium. We denote by $\mathbb{M}$, $\mathbb{S}$, and $\mathbb{N}$ the spaces of real 3×3 matrices, symmetric matrices, and skew-symmetric matrices, respectively. The divergence operator $\operatorname{div} : \mathbb{R}^3 \to \mathbb{R}$ represents the standard divergence for vector fields. It can also be applied to matrix fields, $\operatorname{div} : \mathbb{M} \to \mathbb{R}^3$, by computing the divergence row-wise. We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the $L^2(\Omega)$-norm and inner product, respectively, and use standard notation for Sobolev and Hilbert spaces. We define the spaces

$$H(\operatorname{div};\Omega,\mathbb{R}^3) = \{v \in L^2(\Omega,\mathbb{R}^3) \mid \operatorname{div} v \in L^2(\Omega)\},$$
$$H(\operatorname{div};\Omega,\mathbb{M}) = \{\tau \in L^2(\Omega,\mathbb{M}) \mid \operatorname{div} \tau \in L^2(\Omega,\mathbb{R}^3)\},$$

equipped with the norms, respectively,

$$\|v\|_{\operatorname{div}} = \left(\|v\|^2 + \|\operatorname{div} v\|^2\right)^{1/2} \quad \text{and} \quad \|\tau\|_{\operatorname{div}} = \left(\|\tau\|^2 + \|\operatorname{div} \tau\|^2\right)^{1/2}.$$

The stress-strain relation for a poroelastic material is given by $A\sigma_e = \epsilon(u)$, where $A(x) : \mathbb{S} \to \mathbb{S}$ (extendible to $A(x) : \mathbb{M} \to \mathbb{M}$) is a symmetric, bounded, and uniformly positive definite linear operator representing the compliance tensor.

Here, σ_e is the elastic stress, u the solid displacement, and $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. For a homogeneous, isotropic material, the compliance tensor is

$$A\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + 3\lambda}\mathrm{tr}(\sigma)I\right),$$

where I is the identity matrix, and $\mu > 0, \lambda \geq 0$ are the Lamé coefficients. In this case the elastic stress satisfies $\sigma_e = 2\mu\,\epsilon(u) + (\lambda \operatorname{div} u)I$. The poroelastic stress, which incorporating the fluid pressure p, is $\sigma = \sigma_e - \alpha p I$, where $0 < \alpha \leq 1$ is the Biot–Willis constant. Given a vector field f representing body forces and a source term q, the quasi-static Biot system [6] governing fluid flow within a poroelastic medium is formulated as:

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega \times (0,T], \tag{1}$$

$$K^{-1}z + \nabla p = 0 \quad \text{in } \Omega \times (0,T], \tag{2}$$

$$\partial_t(c_0 p + \alpha \operatorname{div} u) + \operatorname{div} z = q \quad \text{in } \Omega \times (0,T], \tag{3}$$

where z denotes the Darcy velocity, $c_0 > 0$ is the mass storativity coefficient, and K is a symmetric and positive definite tensor representing the permeability of the porous medium divided by the fluid viscosity. The system is supplemented with the boundary conditions:

$$u = g_u \quad \text{on } \Gamma_D^{\mathrm{displ}} \times (0,T], \quad \sigma n = 0 \quad \text{on } \Gamma_N^{\mathrm{stress}} \times (0,T], \tag{4}$$

$$p = g_p \quad \text{on } \Gamma_D^{\mathrm{pres}} \times (0,T], \quad z \cdot n = 0 \quad \text{on } \Gamma_N^{\mathrm{vel}} \times (0,T], \tag{5}$$

along with the initial condition $p(x,0) = p_0(x)$ in Ω, where $\Gamma_D^{\mathrm{displ}} \cup \Gamma_N^{\mathrm{stress}} = \Gamma_D^{\mathrm{pres}} \cup \Gamma_N^{\mathrm{vel}} = \partial\Omega$, and n represents the outward unit normal vector field on $\partial\Omega$.

We now introduce the five-field mixed weak formulation of (1)–(3), as proposed in [9], where stress symmetry is imposed weakly via the Lagrange multiplier $\gamma = \mathrm{Skew}(\nabla u)$, $\mathrm{Skew}(\tau) = \frac{1}{2}(\tau - \tau^T)$: find $(\sigma, u, \gamma, z, p) : [0,T] \mapsto \mathbb{X} \times V \times \mathbb{Q} \times Z \times W$ such that $p(0) = p_0$ and, for a.e. $t \in (0,T)$,

$$\begin{aligned}
&(A(\sigma + \alpha p I), \tau) + (u, \operatorname{div}\tau) + (\gamma, \tau) = \langle g_u, \tau\, n\rangle_{\Gamma_D^{\mathrm{displ}}}, && \forall \tau \in \mathbb{X},\\
&(\operatorname{div}\sigma, v) = -(f, v), && \forall v \in V,\\
&(\sigma, \xi) = 0, && \forall \xi \in \mathbb{Q},\\
&(K^{-1}z, \zeta) - (p, \operatorname{div}\zeta) = -\langle g_p, \zeta \cdot n\rangle_{\Gamma_D^{\mathrm{pres}}}, && \forall \zeta \in Z,\\
&(c_0\,\partial_t p, w) + \alpha(\partial_t A(\sigma + \alpha p I), wI) + (\operatorname{div} z, w) = (q, w), && \forall w \in W,
\end{aligned}$$

where the functional spaces are defined as:

$$\mathbb{X} = \{\tau \in H(\mathrm{div}; \Omega, \mathbb{M}) : \tau\, n = 0 \text{ on } \Gamma_N^{\mathrm{stress}}\}, \quad V = L^2(\Omega, \mathbb{R}^3), \quad \mathbb{Q} = L^2(\Omega, \mathbb{N}),$$
$$Z = \{\zeta \in H(\mathrm{div}; \Omega, \mathbb{R}^3) : \zeta \cdot n = 0 \text{ on } \Gamma_N^{\mathrm{vel}}\}, \quad W = L^2(\Omega).$$

3 Mixed finite element discretization

We now introduce the coupled MSMFE–MFMFE method. We assume that Ω can be covered by a shape-regular cuboid partition $\mathcal{T}_h$, where h is the maximum element diameter. Let $\mathbb{X}_h \times V_h \times \mathbb{Q}_h$ be the triple $(\mathcal{ERT}_0)^3 \times (\mathcal{Q}_0)^3 \times (\mathcal{Q}_1^{\text{cts}})^{3\times 3,\text{skew}}$, where $\mathcal{ERT}_0$ represents the lowest-order enhanced Raviart-Thomas space introduced in [1], and $\mathcal{Q}_k$ denotes the space of polynomials of degree k in each variable. For the Darcy flow discretization, we use the lowest-order $\mathcal{ERT}_0 \times \mathcal{Q}_0$ mixed finite element (MFE) spaces, denoted as $Z_h \times W_h$.

It was established in [1] that the degrees of freedom for $\mathcal{ERT}_0$ can be selected as the values of the normal fluxes at any four points on each face e of E; similarly, for normal stresses in the case of $(\mathcal{ERT}_0)^3$. In this work, we choose these points to be located at the vertices of e for both the velocity and stress spaces, see Fig 1. This choice is motivated by the application of the vertex quadrature rule, defined as follows. For any element-wise continuous vector or tensor functions φ and ψ on Ω, we approximate (φ, ψ) by

$$(\varphi, \psi)_Q = \sum_{E \in \mathcal{T}_h} \frac{|E|}{8} \sum_{i=1}^{8} \varphi(\mathbf{r}_i) \cdot \psi(\mathbf{r}_i),$$

where $\cdot$ denotes the inner product for both vector- and tensor-valued functions. The quadrature rule is applied to the velocity, stress, and stress–rotation bilinear forms.

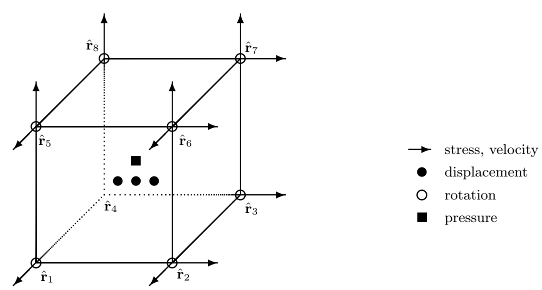

Fig. 1. Degrees of freedom of $\mathbb{X}_h \times V_h \times \mathbb{Q}_h \times Z_h \times W_h$.

Our method, referred to as the MSMFE–MFMFE method, is formulated in its semidiscrete form as follows: find $(\sigma_h, u_h, \gamma_h, z_h, p_h) : [0,T] \mapsto \mathbb{X}_h \times V_h \times \mathbb{Q}_h \times Z_h \times W_h$ such that $p_h(0) = p_{h,0}$ and, for a.e. $t \in (0,T)$,

$$(A(\sigma_h + \alpha p_h I), \tau)_Q + (u_h, \operatorname{div} \tau) + (\gamma_h, \tau)_Q = 0, \quad \forall \tau \in \mathbb{X}_h, \quad (6a)$$

$$(\operatorname{div} \sigma_h, v) = -(f, v), \quad \forall v \in V_h, \quad (6b)$$

$$(\sigma_h, \xi)_Q = 0, \quad \forall \xi \in \mathbb{Q}_h, \quad (6c)$$

$$(K^{-1} z_h, \zeta)_Q - (p_h, \operatorname{div} \zeta) = 0, \quad \forall \zeta \in Z_h, \quad (6d)$$

$$(c_0 \partial_t p_h, w) + \alpha(\partial_t A(\sigma_h + \alpha p_h I), wI)_Q + (\operatorname{div} z_h, w) = (q, w), \quad \forall w \in W_h, \quad (6e)$$

where for simplicity we set $g_u = 0$ and $g_p = 0$. The fully-discrete MSMFE–MFMFE method is based on the backward Euler time discretization.

4 Stability and error analysis

We state the inf–sup stability of the mixed Darcy and elasticity spaces, which is fundamental to the analysis. It is well known [7] that the spaces $Z_h \times W_h$ satisfy the following inf–sup condition:

$$\exists \beta_1 > 0 \text{ such that } \forall w_h \in W_h, \quad \sup_{0 \neq \zeta \in Z_h} \frac{(w_h, \operatorname{div} \zeta)}{\|\zeta\|_{\operatorname{div}} \|w_h\|} \geq \beta_1.$$

The inf–sup stability of the mixed elasticity spaces $\mathbb{X}_h \times V_h \times \mathbb{Q}_h$ with the vertex quadrature has been recently investigated in [11] for cuboid grids. In this analysis, a new auxiliary H(curl)-conforming matrix-valued space is constructed, which forms an exact sequence with the stress space. A matrix-matrix inf-sup condition is shown for the curl of this auxiliary space and the trilinear rotation space. The inf–sup condition is given by:

$$\exists \beta_2 > 0 \text{ such that } \forall v_h \in V_h, \; \xi_h \in \mathbb{Q}_h, \quad \sup_{0 \neq \tau \in \mathbb{X}_h} \frac{(v_h, \operatorname{div} \tau) + (\xi_h, \tau)_Q}{\|\tau\|_{\operatorname{div}} (\|v_h\| + \|\xi_h\|)} \geq \beta_2.$$

Using the above inf-sup conditions, the well-posedness, stability, and error analysis of the semidiscrete and fully-discrete MSMFE–MFMFE methods given in (6) follow the corresponding arguments presented in [4]. The following theorem gives an error bound for the semidiscrete formulation of the method.

Theorem 1. *Assuming that the solution to* (1)–(3) *is sufficiently smooth, there exists a positive constant C independent of h and c_0, such that the solution of* (6) *satisfies*

$$\begin{aligned}
&\|\sigma - \sigma_h\|_{L^\infty(0,T;H(\operatorname{div};\Omega))} + \|u - u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\gamma - \gamma_h\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad + \|z - z_h\|_{L^\infty(0,T;L^2(\Omega))} + \|p - p_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\sigma - \sigma_h\|_{L^2(0,T;H(\operatorname{div};\Omega))} \\
&\quad + \|u - u_h\|_{L^2(0,T;L^2(\Omega))} + \|\gamma - \gamma_h\|_{L^2(0,T;L^2(\Omega))} + \|z - z_h\|_{L^2(0,T;H(\operatorname{div};\Omega))} \\
&\quad + \|p - p_h\|_{L^2(0,T;L^2(\Omega))} \leq Ch.
\end{aligned}$$

5 Reduction to a cell-centered system

In this section, we demonstrate that the fully discrete form of the MSMFE–MFMFE method (6) results in an algebraic system at each time step that can be reduced to a positive definite cell-centered displacement–pressure system.

Let σ denote the algebraic vector corresponding to σ_h^n, with similar notation for the other variables. The algebraic system at each time step takes the form:

$$\begin{pmatrix} A_{\sigma\sigma} & A_{\sigma u}^T & A_{\sigma\gamma}^T & 0 & A_{\sigma p}^T \\ -A_{\sigma u} & 0 & 0 & 0 & 0 \\ -A_{\sigma\gamma} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{zz} & A_{zp}^T \\ A_{\sigma p} & 0 & 0 & -A_{zp} & A_{pp} \end{pmatrix} \begin{pmatrix} \sigma \\ u \\ \gamma \\ z \\ p \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix}. \tag{7}$$

The application of the vertex quadrature rule to the stress and velocity bilinear forms, $(A\sigma_h^n, \tau)_Q$ and $(K^{-1}z_h^n, \zeta)_Q$, respectively, leads to the corresponding matrices $A_{\sigma\sigma}$ and A_{zz} being block-diagonal, with blocks associated with the mesh vertices. Specifically, consider an interior vertex $\mathbf{r}$ that is shared by 12 faces $e_1, \dots, e_{12}$. Let $\zeta_1, \dots, \zeta_{12}$ denote the velocity degrees of freedom associated with this vertex, and let $z_1, \dots, z_{12}$ represent the corresponding normal velocity values. The vertex quadrature rule $(K^{-1}\cdot, \cdot)_Q$ localizes the interaction of basis functions around each vertex, effectively decoupling them from other basis functions. As a result, selecting $\zeta_1, \dots, \zeta_{12}$ yields a local 12×12 linear system. Consequently, the matrix A_{zz} is block-diagonal, with 12×12 blocks corresponding to mesh vertices. Similarly, the matrix $A_{\sigma\sigma}$, which arises from the stress bilinear form, is block-diagonal with 36×36 blocks (see Fig. 1). The blocks of $A_{\sigma\sigma}$ and A_{zz} are positive definite. This property enables efficient elimination of the velocity and stress variables by solving small local linear systems.

Furthermore, the rotation variable can be eliminated as follows. Let $A_{\sigma\gamma}$ denote the matrix corresponding to $(\sigma_h^n, \xi)_Q$. The vertex quadrature rule ensures that $A_{\sigma\gamma}$ is block-diagonal with 3×36 blocks. After eliminating the stress, the rotation matrix simplifies to the product $A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T$. Since $A_{\sigma\sigma}$ is block-diagonal with 36×36 blocks, the resulting matrix $A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T$ is block-diagonal with 3×3 blocks. This structure allows the rotation variable to be eliminated, ultimately reducing system (7) to a cell-centered displacement–pressure system:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}, \tag{8}$$

where,

$$\begin{aligned}
A_{11} &= A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma u}^T - A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T(A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T)^{-1}A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma u}^T, \\
A_{12} &= A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma p}^T - A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T(A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T)^{-1}A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma p}^T, \\
A_{21} &= -A_{\sigma p}A_{\sigma\sigma}^{-1}A_{\sigma u}^T + A_{\sigma p}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T(A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T)^{-1}A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma u}^T, \\
A_{22} &= A_{pp} - A_{\sigma p}A_{\sigma\sigma}^{-1}A_{\sigma p}^T + A_{zp}A_{zz}^{-1}A_{zp}^T \\
&\quad + A_{\sigma p}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T(A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T)^{-1}A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma p}^T.
\end{aligned}$$

The following theorem, which can be established using the arguments in [4, Proposition 7.1], ensures the desirable properties of the reduced system matrix.

Lemma 1. *The matrix of the cell-centered displacement–pressure system* (8) *is block skew-symmetric and positive definite.*

Remark 1. The positive definiteness of the matrix in (8) enables the use of an efficient Krylov subspace iterative solver, such as GMRES, for solving the reduced displacement–pressure system. Furthermore, since the diagonal blocks are symmetric and positive definite, the block-diagonal part of the matrix serves as an effective preconditioner.

6 Numerical results

This section presents numerical experiments to validate the theoretical results discussed in the previous sections. We evaluate the convergence of the proposed fully discrete MSMFE–MFMFE method on cuboid grids. The implementation was performed using the deal.II finite element library [5].

We consider the unit cube $\Omega = (0,1)^3$ as the computational domain and choose the analytical solution for pressure and displacement as follows:

$$p = \cos(t)(x+y+z+1.5),$$
$$u = \sin(t)\begin{pmatrix} 0 \\ -(e^x-1)\left(y-\cos\left(\frac{\pi}{12}\right)\left(y-\frac{1}{2}\right)+\sin\left(\frac{\pi}{12}\right)\left(z-\frac{1}{2}\right)-\frac{1}{2}\right) \\ -(e^x-1)\left(z-\sin\left(\frac{\pi}{12}\right)\left(y-\frac{1}{2}\right)-\cos\left(\frac{\pi}{12}\right)\left(z-\frac{1}{2}\right)-\frac{1}{2}\right) \end{pmatrix}.$$

The permeability tensor is given by:

$$K = \begin{pmatrix} x^2+y^2+1 & 0 & 0 \\ 0 & z^2+1 & \sin(xy) \\ 0 & \sin(xy) & x^2y^2+1 \end{pmatrix}. \tag{9}$$

The other parameters are $\mu = 100.0$, $\lambda = 100.0$, $c_0 = 1.0$, $\alpha = 1.0$, and $T = 10^{-3}$. Pressure and displacement boundary conditions are imposed on the boundary.

In Table 1, we present the relative errors and spatial convergence rates for a sequence of mesh refinements in the norm $\|\cdot\|_{l^2(0,T;L^2(\Omega))}$. A sufficiently small time step, $\Delta t = 10^{-4}$, is chosen to ensure that the time discretization error remains negligible. As predicted by the theory, we observe at least first-order convergence for all variables. We also observe second order superconvergence for the displacement and the pressure at the cell-centers, which will be the topic of further investigation.

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References

1. Ambartsumyan, I., Khattatov, E., Lee, J.J., Yotov, I.: Higher order multipoint flux mixed finite element methods on quadrilaterals and hexahedra. Math. Models Methods Appl. Sci. **29**(6), 1037–1077 (2019)
2. Ambartsumyan, I., Khattatov, E., Nordbotten, J.M., Yotov, I.: A multipoint stress mixed finite element method for elasticity on simplicial grids. SIAM J. Numer. Anal. **58**(1), 630–656 (2020)
3. Ambartsumyan, I., Khattatov, E., Nordbotten, J.M., Yotov, I.: A multipoint stress mixed finite element method for elasticity on quadrilateral grids. Numer. Methods Partial Differential Equations **37**(3), 1886–1915 (2021)

Table 1. Numerical errors and convergence rates.

	$\|\sigma - \sigma_h\|$		$\|\text{div}(\sigma - \sigma_h)\|$		$\|u - u_h\|$		$\|\gamma - \gamma_h\|$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
1/2	5.04E–02	–	3.05E–03	–	4.33E+00	–	1.69E+00	–
1/4	2.22E–02	1.19	1.54E–03	0.99	1.09E+00	1.99	5.44E–01	1.63
1/8	1.01E–02	1.13	7.69E–04	1.00	2.99E–01	1.87	1.61E–01	1.75
1/16	4.86E–03	1.06	3.85E–04	1.00	9.79E–02	1.61	4.73E–02	1.77

	$\|z - z_h\|$		$\|\text{div}(z - z_h)\|$		$\|p - p_h\|$		$\|Q_h^u u - u_h\|$		$\|Q_h^p p - p_h\|$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
1/2	4.28E–02	–	2.46E–01	–	8.22E–02	–	5.30E+00	–	8.85E–05	–
1/4	1.07E–02	2.00	1.23E–01	1.00	4.11E–02	1.00	1.11E+00	2.26	3.46E–05	1.35
1/8	2.68E–03	2.00	6.19E–02	0.99	2.06E–02	1.00	2.65E–01	2.06	1.01E–05	1.78
1/16	6.72E–04	2.00	3.10E–02	1.00	1.03E–02	1.00	6.58E–02	2.01	2.46E–06	2.04

4. Ambartsumyan, I., Khattatov, E., Yotov, I.: A coupled multipoint stress–multipoint flux mixed finite element method for the Biot system of poroelasticity. Comput. Methods Appl. Mech. Engrg. **372**, 113407, 29 (2020)
5. Arndt, D., Bangerth, W., Davydov, D., Heister, T., Heltai, L., Kronbichler, M., Maier, M., Pelteret, J.P., Turcksin, B., Wells, D.: The `deal.II` library, version 8.5. J. Numer. Math. **25**(3), 137–146 (2017)
6. Biot, M.A.: General theory of three-dimensional consolidation. J. Appl. Phys. **12**(2), 155–164 (1941)
7. Boffi, D., Brezzi, F., Fortin, M.: Mixed finite element methods and applications, Springer Series in Computational Mathematics, vol. 44. Springer, Heidelberg (2013)
8. Ingram, R., Wheeler, M.F., Yotov, I.: A multipoint flux mixed finite element method on hexahedra. SIAM J. Numer. Anal. **48**(4), 1281–1312 (2010)
9. Lee, J.J.: Robust error analysis of coupled mixed methods for Biot's consolidation model. J. Sci. Comput. **69**(2), 610–632 (2016)
10. Wheeler, M.F., Yotov, I.: A multipoint flux mixed finite element method. SIAM J. Numer. Anal. **44**(5), 2082–2106 (2006)
11. Yazici, I., Yotov, I.: Multipoint stress mixed finite element methods for elasticity on cuboid grids. arXiv preprint arXiv:2502.00163 (2025)
12. Yi, S.Y.: Convergence analysis of a new mixed finite element method for Biot's consolidation model. Numer. Methods Partial Differential Equations **30**(4), 1189–1210 (2014)